# Topology of Series-Parallel Networks* 

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## 1. Introduction

There is a simple type of electric network termed a series-parallel connection which occurs frequently in both theoretical and applied electrical engineering. One reason for the importance of series-parallel connection stems from the fact that the joint resistance is easily evaluated by the following two rules due to Ohm :
$0_{\mathrm{s}}$. Resistance is additive for resistors in series.
$0_{\mathrm{p}}$. Reciprocal resistance is additive for resistors in parallel.
For example, consider Fig. 1 which is a graph diagram of an electrical


Fig. 1
network with branches $a, b, e, d, f, k$. Let $r_{a}$ denote the resistance of branch $a$. Thus $R_{a}$ is the joint resistance of the network as measured by a battery inserted in branch $a$. Then by repeated application of rules $0_{\mathrm{s}}$ and $0_{\mathrm{p}}$ it is readily found that

$$
\begin{equation*}
R_{a}=r_{a}+r_{k}+\left\{r_{f}^{-1}+\left[r_{e}+\left(r_{b}^{-1}+r_{d}^{-1}\right)^{-1}\right]^{-1}\right\}^{-1} \tag{1}
\end{equation*}
$$

[^0]This example is an instance of the following general definition. A branch $a$ in a finite network is said to be in series-parallel connection if the joint resistance $R_{a}$ through branch $a$ can be evaluated by Ohm's two rules. A network in which every branch is in series-parallel connection shall be termed a series-parallel network.

If the resistors of a network have a nonlinear characteristic then it is difficult to evaluate the current flow. However, if the network has the seriesparallel topology then a great simplification results. This is shown in Section 5 .

Series parallel connections play a prominent role in Shannon's well known application of Boolean algebra to switching circuits [1]. Riordan and Shannon [2] extended some early work of Macmahon [3] on the enumeration of series-parallel networks. Riordan and Shannon proposed two definitions of series-parallel networks. One of these is similar to that given above. Their other definition corresponds to the definition of a confluent network given below.

Some time ago Raoul Bott and the writer gave a method for the synthesis of a given impedance by use of a series-parallel connection of resistors, inductors, and capacitors [4]. The material in this note was developed at that time with the thought that it might throw light on the synthesis problem. Thus it appeared desirable to relate three alternative characterizations of series-parallel networks:
(i) Direct construction by the series operation and the parallel operation.
(ii) The confluence property.
(iii) The absence of an embedded Wheatstone bridge.

The proof that these characterizations are equivalent is not very deep but it seems desirable to have a unified formal treatment such as given here.

Consider the network shown in Fig. 2. Note that this network has a planar


Fig. 2
graph and that all the nodes are on the boundary. Thus it is a direct consequence of Corollary 1 to follow that every branch of this network is in seriesparallel connection.

On the other hand it follows from Theorem 1 that no branch of the Wheatstone bridge network shown in Fig. 3 is in scries-parallel connection.


Fig. 3

## 2. Confluent Networks

It is desirable to express the series-parallel connection as a topological property. To do this familiar geometric terminology and concepts will be employed to study the graph diagram of a network. Thus a branch of a network is depicted in the graph as an edge. An edge is a simple curve and its two endpoints are termed nodes. Then a graph $G$ is defined in this paper as a finite set of edges arbitrarily interconnected at their nodes. A circuit is a sequence of edges forming a closed curve such that no more than two ends meet at each node. A loop is a circuit with only one edge. Each of the edges of a graph is given a direction. Each of the circuits of a graph is given a direction of circulation.

We say that two edges $a$ and $b$ are confluent if there do not exist two circuits $C_{1}$ and $C_{2}$ such that $C_{1}$ meets $a$ and $b$ in the same sense but $C_{2}$ meets $a$ and $b$ in opposite sense. We term a graph confluent if every pair of edges is confluent. A tree is an example of a confluent graph because there are no circuits.

