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Matroids, Generalized Networks, and Electric Network Synthesis

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Matroid theory has been applied to solve problems in generalized assignment, operations research, control theory, network theory, flow theory, generalized flow theory or linear programming, coding theory, and telecommunication network design. The operations of matroid union, matroid partitioning, matroid intersection, and the theorem on the greedy algorithm, Rado's theorem, and Brualdi's symmetric version of Rado's theorem have been important for some of these applications. In this paper we consider the application of matroids to solve problems in network synthesis. Previously Bruno and Weinberg defined a generalized network, which is a network based on a matroid rather than a graph; for a generalized network the duality principle holds whereas it does not hold for a network based on a graph. We use the concept of the generalized network to formulate a solution to the following problem: What are the necessary and sufficient conditions for a singular matrix of real numbers, of order p and rank s, to be realizable as the open-circuit resistance matrix of a resistance p-port network. A simple algorithm is given for carriyng out the synthesis. We then present a number of unsolved problems, included among which is what could be called the four-color problem of network synthesis, namely, the resistance n-port problem.

I. INTRODUCTION

Matroid theory has been applied to solve problems in generalized assignment [1], operations research [2], game theory [3, 4], control theory and network theory [5–7], coding theory [8, 9], flow theory [10], generalized flow theory or linear programming [11], and telecommunication computer network design [12]. As discussed in reference [13], the operations of matroid union, matroid partitioning, and matroid intersection are used in some of these applications, as are the theorems on the greedy algorithm, Rado's matroid generalization of Hall's theorem on systems of distinct representatives, and Brualdi's symmetric version of Rado's theorem. It will be noted by an astute reader that in some of the above applications, matroid theory per se is not used. However, its *insights* and *concepts* are, and this we consider one of the important ways in which matroids have affected applications. This is true, for example, in reference [4] on the Shannon Switching Game, where a precise use of duality is essential, and in Fulkerson's work on frames [11], where the significant concept of elementary vectors and the concept of duality in terms of orthogonality play crucial roles. In this paper we attempt to accelerate applications in a new area; specifically, we introduce the reader to a matroid structure and to matroid theorems designed to solve problems in the synthesis of electric networks.

It is well known that a matroid is a generalization of many structures, included among which are graphs, matrices, and linear codes. Thus matroid theorems can be fruitfully applied to these special cases. Since the proofs of most theorems are much simpler in matroid terms rather than in those of graphs or matrices, a gain is achieved. One may then question why there has been a proliferation of structures that stop just short of the matroid. Examples that come to mind are the anagraph of Duffin and Morley [14], which are a pair of orthogonal vector subspaces, the Kirchhoff space of Trapp and Anderson [15], and the frames of Fulkerson [11]. There are also other definitions of a matroid which are equivalent to the standard definitions or add structure to the general matroid [13]. To mention only two, we have the graphoid and digraphoid of Minty, the latter being essentially a pair of dual regular matroids, and the matroid definition due to Gale [16], where the underlying set of the matroid is an ordered one.

The main part of the answer to our question is that in taking the step of generalization to a matroid, which is a purely combinatorial object, we lose some structure. Another reason is that it is useful, indeed, even essential, to formulate matroid insights and theorems explicitly in terms of an area where its practitioners are intimate with the problems and the applications to be made. This suggests and speeds the applications of matroid theory. A graph used in network or flow theory, for example, has metrical as well as topological aspects. The topological aspect will be covered by the matroid generalization, but unless added structure is given to the matroid, the metrical information will be lost.

The process of making matroid theorems and relations explicit in an area it generalizes has another benefit besides accelerating applications: it often leads to new results. This was true in Minty's study of graphoids and digraphoids, and it is the goal sought by Greene in his proof that the MacWilliams formula for the weight distribution of a code is a special case of the Tutte polynomial of a matroid [8]. Although the Tutte polynomial is just as hard to compute as the weight enumerator of a code, he hopes that something will be gained by linking the two theories.

Ray Fulkerson, in defining the concept of a frame, was well aware of the value of making some of the important concepts of matroid theory explicit in areas where it could have important applications. Although he states that most of his results are implicit in Tutte's work on matroids, he finds it desirable to take over into the theory of vector spaces the concept of an elementary vector and the definition of duality in terms of orthogonality. He presents the frame as the structure obtained just prior to the matroid in making the transition from the matrix to its matroid. More precisely, given a vector space $\mathscr{R}$, he defines its frame $F = F(\mathscr{R})$ as the unique, finite set of lines represented by the elementary vectors. Fulkerson is interested here in the linear programming problem, and in not making the transition to a matroid, he keeps the metrical aspects of the problem intact. He achieved some new results in this context. By letting his vector space be an arbitrary real vector space rather than a regular vector space, he generalized the network flow problem. In graph-theoretic terms, rather than have his matrix represent the vertex–edge incidence matrix of a graph, he defines it as an arbitrary matrix over the real field or, more generally, over an ordered field. He is thus able to generalize Minty's painting theorem for a digraphoid to a vector space over an ordered field. He also shows that the general linear programming problem is equivalent to the generalized flow problem.

It is in this spirit of Fulkerson's work on the application of matroids to linear programming that Bruno and Weinberg defined a generalized network in order to focus the full power of matroid theory on problems in network synthesis. Though formulated in a doctoral thesis in 1969, it was first presented in the journal literature 7 years later in 1976 [17]. The *generalized network* is a network based on a matroid rather than a graph. It was formulated to remove the inadequacies of basing a network on a graph, one of which is the lack of a principle of duality. There are a number of significant synthesis problems which for too long have eluded solution, chief among them being the resistance n-port problem. This problem could well be called the four-color problem of network theory because while it remains unsolved we cannot properly state that we understand network theory. Despite the fact that network synthesis has a long history of good mathematics, including the work of such mathematicians as Brune, Foster, Duffin, Cauer, and Bott, this problem still remains unsolved. Though most mathematicians are not aware of it, Raoul Bott's most famous theorem is the Bott–Duffin theorem on the synthesis of a driving-point function without mutual inductance. For an introduction to synthesis and the most thorough available introduction to the resistance n-port problem the reader should consult Weinberg's book [18]. A later contribution to the resistance n-port problem for the special case of singular matrices is given in [19].

