O-MINIMALITY AND UNIFORMITY IN $n-1$ GRAPHS

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1 Introduction

Disclaimer: This is a first draft of notes intended for a talk. I haven't read over everything, so there may be (grave) mistakes. The results are also not original enough to warrant much circulation, either (they were realized in the span of a single day)- just to sketch a program for finding applications of real algebraic geometry and o-minimality to $n-1$ graphs.

Much of the recent work on inverse problems in electrical networks has been focused on painstakingly constructing families of graphs and specific response matrices A such that the fiber under the Dirichlet-to-Neumann map $\Lambda : (\mathbb{R}^+)^{|E|} \to M_\ell\mathbb{R}$ (given by taking the Schur complement of the Kirchoff matrix of the network) satisfies $|\Lambda^{-1}(A)| = n$ for a given n. This approach is primarily *constructive*, leaving open many questions about how the fibers of the Dirichlet-to-Neumann map in a general $n-1$ graphs behave and what restrictions are there, in general, on the cardinality of these fibers.

In this note I will sketch a program, rooted in real algebraic geometry and its modern incarnation *o-minimality*, to understand generic features of $n-1$ graphs. Among the results proved are:

- Let $X = \Lambda((\mathbb{R}^+)^{|E|})$ be the image of the Dirichlet-to-Neumann map. Then we can partition $X = X_1 \cup \cdots \cup X_m$ into a finite union of *nice* subsets such that $\Lambda : \Lambda^{-1}(X_m) \to X_m$ is a fiber bundle with *definable* bijection $h_m : \Lambda^{-1}(X_m) \to X_m \times F_m$. In particular, since this decomposition is finite, we get that the Dirichlet-to-Neumann map for a graph G admits only finitely many cardinalities of fibers.

- The only cardinals κ such that $|\Lambda^{-1}(A)| = \kappa$ are natural numbers $\kappa = n$ and the cardinality of the continuum $\kappa = |\mathbb{R}| = \mathfrak{c}$.

Both of these results are straightforward applications of two black boxes in the theory of o-minimal structures: the cell-decomposition theorem and the local triviality theorem. However, the proofs of these black-box theorems are highly non-constructive and merely show the existence of certain sets and maps. It would be great to find a suitably large family of graphs for which the fibers of the Dirichlet-to-Neumann map are very nicely behaved so that we can actually produce this partition $X = X_1 \cup \cdots \cup X_m$.

Furthermore, these results were gotten almost purely by using the fact that taking the Schur complement of a matrix is given by a finite family of polynomial equations so that our Dirichlet-to-Neumann map Λ was in fact definable in the o-minimal structure $(\mathbb{R}, +, \times, <, \{c\}_{c\in\mathbb{R}})$. It is a hope of mine that we can somehow use the framework of o-minimality applied to some other o-minimal structure to talk about $n-1$ recovery of *functions* (and not just conductances) in certain nonlinear networks.

2 Languages and Structures

To talk about o-minimality, we must first refresh ourselves on the basics objects of model theory. In a nutshell, model theory is concerned with studying the fine structure of the *definability* of mathematical objects within other ones. The notion of a formal language (and its corresponding interpretation) are needed so that we can precisely state what we mean by definability and also to make it clear that mathematical objects mean nothing a language to probe them.

Definition 1 (Marker)**.** A *language* $\mathcal{L} = \langle \mathcal{C}, \mathcal{R}, \mathcal{F} \rangle$ is a set given by the following data:

1. All logical symbols and variables; e.g., $\wedge$ for conjunction, $\vee$ for disjunction, $\neg$ for negation, $=$ for equality, $\forall$ and $\exists$ for quantifiers, and tuples $\overline{x} = (x_1, \cdots, x_n)$ for tuples of variables.
2. A set of *constant symbols* $\mathcal{C}$ consisting of elements c, giving us the parameters of our language
3. A set of *relation symbols* $\mathcal{R}$ consisting of pairs (R, n_R) where n_R is the *arity* of R.
4. A set of *function symbols* $\mathcal{F}$ consisting of pairs (f, n_f) where n_f is the arity of f.

Remark 1. It suffices to take only countably many variables since we'll be working exclusively with first-order sentences, which by definition only involve finitely many variables.