The Wheatstone bridge (also called the complete four-graph) is shown in Fig. 3 as a square ( $a, k, b, e$ ) with diagonals $d$ and $f$. The edges $a$ and $b$ are taken to be opposite sides of the square. Suppose that the directions of $a$ and $b$
are choosen so that they have the same sense relative to the circuit of the square ( $a, k, b, e$ ). However, it then results that $a$ and $b$ have opposite sense relative to the circuit $(a, d, b, f)$ formed with the diagonals. Thus $a$ and $b$ are not confluent edges. This property is seen to be connected with the well known fact that if a battery is inscrted in branch $a$ of the Wheatstone bridge network then the current in branch $b$ is in one direction if $r_{f} r_{d}>r_{e} r_{k}$ and in the other direction if $r_{f} r_{d}<r_{e} r_{k}$. If $r_{f} r_{d}=r_{e} r_{t}$ the bridge is said to be balanced. This suggests the following theorem.

Theorem 0. Let $a$ and $b$ be branches of $a$ network of resistors. Let a battery be inserted in branch $a$. Then the direction of the current flow through branch $b$ is independent of the resistance values of the branches of the network if and only if $a$ and $b$ are confluent branches.

Proof. If $a$ and $b$ are not confluent let the resistances of all branches be infinite except those in $C_{1}$. Then the current flow will be confined to $C_{1}$ and so will flow through $b$ in a certain direction. Now consider the case when the resistances of all branches are infinite except those in $C_{2}$. Then the current flow through $b$ will be opposite to that in the first case. This proves that if $a$ and $b$ are not confluent the direction of current depends on the resistance values.

Now suppose $a$ and $b$ are confluent and consider the current flow resulting. Let $v_{1}, v_{2}, \cdots$ denote the nodes and let $u_{1}, u_{2}, \cdots$ denote the electric potentials of the nodes. Let $v_{1}$ and $v_{2}$ be the nodes of $a$ and let $v_{3}$ and $v_{4}$ be the nodes of $b$. It may be assumed that $u_{1}>u_{2}$ and $u_{3}>u_{1}$. Thus current is flowing in $b$ from $v_{3}$ to $v_{4}$ so Kirchhoff's first law states that current must leave $v_{4}$ by at least one of the connecting branches. Let such a branch have nodes $v_{4}$ and $v_{i}$. Then $u_{4}>u_{i}$ because each branch is assumed to have some resistance. Again some of the current must leave $v_{i}$ and flow to a neighboring node, say $v_{j}$. Continuing this process leads to a chain of neighboring nodes $v_{i}, v_{i}, v_{j}, \cdots$ such that

$$
u_{4}>u_{i}>u_{j}>\cdots
$$

Similar reasoning shows that there is a chain of neighboring nodes $v_{3}, v_{p}$, $v_{q} \cdots$ such that

$$
u_{3}<u_{p}<u_{q} \cdots
$$

But there are only a finite number of nodes and these inequalities are strict so it follows that these chains must terminate at $v_{1}$ and $v_{2}$. Thus

$$
u_{1}>\cdots>u_{p}>u_{3}>u_{4}>u_{i}>\cdots>u_{2} .
$$

The nodes $v_{1}, \cdots, v_{p}, v_{3}, v_{4}, \cdots, v_{2}$ are distinct because of the strict inequalities. Hence the corresponding edges form a circuit containing $a$ and $b$.

Suppose that some other choice of the resistance values would lead to a reverse flow in branch $b$ so that $u_{4}>u_{3}$. Then similar reasoning would show the existence of a circuit with an ordered sequence of nodes $v_{1}, \cdots, v_{4}, v_{3}, \cdots$, $v_{2}$. This contradicts the confluence property and so the theorem follows.

A subgraph is the graph obtained by performing the operation $Q$ any number of times.
Q. Delete an edge.

As an example of a subgraph consider all the edges of a graph which are on circuits going through a given edge a. All edges except these circuits are deleted. This subgraph is designated as $G_{a}$ and is termed the closure of edge $a$. If there are no circuits $G_{a}$ is empty.

Lemma 0. The joint resistance $R_{a}$ through branch a of a network is a function of $r_{b}$, the resistance of branch $b$, if and only if $b$ is in the closure graph $G_{a}$.

Proof. The proof of this lemma can be given on the same lines as the proof of Theorem 0. The details are omitted.

## 3. Embedded Wheatstone Bridge

We define an embedded graph as the graph obtained by performing any number of the operations $Q$ and $S$.
S. Delete a node between two edges in series.

The node deleted is where exactly two edges join. These two edges are then identified.