In this paper we introduce some of the fundamentals of the generalized network, a structure that is fully discussed in [17], and present some theorems that have a bearing on network synthesis problems. We then use some of these theorems to present a new neat solution to a synthesis problem that was originally solved by Nambiar [20], namely, the realization of singular open-circuit resistance matrices as n-ports. Then in an attempt to interest matroid theorists in significant system problems of some depth we pose a number of unsolved synthesis problems.

All the impedance and admittance matrices considered in this paper are symmetric and the networks contain no ideal or real transformers; thus this is assumed without further comment. Also for brevity short-circuit conductance and open-circuit resistance matrices are often referred to as conductance and resistance matrices, respectively.

We assume the reader is familiar with matroid theory or can consult some references [13, 17, 21, 22]. In addition, we do not repeat the proofs of many of the theorems in Section II; these may be found in the reference [17].

II. Generalized Networks

As in the case of n-port resistance networks based on a graph we consider the generalized network to be an interconnection of two kinds of elements, resistance elements and port elements. The generalized network consists of n elements, p of which are port elements and $n - p$ resistance elements.

Let $\mathscr{M} = (E, \mathscr{C})$ be a regular matroid on the finite set E, where $\mathscr{C}$ is the class of circuits. The set E is partitioned into two sets E_p and E_b. The elements in E_p are the port elements and the elements in E_b the resistance elements. Enumerate the elements of E such that

$$E = E_b \cup E_p$$

where

$$E_b = \{e_1, e_2, \ldots, e_{n-p}\}$$

and

$$E_p = \{e_{n-p+1}, \ldots, e_n\}.$$

With each element e_i in E we associate two variables u_i and w_i (for $i = 1, \ldots, n$). We define the vectors $\mathbf{u}$ and $\mathbf{w}$ as follows:

$$\mathbf{u} = \left[\begin{array}{c}\mathbf{u}_b \\ \hline \mathbf{u}_p\end{array}\right]$$

and

$$\mathbf{w} = \left[\begin{array}{c}\mathbf{w}_b \\ \hline \mathbf{w}_p\end{array}\right]$$

where

$$\mathbf{w}_b^t = [w_1, \ldots, w_{n-p}],$$

$$\mathbf{w}_p^t = [w_{n-p+1}, \ldots, w_n],$$

and

$$\mathbf{u}_b{}^t = [u_1, \ldots, u_{n-p}],$$

$$\mathbf{u}_p{}^t = [u_{n-p+1}, \ldots, u_n].$$

We associate with each member of E_b a positive number d_i $(i = 1, \ldots, n - p)$ and require that

$$\mathbf{w}_b = D\mathbf{u}_b,$$

$$D = \text{diag}[d_1, d_2, \ldots, d_{n-p}].$$

D is called the *resistance-element immittance matrix.*

The next step in defining a generalized network is to write the "topological" constraints for the vectors $\mathbf{u}$ and $\mathbf{w}$. Since $\mathscr{M}$ is regular, there exists a regular vector space $\mathscr{R}$ on E over the field of real numbers such that the supports of the primitive vectors of $\mathscr{R}$ are in 1-1 correspondence with the circuits of $\mathscr{M}$, that is, $\mathscr{M} = \mathscr{M}_{\mathscr{R}}$. (The support of a vector f, denoted by $\|f\|$, is the set of elements e of E for which $f(e)$ has a nonzero value.)

Generalizing Kirchhoff's current law (KCL) and Kirchhoff's voltage law (KVL) we require that $\mathbf{u}^t$ be the representative vector of some member of $\mathscr{R}$ and $\mathbf{w}^t$ be the representative vector of some member of $\perp\mathscr{R}$, the subspace complementary and orthogonal to $\mathscr{R}$. We write the generalized KCL and KVL symbolically as

$$\mathbf{u} \in \mathscr{R}$$

and

$$\mathbf{w} \in \perp\mathscr{R}.$$

We define a generalized network N as a quadruple:

$$N = (\mathscr{M}_{\mathscr{R}}, \mathscr{R}, D; E),$$

where $\mathscr{M}_{\mathscr{R}}$ is a *regular* matroid on a finite set E and $\mathscr{R}$ is a corresponding *regular* vector space on E over the field of real numbers.

The *generalized network equations* are

$$\mathbf{u} \in \mathscr{R}, \tag{4}$$

$$\mathbf{w} \in \perp\mathscr{R}, \tag{5}$$

$$\mathbf{w}_b = D\mathbf{u}_b, \tag{6}$$

where $D = \text{diag}[d_1, \ldots, d_{n-p}]$.

Equations (4) and (5) are the "topological" constraints on $\mathbf{u}$ and $\mathbf{w}$, while Eq. (6) is an Ohm's law constraint.

At this point we will make the appropriate correspondences between the generalized network and the ordinary impedance and admittance formulations of p-port resistance networks. First it is necessary to introduce the p-port resistance network.

Let G be the network graph of a p-port resistance network and partition $E(G)$ according to port and resistance designations. Thus

$$E(G) = E(G)_b \cup E(G)_p ,$$

where

$$E(G)_b = \{e_1 ,..., e_{n-p}\}$$

and

$$E(G)_p = \{e_{n-p+1} ,..., e_n\}.$$

The edges in $E(G)_b$ correspond to the resistances and the edges in $E(G)_p$ correspond to the ports. The quantities $\mathbf{i}^t = [\mathbf{i}_b{}^t\ \mathbf{i}_p{}^t]$ and $\mathbf{v}^t = [\mathbf{v}_b{}^t\ \mathbf{v}_p{}^t]$ are the vectors of resistance and port currents and voltages, respectively, and $Z_b = \mathrm{diag}[z_1 ,..., z_{n-p}]$ and $Y_b = Z_b^{-1}$, where $0 < z_i < \infty$ for $i = 1,..., n-p$ are the resistance-element impedance matrix and admittance matrix, respectively. We have

