Topics in Signed and Nonlinear Electrical Networks

David Jekel

October 23, 2014

Acknowledgements

This paper owes its existence to the summer 2013 math REU at the UW, and some thanks goes to all the faculty, TA's, and students. I especially want to thank Professor Jim Morrow who organized the REU, and Ian Zemke, one of the TAs, both of whom spent a lot of time listening to and critiquing my ideas about annular networks, which first inspired "layerings." After the REU ended, Professor Morrow told me to keep researching electrical networks, and has continued to provide valuable advice. I should also thank the REU students of past years, especially Will Johnson and Konrad Schroder, for the good ideas in their papers which inspired my own.

Remarks

For the most part, past research has focused on electrical networks with positive linear conductances, which model simple circuits with Ohmic resistors. These electrical networks were an analogue to the continuous model for electrical plates, and were used for electrical tomography. On the mathematical side, their combinatorial properties have been connected with graph theory and random walks on graphs. Several REU students have also considered networks with signed and nonlinear conductance or resistance functions. Although nonlinear networks should provide a more accurate model for real-world resistor networks, signed resistors do not have physical applications, at least not to real-world electrical networks.

However, "electrical networks" in this paper are considered as mathematical, not physical, objects and defined in sufficient generality to allow signed and nonlinear resistors. Although this generality is not physically motivated, it is mathematically justified by the interesting results which are true about signed and nonlinear networks. And it takes nothing away from the physical applications of special types of "electrical networks."

For the most part, I develop the theory from the ground up, assuming only undergraduate real analysis, linear algebra, and familiarity with the concepts of groups and manifolds. Sometimes, I refer the reader to outside sources for standard results, or to Curtis and Morrow's *Inverse Problems for Electrical Networks* [1] for results on positive linear electrical networks. Familiarity with [1] is also useful for understanding the motivations and proofs of several theorems in this paper.

I will cite student papers from the UW math REU in order to give credit where credit is due, with the caution that these papers are not polished and may contain errors.

Contents

1 Introduction

1.1 Graphs with Boundary

A *graph* G consists of two sets V (or $V(G)$) and E (or $E(G)$), a function $\iota : E \to V$, and a function $^{-} : E \to E$ with $\overline{e} \neq e$ and $\overline{\overline{e}} = e$. For $e \in E$, $\iota(e)$ is the *initial vertex* of e and $\tau(e) = \iota(\overline{e})$ is the *terminal vertex*. Let E' be the set formed by identifying each e with $\overline{e}$. An *edge* is an element of E' and an *oriented edge* is an element of E. This definition allows multiple edges with the same endpoints.

The *valence* of a vertex p is the cardinality of $\{e : \iota(e) = p\}$. Two vertices p and q are *adjacent* if there is an oriented edge e with $\iota(e) = p$ and $\tau(e) = q$. An edge e and a vertex p are *incident* if p is an endpoint of e. Two edges are incident if they share an endpoint.

A *graph with boundary* consists of a graph together with a subset $B \subset V$. Vertices in B are called *boundary vertices* and vertices in $I = V \setminus B$ are called *interior vertices*. In this paper, I will use "graph" to mean "graph with boundary" and assume that the graphs have no self-loops, that is, for each oriented edge $\iota(e) \neq \tau(e)$. We will also assume V and E are finite.[1]

A graph G' (without boundary) is a *subgraph* of G if $V(G') \subset V(G)$, $E(G') \subset E(G)$, and for $e \in E(G')$, $\iota(e)$ and $\overline{e} \in E(G')$, and the ι and $^{-}$functions for G' are the restrictions of the ι and $^{-}$functions for G. We say a graph with boundary G' is a subgraph of G if

- $V(G') \subset V(G)$, $E(G') \subset E(G)$, $I(G') \subset I(G)$.
- If $e \in E(G')$, then $\iota(e) \in V(S)$ and $\overline{e} \in E(G')$, and they are defined the same for H as for G.
- If $p \in I(G')$ and $e \in E(G')$ with $\iota(e) = p$, then $e \in E(G')$.

For a graph G, a *path* is a sequence of vertices $p_0, \dots, p_K$ and oriented edges $e_1, \dots, e_K$ such that $\iota(e_k) = p_{k-1}$ and $\tau(e_k) = p_k$. We allow a "trivial" path with one vertex and no edges. A path is an *embedded path* if the vertices $p_0, \dots, p_K$ are distinct and the non-oriented edges in the path are distinct. A *boundary-to-boundary path* is an embedded path such that p_0 and p_K are boundary vertices and the other vertices are interior. A *cycle* is a non-trivial path such that the edges are distinct, and the vertices $p_0, \dots, p_{K-1}$ are distinct with $p_K = p_0$.