Lemma 1. An embedded graph of a confluent graph is a confluent graph.
Proof. Let $G$ be an arbitrary graph, and let $G^{\prime}$ be the graph obtained by operation S (or Q ). If $G^{\prime}$ is not confluent then certainly $G$ is not confluent. Repeating this argument a sufficient number of times completes the proof.

Theorem 1. A necessary and sufficient condition that a graph be a confluent graph is that no embedded graph be a Wheatstone bridge.

Proof Let the given graph $G$ be confluent. Then by Lemma 1 an embedded graph cannot be a Wheatstone bridge, because a Wheatstone bridge is not confluent.

Next suppose that $G$ is not a confluent graph. Then according to the
definition there exist two circuits $C_{1}$ and $C_{2}$ which both share two edges $a$ and $b$. Moreover it may be supposed that $C_{1}$ meets $a$ and $b$ in the positive direction and that $C_{2}$ meets $a$ in the positive direction and $b$ in the negative direction. Hence if all edges of $G$ are deleted except those of $C_{1}$ and $C_{2}$, there results a subgraph $G^{\prime}$ which is also not confluent. In the graph $G^{\prime}$ suppose that $C_{2}$ has a node which is not a node of $C_{1}$. Then that node may be deleted by operation $S$, and it is seen that the resultant embedded graph is also not confluent. Thus we are led to consider an embedded graph $G^{\prime \prime}$ in which the nodes of $C_{2}$ are also nodes of $C_{1}$. Let ( $v_{1}, v_{2}, \cdots, v_{n}$ ) denote the nodes of $C_{1}$ ordered in the direction of circulation. Let $v_{n}$ and $v_{1}$ be the nodes of $a$, and let $v_{i}$ and $v_{i+1}$ be the nodes of $b$. The circuit $C_{1}$ may be drawn as a circle as shown in Fig. 4. Then the edges of $C_{2}$ which are not edges of $C_{1}$ may be drawn as chords of this circle. The nodes of $C_{1}$ are divided into the set $M$ containing ( $v_{1}, v_{2}, \cdots, v_{i}$ ) and the set $M^{\prime}$ containing ( $v_{i+1}, \cdots, v_{n}$ ). Leaving point $v_{1}$ on $C_{2}$ there is a first edge of $C_{2}$ which has a node in $M^{\prime}$. Let this be edge $x$ with node $v_{n}$ in $M$ and node $v_{q}$ in $M^{\prime}$. Since $C_{2}$ meets $b$ in the negative direction, it follows that $x$ is not $b$. Thus $x$ is a chord of the circle which divides the circle into two parts: $C_{1 a}$ containing $a$ and $C_{1 b}$ containing $b$. We continue on $C_{2}$ through $v_{i+1}$ to $v_{i}$. After leaving $v_{i}$ on $C_{2}$, there is a first edge $y$ which connects $C_{1 b}$ and $C_{1 a}$. Let $v_{r}$ and $v_{s}$ be the nodes of $y$. Note that $v_{p}, v_{q}, v_{r}, v_{s}$ is a set of ordered nodes of $C_{2}$, and since $C_{2}$ is a simple closed curve, these nodes are distinct. This implies that $v_{r}$ is an interior node of $C_{1 b}$ and that $v_{s}$ is an interior node of $C_{1 a}$. Thus $y$ is a chord which crosses the chord $x$, as is indicated in Fig. 4. Delete all edges of $C_{2}$


Fig. 4
not in $C_{1}$ except $x$ and $y$. Then delete all nodes of $C_{1}$ except $v_{p}, v_{q}, v_{r}, v_{s}$. This leaves an embedded graph which is a Wheatstone bridge.

Corollary 1. If a graph $G$ has a planar map with all nodes on the boundary, then $G$ is a confluent graph.

Proof. Consider any embedded graph of $G$ with exactly four nodes. The map of this embedded graph will also have all its nodes on the boundary. It is obvious that a planar map of the Wheatstone bridge cannot have all nodes on the boundary, so $G$ is confluent.

It is seen that the graph shown in Fig. 2 has all its nodes on the boundary. Thus it follows from Corollary 1 that it is a confluent graph.

## 4. Construction of Series-Parallel Networks

From the point of view of network theory the most interesting graphs are those in which all edges are connected by circuits. Thus of especial concern here are closed confluent graphs, defined to be confluent graphs in which any two edges are common to at least one circuit. If there is only one edge, the graph is a loop. The following lemma is well known.