There are many, many examples of languages that arise in mathematics. A key examples of languages include the language of rings $\mathcal{L}_{\text{ring}} = \langle \{0,1\}, \varnothing, \{(+,2), (\cdot,2), (-,2)\}\rangle$. We will work under the convention that, since all of our languages will come from some natural mathematical object, that we'll omit the stratification of $\mathcal{L}$ into classes and omit the arities of function symbols and relations. For instance, we would write $\mathcal{L}_{\text{ring}}$ as $\langle 0, 1, +, -, \cdot\rangle$. Note that the definition of a language is purely formal; it is simply a triple of sets, existing independently of any semantics. The definition of *interpretations* and *structures* below is what allow us to bootstrap semantics onto syntax.

Definition 2 (Marker)**.** An $\mathcal{L}$ structure $\mathbb{M}$ is given by a set M (also written $|\mathbb{M}|$), the "universe" of $\mathbb{M}$, together with *interpretations* of the language $\mathcal{L}$:

1. A constant $c^{\mathbb{M}} \in M$ for every symbol $c \in \mathcal{C}$
2. A function $f^{\mathbb{M}} : M^{n_f} \to M$ for every symbol $f \in \mathcal{F}$
3. A set $R^{\mathbb{M}} \subseteq M^{n_R}$ for every relation symbol $R \in \mathcal{R}$
4. For every logical symbol, the relativization of it to $\mathbb{M}$; for instance $=^{\mathbb{M}}$ is just set theoretic equality.

Notation-wise we will write M for $\mathbb{M}$ and drop the superscripts as long as it doesn't cause confusion. In the definition, note that the only natural assignment we force is that of the logical symbols; the constants, relations, and function symbols can be assigned to any constants, relations, and functions you want. To illustrate this, consider $\mathcal{L}_{\text{ring}}$ and the two following $\mathcal{L}_{\text{ring}}$ structures Z_1 and Z_2. Let $|Z_1| = |Z_2| = \mathbb{Z}$. In Z_1 let all of the symbols be interpreted in the regular way, but in Z_2 set $0^{Z_2} = 1^{Z_2} = 43479397$ and $+^{Z_2} = -^{Z_2} = \cdot^{Z_2}$ be the zero function. Then both Z_1 and Z_2 are $\mathcal{L}_{\text{ring}}$ structures; Z_1 is the standard interpretation of $\mathcal{L}_{\text{ring}}$ for $\mathbb{Z}$ while Z_2 is something ridiculous. To better restrict the type of interpretations we care about, we define the notions of *satisfaction*, *theory*, and *model*.

To do this we'll also need the notion of a *well-formed formula* (just called a formula) of a language $\mathcal{L}$, but defining it technically is a mess. Essentially, a well-formed formula is a string of symbols that actually makes sense. To get the jist of it, consider the following examples in a language $\mathcal{L} = \langle 3, f\rangle$:

$\vee\vee\vee(\exists(\overline{x})f(\overline{x}) = 3)$ is not a formula while $\exists(\overline{x})\forall(y)(f(y) = x)$ is.

See [Marker] for a more detailed discussion. A variable x_i in a formula $\phi(\overline{x})$ is *free* if at least one occurence of x_i does not occur in the scope of some quantifier, otherwise it is a *bound* variable. For example, with $\phi(x, y) := \exists x(x = y)$, x is bound while y is free. A formula is a *sentence* if it has no free variables.

Definition 3. An $\mathcal{L}$-*theory* T is set of $\mathcal{L}$-sentences.

Given a sentence ϕ, we say that $M \vDash \phi$ just in case ϕ^M is a true statement in M. In this case M *satisfies* ϕ

We say that M is a model of T, written $M \vDash T$, if for all $\phi \in T$, $M \vDash \phi$.

For example, consider $\mathcal{L}_{\text{ring}}$ and let T be the set of axioms of a ring (it's trivial to verify that these are all first order sentences). Then with $M = \mathbb{Q}$, giving $\mathbb{Q}$ the standard interpretation, then $M \vDash T$. Note also that if $M \vDash T$ then M is a ring.

3 Definability and Tame Geometry

We now turn to extensively considering issues of semantics tied to what kind of sets we can define in a given structure.