$$\mathbf{v}_b = Z_b\mathbf{i}_b , \tag{1a}$$

$$\mathbf{i}_b = Y_b\mathbf{v}_b . \tag{1b}$$

The topological constraints on the network voltage and current vectors can be stated as follows. Let G be the network graph of a p-port resistance network and I and V the 1-cycle space and coboundary space, respectively, of G over the field F. Then $\mathbf{i}$ satisfies KCL if and only if $\mathbf{i}^t$ is the representative vector of some member of I and $\mathbf{v}$ satisfies KVL if and only if $\mathbf{v}^t$ is the representative vector of some member of V. Consequently Kirchhoff's laws can be written symbolically as

$$\mathbf{i} \in I \qquad \text{(KCL)} \tag{2}$$

and

$$\mathbf{v} \in V \qquad \text{(KVL)}. \tag{3}$$

Equations (1a) or (1b), (2), and (3) are called the *network equations*.

In order to retain the familiar properties of Z, the *open-circuit* (o.c.) *resistance matrix*, and Y, the *short-circuit* (s.c.) *conductance matrix*, of a p-port resistance network we define an auxiliary port-voltage vector $\mathbf{e}_p = -\mathbf{v}_p$. Then the resistance matrix of a network exists if for *any* prescribed set of port currents $\mathbf{i}_p$ the network equations uniquely determine the response $\mathbf{e}_p$. Similarly the conductance matrix of a network exists if for *any* prescribed set of port voltages $\mathbf{e}_p$ the network equations uniquely determine the response

$\mathbf{i}_p$. If Z exists, then the network operation, viewed from the ports, can be expressed as

$$\mathbf{e}_p = Z\mathbf{i}_p$$

and if Y exists, then

$$\mathbf{i}_p = Y\mathbf{e}_p .$$

Certain matrices are related to p-port resistance networks, and we now give the precise definitions for four of these matrices.

A symmetric matrix of real numbers whose main-diagonal elements are greater than or equal to the sum of the absolute magnitudes of all the other elements in the same row (column) is called a *dominant* matrix. If the dominant matrix has only nonpositive off-diagonal elements, then it is called *hyperdominant*.

A $p \times p$ symmetric matrix of real numbers is called a *paramount* matrix if every principal minor of order r is greater than or equal to the absolute value of any rth-order minor formed from the same rows (columns) for $r = 1,\ldots, p - 1$.

A symmetric matrix each of whose rows sums to zero is called an *indefinite* matrix.

There are two possible ways to make a correspondence between generalized networks and p-port resistance networks.

Consider the following correspondence. Suppose

$$\mathbf{u} = \mathbf{i}. \tag{7}$$

Then it follows that

$$\mathbf{w} = \mathbf{v}, \tag{8}$$

$$\mathscr{R} = I, \tag{9}$$

$$\mathscr{M}_{\mathscr{R}} = \mathscr{P}(G) \tag{10}$$

and

$$D = Z_b , \tag{11}$$

where I is the 1-cycle space of G over the field of real numbers and $\mathscr{P}(G)$ is the polygon matroid of G. Thus the requirement that $\mathbf{u}$ corresponds to $\mathbf{i}$ determines the generalized network

$$N_Z = (\mathscr{P}(G), I, Z_b ; E(G)).$$

If one chooses to have $\mathbf{v}$ correspond to $\mathbf{u}$ the generalized network N_Y is obtained:

$$N_Y = (\mathscr{B}(G), V, Y_b ; E(G)).$$

V is, of course, the coboundary space of G over the field of real numbers and $\mathscr{B}(G)$ is the bond (or cut-set) matroid of G.

The subscripts Z and Y reflect the fact that N_Z will lead to an impedance formulation and N_Y yields an admittance formulation. The correspondences between generalized networks and p-port resistance networks are listed, for future reference, in Table I.

TABLE I

Table of Correspondences

Generalized networks	Network equations
$N = (\mathscr{M}_{\mathscr{R}}, \mathscr{R}, D; E)$	(i) $\mathbf{u} \in \mathscr{R}$ (ii) $\mathbf{w} \in \perp\mathscr{R}$ (iii) $\mathbf{w}_b = D\mathbf{u}_b$
$N_Z = (\mathscr{P}(G), I, Z_b; E(G))$	(i) $\mathbf{i} \in I$ (ii) $\mathbf{v} \in V$ (iii) $\mathbf{v}_b = Z_b\mathbf{i}_b$
$N_Y = (\mathscr{B}(G), V, Y_b; E(G))$	(i) $\mathbf{v} \in V$ (ii) $\mathbf{i} \in I$ (iii) $\mathbf{i}_b = Y_b\mathbf{v}_b$

Having defined a generalized network, the next question to answer is: How does it "work?" In other words, if we specify $\mathbf{u}_p$, how do the network equations determine $\mathbf{u}$ and $\mathbf{w}$. We first introduce some definitions and notation.

A network $N = (\mathscr{M}_{\mathscr{R}}, \mathscr{R}, D; E)$ is called *nondegenerate* if one can specify $\mathbf{u}_p$ arbitrarily and this specification, along with the network equations, uniquely determines $\mathbf{u}$ and $\mathbf{w}$. Let $\mathscr{N}$ denote the class of nondegenerate networks.

Suppose $f \in \mathscr{R}$ and $\mathbf{x}^t$ is a representative vector for f. We define

$$\|\mathbf{x}\| = \|f\|.$$

Also, as was done in the network equations, we write

$$\mathbf{x} \in \mathscr{R}$$

to mean that there exists a vector $f \in \mathscr{R}$ such that $\mathbf{x}^t$ is the representative vector of f. We call $\mathbf{x}$ elementary (primitive) if there exists an elementary (primitive) vector f in $\mathscr{R}$ such that $\mathbf{x}^t$ is the representative vector for f.