Lemma 2. Suppose that the circuits through a certain edge a of a graph go through all other edges. Then any other edge $b$ has the same property.

Proof. Let $C_{1}$ be a circuit containing $a$ and $b$, and let $C_{2}$ be a circuit containing $a$ and some other edge $x$. First suppose $x$ is not on $C_{1}$. Leaving edge $x$ in one direction on $C_{2}$ there is a first node $v_{1}$ which is common to $C_{1}$ and $C_{2}$; leaving in the other direction, let $v_{2}$ be the first common node. It is seen that $v_{1}$ and $v_{2}$ are different because $a$ separates them. Delete all edges except $C_{1}$ and the part of $C_{2}$ from $v_{1}$ to $v_{2}$ containing $x$. The resulting figure is equivalent to a circle with a diameter, and so it is apparent that $x$ and $b$ are on a simple closed curve. If $x$ is on $C_{1}$ this is also true so the proof is complete.

A consequence of this lemma is that any confluent graph can be "decomposed" into closed confluent graphs and to single edges. These graphs may or may not be connected.

Theorem 2. Starting from a loop, apply a sequence of the following operations:
S*. Replace an edge by two edges in series.
$\mathrm{P}^{*}$. Replace an edge by two edges in parallel.

This leads to a closed confluent graph. Moreover, an arbitrary closed confluent graph may be constructed in this way.

Proof. Operation $\mathrm{P}^{*}$ is understood to involve two distinct nodes. A loop has only one node; therefore $\mathrm{P}^{*}$ must not be the first operation. Thefirst part of Theorem 2 is a consequence of Lemma 2 and Theorem 1, as it is easy to see that operation $S^{*}$ or $P^{*}$ could not develop a Wheatstone bridge where one did not exist before.

To treat the second part of the theorem, suppose that graphs with no more than $n$ edges can be so constructed. Then consider a closed confluent graph $G$ with $n-1$ edges. Let $a$ and $b$ be edges with a common node, and consider the graph $G^{\prime}$ composed of all circuits having both $a$ and $b$ as elements. The graph $G^{\prime}$ is a subgraph of $G$, and so Lemma 1 states that $G^{\prime}$ is a confluent graph. It then follows from Iemma 2 that $G^{\prime}$ is a closed confluent graph. By S operations delete all nodes of $G^{\prime}$ where exactly two edges of $G^{\prime}$ meet. This gives an embedded graph $G^{\prime \prime}$ which is also a closed confluent graph.

It may be assumed that at least three edges meet at every node of $G$, for otherwise $G$ could be constructed out of a network of $n$ edges by operation S*. However in $G^{\prime}$ the node where $a$ and $b$ join has only the edges $a$ and $b$. Thus $G^{\prime}$ and $G^{\prime \prime}$ have no more than $n$ edges. By the inductive hypothesis $G^{\prime}$ and $G^{\prime \prime}$ can be constructed through operations $\mathrm{S}^{*}$ and $\mathrm{P}^{*}$. Moreover $G^{\prime}$ can be derived from $G^{\prime \prime}$ by repeated application of operation $S^{*}$ alone. If $G^{\prime \prime}$ is not a loop, then it must have two of its edges, say $k$ and $d$, in parallel. 'This is a consequence of operation $\mathrm{P}^{*}$ in the construction of $G^{\prime \prime}$.

We first suppose that $G^{\prime \prime}$ is not a loop. Then $a, b$, and $d$ are on a circuit $C_{1}$ and $k$ and $d$ are in parallel. This part of $G^{\prime \prime}$ is indicated by the solid lines in Fig. 5. The nodes of $k$ and $d$ are designated as $v_{1}$ and $v_{2}$. Suppose that there


Fig. 5
are no suppressed nodes of $G$ on $k$ or $d$. Let $k$ be deleted from $G$; then it is seen that replacing $k$ constructs $G$ by operation $\mathrm{P}^{*}$ from an $n$ edge closed confluent graph.