A graph is *connected* if for any two vertices p and q, there exists a path from p to q. For any graph, there exist connected subgraphs $G_1, \dots, G_N$, called *components*, such that $V(G_1), \dots, V(G_N)$ are a partition of $V(G)$, and $E(G_1), \dots, E(G_N)$ are a partition of $E(G)$, and $B(G_1), \dots, B(G_N)$ are a partition of $B(G)$.

[1] Infinite electrical networks have been considered, and many of the techniques in this paper generalize to infinite networks.

1.2 Electrical Networks

An *electrical network* $\Gamma = (G, R)$ is a graph with boundary together with a function $R : E \to \mathcal{P}(\mathbb{R}^2)$, which assigns to each edge e, a set $R_e \subset \mathbb{R}^2$, called the *potential-current relationship* (PCR), such that $R_{\overline{e}} = -R_e$.

Often, the relationship R_e will be given in terms of a *conductance function* $\gamma_e : \mathbb{R} \to \mathbb{R}$, by setting

$$R_e = \{(x, \gamma_e(x)) : x \in \mathbb{R}\}.$$

For example, we could use a linear conductance function $\gamma_e(x) = a_e x$ for some $a_e \in \mathbb{R}$. If R_e is given by the conductance function γ_e, then $R_{\overline{e}}$ is given by the conductance function $\gamma_{\overline{e}}(x) = -\gamma_e(-x)$. We can also define R_e in terms of a *resistance function* $\rho_e : \mathbb{R} \to \mathbb{R}$, by setting

$$R_e = \{(\rho_e(y), y) : y \in \mathbb{R}\},$$

and then we must set $\rho_{\overline{e}}(x) = -\rho_e(-x)$. Using a relationship R_e rather than a function allows us to consider both "conductance networks," "resistance networks," some combination of the two, or something even more general.

A *potential* is a function $u : V \to \mathbb{R}$, or equivalently, a vector $u \in \mathbb{R}^V$. The potential at a vertex p will be denoted $u(p)$ or u_p. A *current function* is a function $c : E \to \mathbb{R}$ such that $c_{\overline{e}} = -c_e$ and

$$\sum_{e:\iota(e)=p} c_e = 0 \text{ for each } p \in I.$$

A potential u and current function c are *compatible* if for each edge e,

$$(u_{\iota(e)} - u_{\tau(e)}, c_e) \in R_e.$$

If this holds for an edge e, then it automatically holds for $\overline{e}$ because

$$(u_{\iota(\overline{e})} - u_{\tau(\overline{e})}, c_{\overline{e}}) = -(u_{\iota(e)} - u_{\tau(e)}, c_e) \in -R_e = R_{\overline{e}}.$$

A *harmonic function on G* is a compatible pair (u, c).

For a current function c, the *net current* at a vertex p is $\sum_{e:\iota(e)=p} c_e$. The net current an interior vertex must be zero by the above definition. For each c, there is a function $\psi_c : B \to \mathbb{R}$ mapping each vertex to its net current. For any current function, the net currents on the boundary vertices must sum to zero because

$$\sum_{p\in B} \sum_{e:\iota(e)=p} c_e = \sum_{p\in V} \sum_{e:\iota(e)=p} c_e = \sum_{e\in E} c_e = \frac{1}{2} \sum_{e\in E} (c_e + c_{\overline{e}}) = 0.$$

For an electrical network Γ, the *set of boundary data* is

$$L = \{(\phi, \psi) \in \mathbb{R}^B \times \mathbb{R}^B : \text{ there exist compatible } u \text{ and } c \text{ with } \phi = u|_B, \psi = \psi_c\}.$$

Our primary concern will be the relationship between G, R, and L. In particular, we consider the following questions:

- **The Dirichlet Problem:** Given $\phi \in \mathbb{R}^B$, does there exist a harmonic (u, c) with $u|_B = \phi$? Is it unique?
- **The Neumann Problem:** Given $\psi \in \mathbb{R}^B$, does there exist a harmonic (u, c) with $\psi = \psi_c$? Is it unique?
- **Regularity:** What conditions on R_e (or γ_e or ρ_e) and on G will guarantee that L is a smooth manifold? Does L depend "nicely" on R_e?
- **Mixed Problems:** Does there exist a harmonic function which has given potentials and given currents on a given subset of B? How does this relate to the structure of the given graph?
- **The Inverse Problem:** For a network (G, R), is R uniquely determined by G and L?

If we allow arbitrary PCR's, the inverse problem usually cannot be solved. Thus, we will generally restrict our attention to a certain set $\mathcal{R}$ of R's. (One example would be the set of R's where each R_e is given by a bijective conductance function, but the best set of R's to consider depends on the situation.) We say a network (G, R) is *recoverable over* $\mathcal{R} \subset \mathcal{P}(\mathbb{R}^2)^E$ if $R \in \mathcal{R}$ and there is no other $R' \in \mathcal{R}$ such that $L_{(G,R')} = L_{(G,R)}$. We say the *graph* G is recoverable over $\mathcal{R}$ if this holds for any $R \in \mathcal{R}$, that is, if the map $R \mapsto L$ is injective on $\mathcal{R}$.