The next theorem is important because it characterizes, in terms of matroid structure, those generalized networks which are nondegenerate. Moreover, in the course of proving (II.1) in reference [17] we derive explicit expressions for the "response" of a nondegenerate generalized network to an arbitrary port vector $\mathbf{u}_p$.

THEOREM II.1. *A network* $N = (\mathscr{M}_{\mathscr{R}}, \mathscr{R}, D; E)$ *is in* $\mathscr{N}$ *if and only if* E_p *contains no circuit of* $\mathscr{M}_{\mathscr{R}}^*$.

In this theorem $\mathscr{M}_{\mathscr{R}}^*$ denotes the dual matroid of $\mathscr{M}_{\mathscr{R}}$.

An immediate corollary of (II.1) is

THEOREM II.2. *Let* $N = (\mathscr{M}_{\mathscr{R}}, \mathscr{R}, D; E) \in \mathscr{N}$ *and* R^* *be a representative matrix for* $\perp\mathscr{R}$. *Partition* R^* *as* $R^* = [R_b^* \mid R_p^*]$, *where* R_b^* *and* R_p^* *correspond to the resistance and port elements, respectively. Then*

$$\mathbf{w} = \left[\begin{array}{c}\mathbf{w}_b \\ \hline \mathbf{w}_p\end{array}\right] = \left[\begin{array}{c}-R_b^{*t}[R_b^*D^{-1}R_b^{*t}]^{-1}\, R_p^* \\ \hline -R_p^{*t}[R_b^*D^{-1}R_b^{*t}]^{-1}\, R_p^*\end{array}\right] \mathbf{u}_p$$

and

$$\mathbf{u} = \left[\begin{array}{c}\mathbf{u}_b \\ \hline \mathbf{u}_p\end{array}\right] = \left[\begin{array}{c}-D^{-1}R_b^{*t}[R_b^*D^{-1}R_b^{*t}]^{-1}\, R_p^* \\ \hline 1_p\end{array}\right] \mathbf{u}_p .$$

The immittance matrix X_N is defined as

$$X_N = R_p^{*t}[R_b^*D^{-1}R_b^{*t}]^{-1}\, R_p^*.$$

Therefore X_N characterizes the "operation" of the generalized network in terms of a port description, that is,

$$\mathbf{w}_p = -X_N\mathbf{u}_p .$$

An alternate characterization of a nondegenerate network is given by the following theorem which is a consequence of (II.1).

THEOREM II.3. *Let* $N = (\mathscr{M}_{\mathscr{R}}, \mathscr{R}, D; E)$ *and* R^* *be a representative matrix for* $\perp\mathscr{R}$. *Partition* R^* *as* $R^* = [R_b^* \mid R_p^*]$, *where* R_b^* *and* R_p^* *correspond to the resistance and port elements, respectively. Then* N *is in* $\mathscr{N}$ *if and only if* $\operatorname{rank}(R_b^*) = \operatorname{rank}(R^*)$.

The beauty of matroid theory becomes apparent as one realizes that the matroid structure allows one to visualize the "interconnection" of the elements in E of a generalized network $N = (\mathscr{M}_{\mathscr{R}}, \mathscr{R}; D; E)$. Theorem II.1 is an excellent example of this since it gives the existence of X_N in terms of the matroid structure. Also matroid theory eliminates the necessity of thinking in terms of admittance or impedance and thus focuses attention on the essential features of the analysis of p-port resistance networks. In addition, the different matroid classes allow us to distinguish in a precise way the *differences* between the admittance and impedance formulations of p-port resistance networks.

Let us return now to Table I and interpret X_{N_Z} and X_{N_Y}. It is easy to see that $X_{N_Z} = Z$, the o.c. impedance (or resistance) matrix for the resistance network and $X_{N_Y} = Y$, the s.c. admittance (or conductance) matrix. This is shown in Table II.

TABLE II

Table of Correspondences

$N = (\mathscr{M}_{\mathscr{R}}, \mathscr{R}, D; E)$	$N_Z = (\mathscr{P}(G), I, Z_b; E(G))$	$N_Y = (\mathscr{B}(G), V, Y_b; E(G))$
$\mathbf{w}_p = -X_N\mathbf{u}_p$	$-\mathbf{v}_p = X_{N_Z}\mathbf{i}_p$	$\mathbf{i}_p = X_{N_Y}(-\mathbf{v}_p)$
	$X_{N_Z} = Z$	$X_{N_Y} = Y$

One can now obtain the known results on the existence of Z and Y as special cases of (II.1). In the statement of the next theorem the symbol $G \times S$ and $G \cdot S$ denote the contraction and reduction of G to S, respectively, where $S \subseteq E(G)$.

THEOREM II.4. *Let G be the network graph of a p-port resistance network. Then $Z(Y)$, the o.c. impedance (s.c. admittance) matrix, exists if and only if $G \times E(G)_p$ $(G \cdot E(G)_p)$ contains no bonds (polygons).*

We now turn to a special case of interest, namely, a generalized network satisfying $\alpha(E_p) = r(\mathscr{M}_{\mathscr{R}}^*)$, where $\alpha(\cdot)$ denotes the cardinality of the quantity in the parentheses and $r(\cdot)$ denotes the rank of the matroid in the parentheses.

These networks have special significance in the case of N_Z and N_Y. For instance, if $N = N_Z$, then $\alpha(E_p) = r(\mathscr{M}_{\mathscr{R}}^*)$ becomes $\alpha(E_p) = r(\mathscr{B}(G))$. Thus the number of port elements coincides with the number of elements in a spanning coforest of G. If, moreover, $N_Z \in \mathscr{N}$, then E_p contains no bond of G, and consequently E_p is a spanning coforest of G. If $N = N_Y \in \mathscr{N}$ and $\alpha(E_p) = r(\mathscr{P}(G))$, then E_p is a spanning forest of G.

The above illustrations are encompassed by the following theorem.

THEOREM II.5. *Let $N = (\mathscr{M}_{\mathscr{R}}, \mathscr{R}, D; E) \in \mathscr{N}$; then $r(\mathscr{M}_{\mathscr{R}}^*) = \alpha(E_p)$ if and only if E_p is a base of $\mathscr{M}_{\mathscr{R}}^*$.*

For this class of networks we can also obtain a decomposition of X_N.