The problem has now been reduced to the case that there is a suppressed node, say $v_{3}$, on edge $k$. Since $G$ is a closed graph, there is a circuit $C_{0}$ through edge $a$ and an edge $e$ at $v_{3}$. The circuit $C_{0}$ is indicated as a dotted line in Fig. 5. Starting from $v_{3}$ on $C_{0}$ there is a first node $v_{4}$ where $C_{0}$ touches $C_{1}$. This node $v_{4}$ must actually be the same as $v_{1}$ or $v_{2}$, for otherwise there would be a Wheatstone bridge embedded graph with nodes $v_{1}, v_{2}, v_{4}$, and another node on $k$. From $v_{3}$ on $C_{0}$ in the reverse direction there is a first node $v_{5}$ where $C_{0}$ first meets $C_{1}$. Again $v_{5}$ is the same as $v_{1}$ or $v_{2}$. Now $v_{4}$ and $v_{5}$ are not identical since they are nodes on the circuit $C_{0}$ which are separated by $v_{3}$ and a node of $a$. Thus $v_{4}=v_{2}$ (or $v_{1}$ ) and $v_{5}=v_{1}$ (or $v_{2}$ ). However this implies that there is a composite circuit $C_{10}$ made up of $C_{1}$ and $C_{0}$ and going through $v_{1}, v_{2}, v_{3}, a, b$, and $e$. From the definition of $G^{\prime}$ this means that $e$ is in $G^{\prime}$. This is a contradiction.

The question is now reduced to the case that $G^{\prime \prime}$ is a loop. Thus $G^{\prime}$ consists of a single circuit $C_{1}$. Let $b_{1}, b_{2}, \cdots$, denote the edges at one node of $a$ and let $h_{1}, h_{2}, \cdots$ denote the edges at the other node of $a$. By applying the argument given above to the pair of edges $\left(a, b_{i}\right)$ it follows that there is exactly one circuit, say $C_{i}$, through the edges $a$ and $b_{i}$. Likewise the pair of edges $\left(h_{2}, a\right)$ defines a circuit $C_{i}{ }^{*}$. It may be assumed that circuits $C_{i}$ and $C_{i}{ }^{*}$ are in correspondence. Thus the circuit $C_{i}$ is defined by the triple ( $h_{i}, a, b_{i}$ ). Let $v_{0}$ be a node on the circuit $C_{1}$ which is not a node of a. Then there is another circuit, say $C_{2}$, which goes through $v_{0}$ because every node of $G$ has at least three edges. In Fig. 6 the circuit $C_{1}$ is shown in full lines. The circuit


Fig. 6
$C_{2}$ is shown in dotted lines. Let $v_{1}$ be the node joining $a$ and $h_{2}$. Let $v_{2}$ be the first node of $C_{1}$ encountered when leaving $\tau_{1}$ along $C_{2}$ and going through $h_{2}$. Let $C_{12}$ be the composite circuit made up of the part of $C_{2}$ traversed above and the part of $C_{1}$ between $v_{1}$ and $v_{2}$ which contains $a$ and $b_{1}$. But the circuit $C_{12}$ is not identical with $C_{1}$ becausc it contains $h_{2}$. This contradicts the assumption that there was only one circuit containing both edge $a$ and edge $b_{1}$. This completes the proof of Theorem 2.

Corollary 2. Every confluent graph is planar, and the resulting map can be colored in three colors.

Proof. It is sufficient to consider a closed graph. Assume that the corollary is true for all graphs with $n$ edges. The planar map of such a graph is shown in Fig. 7(a). An edge is shown separating two regions of colors $x$ and $y$. Then operation $\mathrm{P}^{*}$ applied to this edge gives a planar map as shown in Fig. 7(b). The new region is colored $z$. The operation $S^{*}$ does not intro-


Fig. 7a


Fig. 7h
duce any new region. Hence the graphs with $n+1$ edges are planar and can be colored in three colors $x, y$, and $z$.

Corollary 3. The dual graph of a closed confluent graph with two or more edges is also a closed confluent graph.

Proof. The nodes of a dual graph are located in the regions of the primal graph, and there is a one-to-one correspondence between the edges of the two graphs determined by their crossing. The dual of a graph with two edges in parallel also is a graph with two edges in parallel.

Now proceed by induction as in the proof of Corollary 2. The dotted line
in Fig. 7(a) is an edge of the dual graph connecting the region colored $x$ with the region colored $y$. In Fig. 7(b) this edge has been divided and a new node has been inserted in the region colored $z$. This operation $P^{*}$ on the primal graph induces an operation $\mathrm{S}^{*}$ on the dual graph. Likewise the operation $\mathrm{S}^{*}$ on the primal graph induces an opcration $\mathrm{P}^{*}$ on the dual graph. Thus the induction argument is complete.

