DERIVATIVE OF Λ

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Let Λ_γ denote the Dirichlet-to-Neumann map for an electrical network with conductivity γ. By way of the Kirchhoff matrix $K = (\kappa_{ij})$, consider the space of conductivities to be a subset of $\mathbb{R}^{N\times N}$, where N is the number of vertices, We donote the map from γ to Λ_γ by L. In this note we compute the directional derivative $D_\epsilon L$ of L.

A direction in this context is represented by a matrix ϵ, with arbitrary real entries, which is symmetric and has row sum 0.

Lemma 0.1. *Let*

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix},\ \epsilon = \begin{bmatrix} \epsilon_A & \epsilon_B \\ \epsilon_B^T & \epsilon_C \end{bmatrix},$$

where ϵ is symmetric, has row sum 0, and K is a Kirchhoff matrix. Let

$$K(t) = \begin{bmatrix} A + t\epsilon_A & B + t\epsilon_B \\ B^T + t\epsilon_B^T & C + t\epsilon_C \end{bmatrix},$$

and

$$\Lambda(t) = \left[A + t\epsilon_A - (B + t\epsilon_B)(C + t\epsilon_C)^{-1}(B^T + t\epsilon_B^T)\right].$$

Then

$$D_\epsilon L = \Lambda'(0) = \epsilon_A - \epsilon_B C^{-1} B^T - BC^{-1}\epsilon_B^T + BC^{-1}\epsilon_C C^{-1} B^T.$$

Proof. Let $C(t) = C + t\epsilon_C$. Notice $C(t)(C(t))^{-1} = I$. By the product rule $C'(t)C(t) + C(t)(C(t)^{-1})' = 0$, hence $(C^{-1})'(0) = -C^{-1}\epsilon_C C^{-1}$. Using the product rule again

$$D_\epsilon L = \Lambda'(0) = \epsilon_A - \epsilon_B C^{-1} B^T - BC^{-1}\epsilon_B^T + BC^{-1}\epsilon_C C^{-1} B^T.$$

□

Corollary 0.1. *Let ϕ, ψ be boundary functions. Let u, v be γ harmonic functions with boundary values ϕ, ψ. Then*

$$\phi^T D_\epsilon L \psi = \sum_{i\neq j} \epsilon_{ij}(u_i - u_j)(v_i - v_j).$$

Proof. The interior values of the γ-harmonic function with boundary values ϕ is $-C^{-1}B^T\phi$ and with boundary values ψ is $-C^{-1}B^T\psi$. Hence by computation

$$\phi^T D_\epsilon L \psi = \begin{bmatrix} \phi^T & -\phi^T BC^{-1} \end{bmatrix} \begin{bmatrix} \epsilon_A & \epsilon_B \\ \epsilon_B^T & \epsilon_C \end{bmatrix} \begin{bmatrix} \psi \\ -C^{-1}B^T\psi \end{bmatrix} = \sum_{i\neq j} \epsilon_{ij}(u_i - u_j)(v_i - v_j)$$

□