Definition 4. Let M be an $\mathcal{L}$ structure. A set $X \subseteq M^n$ is definable (without parameters) if there is a formula $\phi(\overline{x})$ such that

$$X = \{\overline{b} \in M^n \mid M \vDash \phi(\overline{b})\}$$

Note here that we don't suppose that $\overline{b}$ is a tuple of constants in our language; $\overline{b}$ are just elements of M^n, while constants are elements of M^n that can appear in ϕ. Note also that if we want to look at things definable *from* some element $a \in M$ (that is, by taking formulas $\phi(x, \overline{y})$ and substituting a in for x), then we can simply add $\{a\}$ to our language $\mathcal{L}$.

In general, the structure of definable sets can be incredibly wild or quite tame. For example, with $\mathbb{N}$ equipped with the standard interpretations of $\langle 0, 1, +, \cdot \rangle$, it can be shown that many subsets are uncomputable (by turning a halting problem into an arithmetical statement), so that the definable sets are very hard to describe. On the polar opposite, though, it turns out that in $\mathcal{L}_{\text{ring}}$, the definable sets of $\mathbb{C}$ are simply boolean combinations of Zariski closed sets, which are relatively easy to write down.

We also wish to consider how definable sets of the same structure can be morphed into each other; for instance, in $\langle \mathbb{Z}, +, -, 0, 1, \geq \rangle$ (under the standard interpretations) we have that the sets $A = \{a \in \mathbb{Z} \mid a \geq 0\}$ and $B = \{b \in \mathbb{Z} \mid b \geq 1\}$ have a bijection given by the mapping $f : A \to B$ sending $x \mapsto x + 1$. That this function was expressible within the language is important- if we allow any old bijection to make two sets "equivalent" relative to M then we lose all of our model-theoretic structure. Indeed, this leads to the definition of a definable function:

Definition 5. Let $X \subseteq M^n$, $Y \subseteq M^m$ be definable sets. Then a function $f : X \to Y$ is said to be definable if the graph $\Gamma(f) \subseteq M^{n+m}$ is definable. If f is a bijection then we call f a *definable bijection*.

In the motivating case above, $\Gamma(f) = \{(a, b) \mid (a \geq 0) \wedge (b \geq 1) \wedge (b = a + 1)\} \subseteq \mathbb{Z}^2$ is clearly definable, so that f is a definable bijection.

One of the main obstructions to understanding the structure of definable sets is the presence of quantifiers. In the case of $\mathbb{N}$ above, the sets definable without quantifiers are solution sets of multivariate diophantine polynomials, which by Matiyasevich's theorem [Matiyasevich] implies that they are computable. But as mentioned above, general definable subsets of $\mathbb{N}$ in the language of monoids can be uncomputable, so that the extra complexity comes precisely from the fact that we can't force every formula $\psi \in \mathcal{L}_{\text{monoid}}$ to be equivalent to a quantifier free formula ϕ, assuming $\text{Th}(\mathbb{N}, \mathcal{L}_{\text{monoid}})$. This gives rise to the following definition:

Definition 6. An $\mathcal{L}$ theory T *eliminates quantifiers* if for every formula $\psi \in \mathcal{L}$ there exists a formula $\phi \in \mathcal{L}$ with no quantifiers such that for all models $M \vDash T$, $M \vDash \psi \leftrightarrow \phi$. Equivalently, T admits elimination of quantifiers if for every $\psi \in \mathcal{L}$ there exists a quantifier-free ϕ such that in every model M of T, ϕ and ψ define the same set.

Quantifier elimination of the theory of the real ordered field $(\mathbb{R}, +, \times, <, \{c\}_{c\in\mathbb{R}})$ was a very big deal when it was proven:

Theorem 1 (Tarski-Seidenberg)**.** *The theory of* $(\mathbb{R}, +, \times, <, \{c\}_{c\in\mathbb{R}})$ *admits quantifier elimination.*

In particular, this means that every formula with quantifiers ϕ is equivalent to a formula ψ without quantifiers; hence every formula "is" after putting it in a standard form a formula that is a boolean combination of solution sets of polynomials and polynomial inequalities. In particular, every subset of $\mathbb{R}$ itself is a finite union of points (gotten from solutions of a univariate polynomial) and intervals (given by the inequalities). It turns out that this condition: having every subset of 1-dimensional space be a finite union of points and intervals guarantees an incredibly rich structure theory for general definable subsets.