Corollary 4. A graph is a closed confluent graph if and only if it can be reduced to a loop by a suitable sequence of the following operations:
S. Delete a node between two edges in series.
P. Delete an edge parallel to another edge.

Moreover the sequence of operations can be choosen so that a given edge a is not involved except for the last operation of the sequence.

Proof. The operations $S$ and $P$ are the inverse of the operations $S^{*}$ and $P^{*}$. Then a sequence of the operations $S$ and $P$ may be choosen to undo the construction of the graph by the operations $S^{*}$ and $P^{*}$. This proves the first part of Corollary 4.

Suppose the last statement of Corollary 4 is true for graphs with $n$ edges and consider a graph $G$ with $n+1$ edges. It may be supposed that $n \geqslant 3$. If $a$ is the given edge suppose that edge $b$ is in series with $a$ (or in parallel with $a$ ). Then applying operation S (or P ) in which edges $a$ and $b$ are replaced by a single edge $a^{\prime}$. This gives a graph $\mathrm{G}^{\prime}$ with $n$ edges. Thus there is a sequence of operations which reduce $G^{\prime}$ to a graph which is a circuit with two edges, $a^{\prime}$ and $x$. Now $a^{\prime}$ is replaced by $a$ and $b$ in series (or $a$ and $b$ in parallel). Then S (or P ) is applied to $x$ and $b$. Finally apply S and the graph is reduced to a loop.

Theorem 3. A network is of series-parallel type if and only if it is confluent.

Proof. Let us consider the joint resistance $R_{a}$ as determined by a battery inserted in branch $a$ of resistance $r_{a}$. According to Lemma 0 it is sufficient to consider the closed subnetwork $G_{a}$ of which $a$ is a part. Of course if $a$ is not part of a closed network, then the joint resistance is infinite.

First suppose that edge $a$ is in a closed confluent network $G_{a}$. The operations S and P of Corollary 4 are to be carried out so as to reduce $G_{a}$ to a loop. Suppose operation S is carried out on two edges $d$ and $k$ which are in series. Then the new edge may be termed $d^{\prime}$, and it is given the resistance $r_{d^{\prime}}=r_{d}+r_{k}$. This procedure would be followed even if $d$ were $a$. In any case $R_{a}$ has the same value for the reduced network.

If the operation P is carried out on two edges $d$ and $k$ which are in parallel, then the new edge $d^{\prime}$ is given the resistance $r_{a^{\prime}}=\left(r_{d^{-1}}^{-1}-r_{k^{-1}}^{-1}\right)^{-1}$. If $d$ or $k$ is not $a$, then $R_{a}$ has the same value for the reduced network. This would not be true if $d$ were $a$. However according to the last statement of Corollary $4 a$ is involved only in the last operation. This can be operation S. This proves that a confluent network is series-parallel.

Conversely suppose that branch $a$ is in series-parallel connection. Then the operations S and P are defined by the process of evaluating $R_{a}$. These operations reduce $G_{" \prime}$ to a loop and the proof is completed by Corollary 4.
(The writer is indebted to A. F. Kaupe and to the referee for pointing out a relationship of the present investigation with work of Dirac $[5,6]$ on chromatic graphs. In particular 'Theorem 14.3 .7 quoted in the book by Ore [7] would furnish an alternative proof to some of the questions treated here. This theorem, when translated into the terminology of this paper, states that a closed graph without series or parallel edges must have an embedded Wheatstone Bridge.)

## 5. Nonlinear Networks

In a theory of nonlinear networks developed by the writer [8] the linear relation of Ohm $y=r x$ between current $x$ and voltage $y$ is replaced by the relation $y=\rho(x)$ where the function $\rho(x)$ satisfies the conditions: (1) $\rho(x)$ is continuous and increasing, (2) $\rho(x)$ is unbounded for $x= \pm \infty$, and (3) $\rho(0)=0$. The function $\rho(x)$ may be termed the resistance function. If $x=\rho^{-1}(y)$ the function $\rho^{-1}(y)$ may be termed the conductance function. Resistors with these properties are termed monotone resistors. The uniqueness and existence theorems for a network of monotone resistors were found to be essentially the same as for a network of Ohmic resistors.