THEOREM II.6. *Let* $N = (\mathcal{M}_{\mathscr{R}}, \mathscr{R}, D; E) \in \mathcal{N}$ *and* $r(\mathcal{M}_{\mathscr{R}}{}^*) = \alpha(E_p)$. *Then* $X_N = ADA^t$, *where* A *is a totally unimodular matrix.*

In (II.6) the term *totally unimodular matrix* denotes a matrix each of whose minors (including the elements themselves) is equal to ± 1 or 0.

Under the hypothesis of (II.6) it follows that X_N is a paramount matrix [18].

As is well known in the case of p-port resistance networks, when an immittance matrix is singular, the linear dependence of the columns (or rows) contains information on the port structure of the network. We now show how the linear dependence of the columns of X_N is reflected in the structure of the matroid $\mathcal{M}_{\mathscr{R}}$ associated with N. More precisely, we show that the circuits of $\mathcal{M}_{\mathscr{R}} \times E_p$ are in 1–1 correspondence with the sets of minimal dependent columns of X_N. We also prove a result on singular paramount matrices which has direct bearing on the synthesis problem. We show that the linear dependence of the columns of a singular paramount matrix cannot be arbitrary, and in fact its null space must be regular.

THEOREM II.7. *If* $N = (\mathcal{M}_{\mathscr{R}}, \mathscr{R}, D; E) \in \mathcal{N}$, *then*

$$\mathbf{u}_p{}^t X_N \mathbf{u}_p = \mathbf{u}_b{}^t D \mathbf{u}_b .$$

Proof. Since $\mathbf{u} \in \mathscr{R}$ and $\mathbf{w} \in \perp\mathscr{R}$, it follows that $\mathbf{u}_b{}^t\mathbf{w}_b + \mathbf{u}_p{}^t\mathbf{w}_p = 0$. Using $\mathbf{w}_b = D\mathbf{u}_b$ and $\mathbf{w}_p = -X_N\mathbf{u}_p$, the theorem follows. ∎

The next result relates the minimal dependent columns of X_N to the structure of the matroid $\mathcal{M}_{\mathscr{R}}$.

THEOREM II.8. *Let* $N = (\mathcal{M}_{\mathscr{R}}, \mathscr{R}, D; E) \in \mathcal{N}$. *Then a set of columns of* X_N *forms a minimal dependent set if and only if the corresponding set of elements in* E_p *is a circuit of* $\mathcal{M}_{\mathscr{R}}$.

Proof. Let $C \subseteq E_p$ be a circuit of $\mathcal{M}_{\mathscr{R}}$. Then there exists an elementary vector $\mathbf{u}' \in \mathscr{R}$ such that $\| \mu' \| = C$. Clearly the pair $\mathbf{u} = \mathbf{u}'$ and $\mathbf{w} = \mathbf{0}$ satisfy the network equations, and since $N \in \mathcal{N}$ it follows that

$$X_N \mathbf{u}_p{}' = \mathbf{0}, \tag{1}$$

where

$$\mathbf{u}' = \left[\begin{array}{c}\mathbf{0}\\ \hline \mathbf{u}_p{}'\end{array}\right].$$

We claim that the columns of (1) which are linearly dependent form a minimal dependent set.

Assume there exists a nonzero vector $\mathbf{u}_p''$ such that

$$X_N \mathbf{u}_p'' = \mathbf{0}$$

and

$$\| \mathbf{u}'' \| \subset \| \mathbf{u}' \|, \tag{2}$$

where

$$\mathbf{u}'' = \begin{bmatrix} \mathbf{0} \\ \hline \mathbf{u}_p'' \end{bmatrix}$$

and $\subset$ denotes a proper subset. Since $N \in \mathscr{N}$, $\mathbf{u}_p$ can be specified arbitrarily and therefore by (II.7) $\mathbf{u}'' \in \mathscr{R}$. But then (2) contradicts the hypothesis and accordingly the dependent columns in (1) form a minimal dependent set.

To show necessity suppose

$$X_N \mathbf{u}_p = \mathbf{0} \tag{3}$$

and that the dependent columns of (2) form a minimal dependent set. Again since $N \in \mathscr{N}$, $\mathbf{u}_p$ can be specified arbitrarily and therefore by (II.7) the vector

$$\mathbf{u} = \begin{bmatrix} \mathbf{0} \\ \hline \mathbf{u}_p \end{bmatrix}$$

satisfies $\mathbf{u} \in \mathscr{R}$ and $\| \mathbf{u} \| \subseteq E_p$.

Assume $\mathbf{u}$ is not elementary. Then there exists a nonzero vector $\mathbf{v}^t = [\mathbf{0}^t \; \mathbf{v}_p{}^t]$ satisfying

$$\| \mathbf{v} \| \subset \| \mathbf{u} \|$$

and

$$X_N \mathbf{v}_p = \mathbf{0}. \tag{4}$$

However, (4) contradicts the hypothesis and accordingly $\mathbf{u}$ is elementary. Therefore there exists a circuit $C \subseteq E_p$ such that $C = \| \mathbf{u} \|$. ▮

Theorem II.8 shows that, in the case of the generalized network, matroid theory allows a geometric interpretation of the singular immittance matrices. For the cases N_Z and N_Y, Theorem II.8 specializes to the following well-known result.