Definition 7. A structure M in a language expanding $\{<\}$ is o-minimal if $<^M$ is a total order and every definable subset of M is a finite union of points and intervals. A theory T is o-minimal if every $M \vDash T$ is o-minimal.

As such, the Tarski-Seidenberg result has the following corollary:

Corollary 1. *The theory of* $(\mathbb{R}, +, \times, <, \{c\}_{c\in\mathbb{R}})$ *is o-minimal.*

With this result in mind, it's time to investigate the consequences of o-minimality. The first that we will try to understand is the statement of cell-decomposition, which shows that definable subsets of $\mathbb{R}^n$ are particularly nicely behaved. To have a cell decomposition, of course, we need a definition of a cell!

Definition 8 (vdD 2.2.3)**.** We define an m-cell in $\mathbb{R}^\ell$ inductively.

- A 0-cell in $\mathbb{R}$ is a point $* \in \mathbb{R}$
- A 1-cell in $\mathbb{R}$ is in interval $(a, b) \subseteq \mathbb{R}$
- Given an i-cell $C \subseteq \mathbb{R}^\ell$, we can construct cells in $\mathbb{R}^{\ell+1}$ as follows:

  - If $f : C \to \mathbb{R}$ is a definable function then the graph $\Gamma(f) = \{(x, f(x)) \subseteq \mathbb{R}^\ell \times \mathbb{R} \,|\, x \in C\}$ is an i-cell

  - If $f, g : C \to \mathbb{R}$ are definable functions (or $f = \pm\infty$) with $f < g$ then the set $\{(x, s) \,|\, x \in C, f(x) < s < g(x)\}$

Now we can define a cell decomposition as follows:

Definition 9 (vdD 2.2.10)**.** A cell decomposition of $\mathbb{R}^n$ is a partition of $\mathbb{R}^n = C_1 \cup \cdots \cup C_n$ where the C_i disjoint cells satisfying the following inductive definition:

- A cell decomposition of $\mathbb{R}$ is a partition $\mathbb{R} = C_1 \cup \cdots \cup C_n$ where the C_i are disjoint intervals or points.

- A cell decomposition of $\mathbb{R}^n = C_1 \cup \cdots \cup C_n$ into cells such that $\pi(C_1) \cup \cdots \cup \pi(C_n)$ (where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$) is projection) is a cell decomposition.

A cell decomposition subordinate to a definable $S \subseteq \mathbb{R}^n$ is a cell decomposition of $\mathbb{R}^n$ such that each cell C_i occuring in the decomposition is such that either $C_i \subseteq S$ or $C_i \subseteq \mathbb{R}^n \setminus S$.

We're now in the position to state the cell decomposition theorem (which holds generally for an o-minimal theory or structure)

Theorem 2. *Given a definable $S \subseteq \mathbb{R}^n$, there is a cell decomposition of $\mathbb{R}^n$ subordinate to S.*

We'll see an application of the cell decomposition theorem shortly. It is also one of the key technical theorems used to prove refined uniformity results about families of definable sets. A key example of this, useful to our consideration of $n-1$ graphs, is the so-called *trivialization* theorem. It states roughly that given a surjective definable map $f : S \to T$ we can partition T into definable subsets T_i such that we can *definably* parametrize the fibers.

We begin with the definition of a *trivial* definable map.

Definition 10. Let $f : S \to T$ be a definable map. f is said to be *definably trivial* if the following diagram commutes

$$\begin{tikzcd} S \arrow[r, "(f,\lambda)", "\sim"'] \arrow[d, "f"] & T \times F \arrow[dl, "\pi_T"] \\ T & \end{tikzcd}$$

Where $F \subseteq \mathbb{R}^\ell$ and $\lambda : S \to F$ are definable.

Note, in particular, that $\lambda : S \to F$ is a surjective definable map, meaning that λ gives us a formula for generating the fibers To see this, let $f^{-1}(t)$ be a fiber. Then as the diagram commutes, we have that $\pi_T^{-1}(t) = \{t\} \times F$ and so we get a definable bijection $f^{-1}(a) \to f^{-1}(t)$. As such, this definable map λ encodes a great deal of information about the fibers of f and indeed is (part of a) parametrization of the fibers as we vary over the elements of T.