It was proved that the joint resistance function $R_{a}(x)$ is also a monotone increasing function. However, to evaluate $R_{u}(x)$ is usually a very difficult problem, see [9-11]. It is to be brought out here that there is a great simplification in the special case of series-parallel networks. (These results are a joint work of Raoul Bott and the writer.)

Shown in Fig. 8 are the resistance functions $\rho_{b}(x)$ and $\rho_{d}(x)$ of monotone resistors $b$ and $d$. Let $\rho_{s}(x)$ be the resistance function of $b$ and $d$ in series. Then

$$
\rho_{\mathrm{s}}(x)=\rho_{b}(x)+\rho_{a}(x) .
$$

The function $\rho_{s}(x)$ is determined graphically by adding ordinates as shown in Fig. 4. Let $\rho_{p}(x)$ be the resistance function for $b$ and $d$ in parallel. Thus

$$
\rho_{y}^{-1}(y)=\rho_{b}^{-1}(y)-\rho_{d}^{-1}(y) \quad \text { and so } \quad \rho_{p}(x)=\left(\rho_{b}^{-1} \quad \rho_{d}^{-1}\right)^{-1}(x) .
$$

The function $\rho_{p}(x)$ is determined graphically by adding abcissas as shown in Fig. 4. Thus we may say that the resistance function for two resistors (in series or in parallel) is given by an explicit formula involving only the operation of addition and the operation of inversion.


Fig. 8

Theorem 4. Given a confluent network of monotone resistors. Let a generator of voltage $e$ be inserted in branch a of the network. Then for each node $v_{i}$ of the network there is an explicit formula $u_{i}=U_{i a}(e)$ giving the node potential $u_{i}$ as a function of the generator voltage. Also for each branch $b$ there is an explicit formula $x_{b}=X_{b a}(e)$ giving the branch current $x_{b}$ as a function of the generator voltage. These formulas are constructed using only the operations of inversion and the operation of addition (subtraction and multiplication are not used).

Proof. If the generator voltage $e$ is held constant then effective values of resistance can be defined as the ratio of potential drop to current flow. These effective values will be positive. Thus the same arguments can be applied as were used for networks of ohmic resistance. In particular it follows from Lemma 0 that no current will flow in a branch not on a circuit with $a$.

To make the potential definite one node of branch $a$ is assigned zero potential. If the network is not connected all nodes not connected to $a$ are assigned zero potential.

Now proceed by induction. The theorem is true if there is only one branch so suppose it to be true for all networks with $n$ branches and consider a network $N$ with $n+1$ branches. Let $N_{a}$ be the subnetwork of $N$ consisting of all branches on circuits through $a$. If $N_{a}$ is empty all nodes of $N$ are given the potential $e$ or the potential zero depending on which node of $a$ they are
connected. If $a$ is a loop then all nodes are given the potential zero and $x_{a}=\rho_{a}^{-1}(e)$.

Now we consider the case that $N_{a}$ has at least two branches. Then the corresponding graph $G_{a}$ is a closed confluent graph. It follows that edges can be given a direction such that if $a$ and $b$ are in a circuit then $a$ and $b$ have the same sense relative to the circuit. By virtue of Corollary 4 there are either two edges of $G_{\eta}$ in series or else there are two edges in parallel.

First suppose edges $b$ and $d$ are in series and that $b$ has nodes $\left(v_{1}, v_{2}\right)$ and that $d$ has nodes $\left(v_{2}, v_{3}\right)$ and $\rho_{3}(x)=\rho_{b}(x)+\rho_{d}(x)$. This gives a network $N^{\prime}$ with $n$ branches. Let the currents and potentials be determined for $N^{\prime}$. Let the current flow through $s$ be $x_{\mathrm{s}}$ in the direction 1 to 3 . Let the potentials at $v_{1}$ and $v_{3}$ be $u_{1}$ and $u_{3}$. Now let the old branches be restored the current flow through $b$ and $d$ is taken to be $x_{\mathrm{s}}$ also. The potentials $u_{1}$ and $u_{3}$ are unchanged. The potential at $v_{2}$ is determined by the formula

$$
u_{2}=u_{3} \div \rho_{d}^{-1}\left(x_{s}\right) .
$$

With these choices it is clear that Kirchhoff's laws are satisfied for the network $N$.