THEOREM II.9. *Let $Z(Y)$ be the o.c. impedance (s.c. admittance) matrix of a resistance network whose network graph is G. Then the minimal dependent columns of $Z(Y)$ are in a 1–1 correspondence with the polygons (bonds) of G which are contained in E_p.*

Previously we have used a primitive (elementary) representative vector $\mathbf{x}^t$ with respect to some vector space $\mathscr{R}$. It should be clear that if U is a collection of n-tuples $\mathbf{x}$, then we can use the term primitive (elementary) vector in U without reference to a vector space $\mathscr{R}$. Moreover if U is closed under addition of n-tuples and multiplication by a member of F, then we call U a vector space of n-tuples on E over the field F. The reference to a set E is necessary if, for some $\mathbf{x} \in U$, the notation $\| \mathbf{x} \|$ is to have meaning. U (a vector space of n-tuples) is called regular if F is the field of real numbers and corresponding to each elementary vector $\mathbf{x} \in U$ there exists a primitive vector $\mathbf{x}' \in U$ satisfying

$$\| \mathbf{x}' \| = \| \mathbf{x} \|.$$

In the next theorem we characterize the null space of any paramount matrix. If Q is a $p \times p$ matrix, the *null space* $N(Q)$ of Q is the set of all p-tuples $\mathbf{x}$ which satisfy $Q\mathbf{x} = \mathbf{0}$:

$$N(Q) = \{\mathbf{x} \mid Q\mathbf{x} = \mathbf{0}\}.$$

THEOREM II.10. *Let Q be a $p \times p$ paramount matrix; then $N(Q)$, the null space of Q, is a regular vector space of p-tuples on $E_p(\alpha(E_p) = p)$.*

Proof. Without loss of generality, assume the first r columns of Q form a minimal dependent set.

Assume there exists a principal minor $Q\binom{i_1, \ldots, i_{r-1}}{i_1, \ldots, i_{r-1}} = 0$, where $1 \leqslant i_1 < \cdots < i_{r-1} \leqslant r$. Since Q is paramount, then any $(r-1)$th-order minor using columns $i_1, \ldots, i_{r-1}$ is zero. Accordingly columns $i_1, \ldots, i_{r-1}$ are linearly dependent; but this contradicts the hypothesis. Therefore every $(r-1)$th-order principal minor formed from the first r columns is nonzero.

Let Q_r be the submatrix formed from the first r rows and columns of Q; by hypothesis $\det[Q_r] = 0$. If we let Δ_{ij} be the cofactor obtained from Q_r by crossing out row i and column j, it follows from Jacobi's theorem [23] that

$$\Delta_{ii}\Delta_{jj} = \Delta_{ij}\Delta_{ji}. \tag{1}$$

However, Q is paramount and consequently

$$\Delta_{kk} \geqslant | \Delta_{kh} | = | \Delta_{hk} |, \tag{2}$$

for all $1 \leqslant k \leqslant r$ and $1 \leqslant h \leqslant r$. Using (1) and (2) and the fact that $\Delta_{kk} \neq 0$ for $1 \leqslant k \leqslant r$, we conclude that all the first cofactors of Q_r are equal in absolute value.

It follows from the above analysis that the coefficients of the linear relation of the first r columns of Q can be chosen to be ± 1.

Since the first r columns form a minimal dependent set, the vector $\mathbf{x}$, whose coordinates are the coefficients of this linear relation, is elementary in $N(Q)$. Moreover, we have shown that there exists a primitive vector $\mathbf{x}'$ such that $\| \mathbf{x}' \| = \| \mathbf{x} \|$ and $Q\mathbf{x}' = \mathbf{0}$. ∎

Theorem II.10 enables one to exhibit a paramount matrix in a very revealing form [24].

THEOREM II.11. *Let Q be a $p \times p$ paramount matrix of rank s satisfying $Q\binom{1,\ldots,s}{1,\ldots,s} \neq 0$. Then Q can be expressed as*

$$Q = B^t Q_s B,$$

where B is an $s \times p$ totally unimodular matrix and Q_s is the submatrix formed from the first s rows and columns of Q.

Proof. Partition Q as

$$Q = \left[\begin{array}{c|c} Q_s & Q_{12} \\ \hline Q_{12}^t & Q_{22} \end{array}\right], \tag{1}$$

where

$$\begin{aligned} Q_s &= s \times s, \\ Q_{12} &= s \times (p - s), \\ Q_{22} &= (p - s) \times (p - s). \end{aligned}$$

Set

$$T = \left[\begin{array}{c|c} Q_s^{-1} & 0_{s \times (p-s)} \\ \hline -Q_{12}^t Q_s^{-1} & 1_{p-s} \end{array}\right]$$

and form TQ:

$$TQ = \left[\begin{array}{c|c} 1_s & Q_s^{-1} Q_{12} \\ \hline 0_{(p-s)\times s} & Q_{22} - Q_{12}^t Q_s^{-1} Q_{12} \end{array}\right].$$

Since $\det[T] \neq 0$, the rank of TQ is s and accordingly

$$Q_{22} - Q_{12}^t Q_s^{-1} Q_{12} = 0_{(p-s)\times(p-s)}\,. \tag{2}$$

Setting $B = [1_s \mid Q_s^{-1} Q_{12}]$ and using (1) and (2) we can express Q as

$$Q = B^t Q_s B. \tag{3}$$

Let $\mathbf{x}$ be a p-tuple satisfying $Q\mathbf{x} = \mathbf{0}$. Then

$$(B^t Q_s)(B\mathbf{x}) = \mathbf{0}. \tag{4}$$

The matrix $B^t Q_s$ is $p \times s$ and of rank s and the matrix $B\mathbf{x}$ is $s \times 1$. Accordingly (4) implies

$$B\mathbf{x} = \mathbf{0}.$$

Conversely, if $\mathbf{x}$ is a p-tuple satisfying $B\mathbf{x} = \mathbf{0}$, then $Q\mathbf{x} = \mathbf{0}$. The above analysis shows that

$$N(Q) = \{\mathbf{x} \mid B\mathbf{x} = \mathbf{0}\}. \tag{5}$$

It should be clear from the construction of B and Eq. (5) that the row space of B^* is precisely the transpose of the vectors in $N(Q)$, where

$$B^* = [-Q_{12}^t Q_s^{-1} \mid 1_{p-s}].$$

(Note that $B^*B^t = 0_{(p-s)\times s}$.) By (II.10) $N(Q)$ is regular and therefore the theorem follows using two well-known theorems, which are stated in the footnote.[1] ∎