Theorem 3 (vdD 9.1.2)**.** *Suppose that $S \subseteq \mathbb{R}^n$, $T \subseteq \mathbb{R}^m$ are definable sets and $f : S \to T$ is a continuous definable map. Then there is a partition $T = T_1 \cup \cdots \cup T_\ell$ such that $f : f^{-1}(T_i) \to T_i$ is a definably trivial map.*

4 Applications to $n-1$ Graphs

We now state and prove our results above:

Theorem 4. *The only cardinals κ such that $|\Lambda^{-1}(A)| = \kappa$ are natural numbers $\kappa = n$ and the cardinality of the continuum $\kappa = |\mathbb{R}| = \mathfrak{c}$.*

Proof. The set $F = \Lambda^{-1}(A) \subseteq (\mathbb{R}^+)^{|E|}$ is definable, being the solution set of a polynomial map. Hence we can decompose $F = F_1 \cup \cdots \cup F_n$ into finitely many disjoint cells. If $\dim F_i > 0$ for any i then by the inductive criterion for cells, F_i has cardinality $\mathfrak{c}$. If $\dim F_i = 0$ for all i then F is the finite union of points. Hence $|F|$ is either finite or $\mathfrak{c}$. $\square$

Theorem 5. *Let $X = \Lambda((\mathbb{R}^+)^{|E|})$ be the image of the Dirichlet-to-Neumann map. Then we can partition $X = X_1 \cup \cdots \cup X_m$ into a finite union of definable subsets such that $\Lambda : \Lambda^{-1}(X_m) \to X_m$ is a fiber bundle with a bijective semialgebraic map $h_m = (\Lambda, h) : \Lambda^{-1}(X_m) \to X_m \times F_m$.*

Proof. As Λ is definable, the local triviality theorem guarantees that we can partition $T = T_1 \cup \cdots \cup T_n$ such that, with F_i being a representative fiber of T_i, the following diagram commutes:

$$\begin{tikzcd} \Lambda^{-1}T_i \arrow[r, "(\Lambda,h)", "\sim"'] \arrow[d, "\Lambda"] & T_i \times F_i \arrow[dl, "\pi_T"] \\ T_i & \end{tikzcd}$$

□

This fiber bundle result immediately implies the following corollary:

Corollary 2. *For every graph G, the set $\{n \mid \exists A \in M_\ell\mathbb{R} \; |\Lambda^{-1}(A)| = n\}$ is finite.*

Proof. Since $X = \Lambda((\mathbb{R}^+)^{|E|})$ can be decomposed into finitely many definable subsets over which Λ is a fiber map, and since the cardinality of a fiber bundle is constant, there must be finitely many definable subsets □

5 Further Directions

This section lists a few problems that I think would be very interesting to consider. It's also possible that none of these avenues will pan out at all, but that'd be good to know too!

- The proof of cell decomposition is long, nonconstructive, and horribly technical. (It took about 3 lectures for me to prove it to Jim last year!) Because of this, I'm led to the following probably ill-posed question: What conditions on the graph G (or on the Dirichlet-to-Neumann map Λ) would ensure that the cell decomposition can be done in a suitably algorithmic way?
- The proof of the trivialization theorem (the local fiber-bundle theorem) depends on the cell-decomposition theorem, and beyond that it's unclear to me how the actual definable bijection $h_m : \Lambda^{-1}(X_m) \to X_m \times F_m$ is defined in the proof (the argument is really detailed and it's hard to see the forest through the trees). If we can see how to define this definable h_m given a computable cell decomposition, that'd be great because then we'd have an algorithmic method of parametrizing conductances that give rise to $|F_m| - 1$ graphs.
- Can the (cell decomposition of) X admit isolated points?
- Can we use o-minimality to get good upper bounds for the cardinality of a fiber based on some numerical invariant attached to the graph G "detectable" by the o-minimal structure?
- o-minimality is not unique to the real field $\mathbb{R}$. In fact, any *real closed field* is o-minimal, and this includes certain spaces of functions such as the *Puiseux series field.* It's conceivable that under certain circumstances the Dirichlet-to-Neumann map of a nonlinear electrical network is definable with respect to an o-minimal structure and we would get the same results as above.

To attack any of these problems, it would be a good idea to take a look at van den Dries' book "Tame Topology and o-minimal Structures." (Please don't recall it from the library though- I currently have it and don't want it recalled while I'm in LA!) I'll be gone to LA for three weeks and can meet via Skype and email if you want to talk with me more about this project.

6 Bibliography

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