Now suppose branches $b$ and $d$ are in parallel in the network $N$ and that they have nodes $\left(v_{1}, v_{2}\right)$. These branches are replaced by a single branch $p$ and $\rho_{j}^{-1}=\rho_{b}^{-1}(y)+\rho_{d}^{-1}(y)$. For this new network the currents and potentials are determined. Let $x_{p}$ be the current through $p$ directed from $v_{1}$ to $v_{2}$. Now the old branches are restored. Then the potentials $u_{1}$ and $u_{2}$ are unchanged so the currents $x_{b}$ and $x_{d}$ must satisfy

$$
\rho_{p}\left(x_{p}\right)=\rho_{b}\left(x_{b}\right)=\rho_{d}\left(x_{d}\right) .
$$

Hence $x_{b}$ and $x_{d}$ are determined by the formulas

$$
x_{b}=\rho_{b}^{-1}\left(\rho_{p}\left(x_{p}\right)\right), \quad x_{d}=\rho_{d}^{-1}\left(\rho_{p}\left(x_{p}\right)\right)
$$

Again Kirchhoff's laws are seen to be satisfied. Moreover the new formulas involve only operations of addition and inversion so this proof is complete.

Attention has been confined here to monotone resistors because in this case a mathematical solution exists and is unique. It is apparent however from the method of proof that arbitrary nonlinearity in series-parallel networks can be treated by use of multiple-valued functions.

To give an example of Theorem 4 suppose that Fig. 1 now represents a nonlinear network. Let $R_{a}(x)$ be the joint resistance function of branch $a$. Thus if a generator of voltage $e$ is inserted in this branch, giving rise to a current $x=R_{a}^{-1}(e)$ in this branch, then the explicit formula asserted by Theorem 4 is

$$
\begin{equation*}
R_{a}(x)-\rho_{a}(x)+\rho_{k}(x)+\left[\rho_{f}^{-1}+\left\{\rho_{\rho}+\left(\rho_{b}^{-1}+\rho_{d}^{-1}\right)^{-1}\right\}^{-1}\right]^{-1}(x) . \tag{2}
\end{equation*}
$$

The proof of formula (2) and the corresponding linear formula (1) follows from the proof of Theorem 3. It is worth noting that formula (2) can be evaluated graphically on a single $(x y)$ plot. It is simply necessary to plot the six functions $\rho_{a}(x), \rho_{b}(x), \rho_{d}(x), \rho_{e}(x), \rho_{k}(x), \rho_{f}(x)$ and to add ordinates or abscissas as is indicated in relation (2).

## 6. Duality

George Minty has recently introduced a postulational structure termed a graphoid [12] in which prime emphasis is placed on the duality properties of matroids, graphs, and electrical networks. Minty makes the following definitions:
$M_{p}$. Two edges are in parallel if they form a circuit.
$M_{s}$. Two edges are in series if they form a cocircuit.
In the case of a graph a cocircuit is also called a cut and is defined to be a minimal set of edges which separates two nodes. In particular two edges of a closed graph are in series according to definition $M_{s}$ if they have a common node which is not a node for any other edge. It is worth noting that only this particular case was concerned in operations $S$ and $S^{*}$.

The definitions $M_{p}$ and $M_{s}$ are dual in the sense of Corollary 3. Thus if edges $b$ and $e$ are in parallel (series) in the primal graph, then the corresponding edges $b^{\prime}$ and $e^{\prime}$ are in series (parallel) in the dual graph. This follows because the circuits in the primal graph correspond to cocircuits in the dual graph.

The duality principle then suggests the following definitions: We say that edge $a$ and edge $b$ are equipollent if there do not exist two cocircuits $D_{1}$ and $D_{2}$ such that $D_{1}$ meets $a$ and $b$ in the same sense but that $D_{2}$ meets $a$ and $b$ in opposite sense. We term a graph equipollent if every pair of edges is equipollent. This last definition gives another characterization of a series-parallel network because of the following theorem.

Theorem 5. Two edges are equipollent if and only if they are confluent. This theorem is true for graphoids as well as graphs. The proof is omitted.

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