The implications of Theorem II.11 for immittance matrices should be clear. If Q is the conductance matrix of a p-port network whose graph is G, then by Theorem II.9 we see that the matrix B^* is the fundamental cut-set matrix of the directed port graph of G, where the cut-set matrix is based on the spanning rose of port edges corresponding to the first s columns, that is, the port edges of G_s, the graph of the network realizing Q_s. Thus from

$$B^*B^t = \mathbf{0}$$

and the fact that B contains a unit matrix of order s, we conclude that B is the fundamental polygon matrix of the directed port graph and corresponds to the same spanning rose. The dual statement holds when Q is a resistance matrix. We therefore have

COROLLARY 1. *Let Q be the conductance (resistance) matrix of a p-port network whose graph is G. Let S be the set of s port edges of G corresponding to the first s columns of Q and $T = E(G)_p - S$, the complementary set of port edges. Then B is the polygon matrix (cut-set matrix) of the directed port graph of G, that is, the graph given by $G \times E(G)_p$ $(G \cdot E(G)_p)$, the contraction (reduction) of G to the port edges, and G_s, the graph of the network realizing Q_s, is given by*

$$G \times (E(G) - T)(G \cdot (E(G) - T)).$$

In the next section we apply the above theory to a synthesis procedure.

[1] THEOREM. *Let $\mathscr{V}$ be a vector space on E over F, the field of real numbers, and R be a standard representative matrix for $\mathscr{V}$. Then $\mathscr{V}$ is a regular vector space if and only if R is a totally unimodular matrix.*

THEOREM. *If $\mathscr{M}$ is a regular (binary) matroid, then its dual $\mathscr{M}^*$ is a regular (binary) matroid.*

III. Synthesis of Singular Resistance Matrices

As an application of the preceding we treat the problem of realizing a given singular matrix as a resistance matrix. This will lead in a natural way to an understanding of the unsolved problem of the conductance matrix. Thus suppose that Z is a given pth-order matrix of rank s to be realized as the resistance matrix of a p-port network. For reasons to be made clear in a moment, we require that the port graph of the realization be connected. We assume that Z is arranged as in Theorem II.11 with the nonsingular Z_s occupying the first s rows and columns. We relate Z to a generalized network by identifying it with X_{N_Z}. The use of the theorems then allows us to obtain a set of necessary and sufficient conditions for formulating an algorithm to determine whether a network is realizable and to synthesize the network if it is realizable.

First we have by Theorem II.9 and Corollary 1 the following three necessary conditions for realizability.

(a) A minimal dependent set of Z is a circuit of $\mathscr{M}_{\mathscr{R}}$, that is, the set corresponds to a polygon of the port graph (and hence to a polygon of a graph G containing port edges and resistance edges).

(b) The maximal independent set of s columns corresponds to a spanning tree of the port graph.

(c) The matrix B defined in (II.11) is a cut-set matrix of the *directed* port graph in normal form. (In other words, B must be interpreted over the field of real numbers, not the field modulo 2.)

Now we would like to apply Theorems II.5 and II.6 to our realization problem. There is a well-known synthesis procedure due to Cederbaum, which is essentially a method for decomposing a paramount matrix into the congruence transformation given in (II.6). However, we must first show that the s ports of G_s form a spanning tree not only of the port graph of G but also of G_s. This is easily done. If $Y_s = Z_s^{-1}$ is realizable, its graph G_s may have more than the $s + 1$ vertices of the tree of s ports. If it does, there is a so-called star-mesh transformation [18] that deletes the excess vertices. Thus a realization exists for which the tree of ports is then a spanning tree, and hence for Y_s, whose corresponding matroid is a bond matroid, the condition of (II.6), $\alpha(E_p) = r(\mathscr{M}_{\mathscr{R}}^*) = r(\mathscr{P}(G))$, is satisfied. Thus we add a fourth condition to the set which becomes necessary and sufficient for specifying an algorithm.

(d) $Y_s = Z_s^{-1}$ must be realizable as the conductance matrix of a resistance network on $s + 1$ vertices whose port graph is a spanning tree of the network. This spanning tree must be identical to the spanning tree of the port graph B. In other words, to satisfy the dependence relations of the other

$p - s$ ports on the s independent ports, it must be possible to add the $p - s$ directed port edges to obtain an augmented graph G with $s + 1$ vertices and all the port and resistance edges. When the graph G is reduced to the p port edges, that is, all the resistance edges are open circuited, then this subgraph must be isomorphic as a directed labeled graph to the directed port graph realization of B.

As previously mentioned, a synthesis procedure for Z was given by Nambiar. However, the description of the network to be realized given by (c) and (d) above is a new result.

It should be clear why we required the port graph to be connected. Though there is a synthesis procedure for realizing Y_s based on a tree of port edges, there is no known procedure for realizing Y_s by a network whose port graph is unconnected.

We now state the following theorem.

REALIZABILITY THEOREM. *Let Z be a singular matrix of real numbers of order p and rank s. The necessary and sufficient condition for Z to be realizable as the resistance matrix of a resistive p-port network N, whose graph is G and whose directed port graph is connected and given by $G \cdot E_p$, is that Z be of the form*

$$Z = B^t Z_s B$$

where:

(a) *Z_s is a nonsingular principal submatrix of Z which is realizable by a resistive s-port network N_s, whose graph is G_s, containing precisely $s + 1$ vertices, where the set of port edges of G_s, denoted by S, form a spanning tree of G_s. If we let $T = E_p - S$, then $G_s = G \cdot (E - T)$.*

(b) *B is realizable as the fundamental cut-set matrix of $G \cdot E_p$, the directed port graph of N, where the cut-set matrix is defined with respect to S, the same spanning tree of port edges as in G_s.*

For realizing the required network we follow the steps in the algorithm below.

ALGORITHM. We are given a pth-order matrix Z of rank s. We assume a nonsingular matrix Z_s occupies the first s rows and columns. (If it does not, bring this about by transposing columns and corresponding rows.)

(1) [Realization of B.] Determine B from Z as

$$B = [1_s \mid Z_s^{-1} Z_{12}]$$

and realize this as the cut-set matrix of a directed graph. This is the required port graph. (The Löfgren procedure can be used for the realization [25] or

Tutte's procedure [26]. If B is not realizable, for example, B is not totally unimodular because an element is not ± 1 or 0, then Z is unrealizable.)

(2) [Realization of Z_s.] Realize Z_s by a resistance network containing precisely $s + 1$ vertices for which the set of port edges is a spanning tree. (The Cederbaum algorithm is a general procedure that may be used to realize Z_s^{-1} as the conductance matrix of such a network [27]. If there is more than one port structure for the realization, we choose the one that conforms to the realization of B, that is, has an isomorphic spanning tree. If Z_s is not realizable in the required form of a network whose graph has rank s, then Z is not realizable.

(3) [Addition of dependent ports to realization of Z_s.] Insert each of the dependent ports across a pair of vertices of G_s to obtain G so that $G \cdot E_p$ is isomorphic to the realization of B as *directed labeled* graphs, thus satisfying the dependence relations. If this step cannot be satisfied, then Z is unrealizable.)

It should be noted that there are different ways in which the algorithm can fail, that is, the unrealizability of a given matrix can be made evident: there are topological constraints and metrical constraints. Failure to satisfy any one of the three steps leads to unrealizability of Z. Also, we note that the order of steps 1 and 2 may be reversed since they are independent.

IV. Unsolved Synthesis Problems

We strongly believe that the use of matroids and the generalized network will lead to the solution of unsolved synthesis problems. Perhaps it may even help solve the classic resistance n-port problem. We describe a few problems briefly. These can be easily formulated but are difficult nonetheless.

First there is an unsolved problem related to the one treated in this paper, namely, the realization of a singular matrix as a *conductance* matrix. It should be observed that no appeal to duality can be made based on the solution of the resistance matrix case since our proofs did not start from X_N. If they had, then we would have obtained two dual theorems. We started with X_{N_Z} because we could then use a graph-theoretic theorem based on *vertices*; unfortunately, no corresponding concept exists in the matroid generalization of a graph. The theorem we used was the star–mesh theorem; the converse mesh–star theorem does not exist.

This problem has been solved, however, for the fourth-order matrices in [19], where it is shown that the necessary and sufficient condition for realizability is paramountcy. It was possible to do this without recourse to matroids because a proof could be based on the known condition for nth-order nonsingular matrices for $n \leqslant 3$, namely, paramountcy. For $n > 3$

paramountcy is of course still necessary but it is no longer sufficient. It should be noted, furthermore, that matrices of arbitrary order but of rank three may be realized as conductance matrices by the procedure in this paper.

Leaving the singular case, we come to the important problem of the realization of an nth-order matrix as an n-port, where $n > 3$. This has remained unsolved for both the resistance and the conductance matrix. Thus we actually have two problems, whose conditions for realizability will be different and may require different methods of proof.

A third problem that should be mentioned is in terms of generalized networks. Paramount matrices are known that are unrealizable by resistance networks [18]. However, suppose we are given a paramount matrix which we identify with X_N. It is conjectured that there always exists a generalized network $N = (\mathscr{M}_{\mathscr{R}}, \mathscr{R}, D; E)$, but this has neither been proved nor disproved.

For a statement of the resistance n-port problem in another form, where an approach to solving it is given in terms of matrix decompositions, the reader is referred to [28].

We can turn from resistance networks where unsolved problems exist for n-ports with $n > 3$, to RLC networks. When we allow energy-storage elements, namely, inductors (L) and capacitors (C), then we encounter problems that are still unsolved even for a two port. In fact, after Brune gave a solution to the one-port RLC problem that required transformers, the problem of a transformerless solution was not found for about 20 years until Bott and Duffin shocked network theorists with their innocent-appearing letter [29] that contained a complete solution. This is the famous Bott–Duffin theorem we referred to previously.

The RLC n-port problem can be related to the generalized network by permitting the diagonal matrix D to have three types of terms: k_μ, $k_\nu s$, and $1/(k_\rho s)$, where the k's are positive numbers and s is a complex variable.

The two-port problem is even unsolved for the RC case, that is, for a network containing only resistors and capacitors. This is equivalent to the LC two-port problem in that a solution for one solves the other by use of a simple transformation. The conditions are also not known when the two-port is a grounded one, that is, is restricted to a three-terminal network. Thus the RC problem is unsolved for $n \geqslant 2$ for n-ports and $n \geqslant 3$ for n-terminal networks.

For RLC networks the same situation exists as for RC networks.

It is just not well understood why these problems should become so difficult when ideal transformers are not allowed. When they are allowed, the general problem becomes simple: the necessary and sufficient condition is that the matrix be positive real [18]. This applies for R networks, RC, LC, and RLC networks, and for n-ports with arbitrary n. For example, in the resistance case, the matrix

$$\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$$

is positive real, that is, positive definite, but it is not paramount, since $3 > 2$. It therefore requires ideal transformers in its realization as a two-port resistance or conductance matrix. It seems that the ideal transformer acts like a veritable *deus ex machina* of Greek tragedy, to give contrived solutions to intricate problems and to extricate us from difficulties. But perhaps the use of matroid theory and generalized networks will change all this.

V. Conclusion

It appears that the ordinary definition of a network as based on a graph is not adequate for solving some difficult synthesis problems. To remedy this situation we have introduced in this paper the concept of a generalized network, that is, a network based on a matroid, and discussed some of its properties and derived matroid theorems. Some of these theorems were then applied to solve the problem of realizing a given singular matrix as the resistance matrix of a p-port network. The conductance matrix case still remains unsolved. Other unsolved synthesis problems were then formulated, included among which is what could be called the four-color problem of network theory, namely, the realization of a given real matrix as the conductance or resistance matrix of an n-port network.

When we consider resistance–capacitance or equivalently inductance–capacitance networks, even the case of $n = 2$ is unsolved. It is believed that matroids and generalized networks will help solve these synthesis problems.

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