It will become clear that, although the Dirichlet and Neumann problems require more analysis than graph theory, the inverse problem, mixed problems, and to some extent regularity depend crucially on the structure of the graph.

2 Subgraphs and Subnetworks

A *subgraph partition* of G is a collection of subgraphs $G_1, \dots, G_N$ such that

- $V(G) = \bigcup_{n=1}^N V(G_\alpha)$.
- $E(G) = \bigcup_{n=1}^N E(G_\alpha)$.
- $E(G_i) \cap E(G_j) = \varnothing$ for $i \neq j$.
- $I(G_i) \cap V(G_j) = \varnothing$ for $i \neq j$.

If S is a subgraph of G, then we define $G \setminus S$ by

- $V(G \setminus S) = V(G) \setminus I(S)$.
- $E(G \setminus S) = E(G) \setminus E(S)$.
- $I(G \setminus S) = I(G) \setminus V(S)$.

Then $G \setminus S$ is a subgraph of G. S and $G \setminus S$ form a subgraph partition of G, and S is a subgraph of $G \setminus (G \setminus S)$; however, they may not be equal, so this is not a complement in the set-theoretic sense.

A *subnetwork* of an electrical network $\Gamma = (G, R)$ is a network Σ on a subgraph S of G, such that the PCR of an edge in Σ is the same as its PCR in Γ; that is, $\Sigma = (S, R|_{E(S)})$. A *subnetwork partition* of Γ is a family of subnetworks $\Sigma_1, \ldots, \Sigma_n$ such that the underlying graphs form a subnetwork partition of G. If (u, c) is harmonic on Γ, then $(u|_{V(S)}, c|_{E(S)})$ is harmonic on Σ.

The following results generalize principles which have often been observed (see [3]):

Proposition 2.1. *Let $\Sigma_1, \ldots, \Sigma_N$ be a subnetwork partition of a network Γ. Let L be the boundary data of Γ and L_n be the boundary data of Σ_n. Then L is uniquely determined by $L_1, \ldots, L_N$.*

Proof. Let $B' = \bigcup_{n=1}^N B(S_n)$. Let $T \subset \prod_{n=1}^N L_n$ consist of all points $\prod_{n_1}(\phi_n, \psi_n)$ such that

1. If $p \in B(S_j) \cap B(S_k)$, then $(\phi_j)_p = (\phi_k)_p$.
2. If $p \in B' \cap I(G)$, then
$$\sum_{n:p \in B(S_n)} (\psi_n)_p = 0.$$

Define $F : T \to \mathbb{R}^B \times \mathbb{R}^B$ by $\prod_{n=1}^N (\phi_n, \psi_n) \mapsto (\phi, \psi)$, where

- $\phi_p = (\phi_n)_p$ whenever $p \in B(G) \cap B(S_n)$. This makes sense because $B(G) \subset \bigcup_{n=1}^N B(S_n)$, and it is well-defined by our definition of T.
- $\psi_p = \sum_{n:p \in B(G_n)} (\psi_n)_p$. Again, this works because the sum has finitely many nonzero terms.

I claim that $L = F(T)$. If $(\phi, \psi) \in L$, then there exists a harmonic (u, c) with $u|_B = \phi$ and $\psi_c = \psi$. Then $(u|_{V(S_n)}, c|_{E(S_n)})$ is harmonic on Σ_n, and its boundary data (ϕ_n, ψ_n) is in L_n. Also, the (ϕ_n, ψ_n)'s will satisfy conditions (1) and (2), so that $\prod_{n=1}^N (\phi_n, \psi_n) \in T$. Thus, $(\phi, \psi) = F(\prod_{n=1}^N (\phi_n, \psi_n)) \in F(T)$. Conversely, if (u_n, c_n) is harmonic on Σ_n with boundary data (ϕ_n, ψ_n) and $\prod_{n=1}^N (\phi_\alpha, \psi_\alpha) \in T$, then conditions (1) and (2) will guarantee that they can be glued together to a harmonic function (u, c) on Γ, so that $F(\prod_{n=1}^N (\phi_n, \psi_n)) \in L$. Since T and F only depend on $L_1, \ldots, L_N$, the proof is complete. ☐

Definition. Two networks Γ and Γ' are *electrically equivalent* if $B(G) = B(G')$ and $L_\Gamma = L_{\Gamma'}$.

Corollary 2.2 (Subnetwork Splicing)**.** *Let $\Sigma_1, \ldots, \Sigma_N$ and $\Sigma'_1, \ldots, \Sigma'_N$ be subnetwork partitions of Γ and Γ' respectively. If $B(G) = B(G')$ and Σ_n is electrically equivalent to Σ'_n, then Γ and Γ' are electrically equivalent.*

Definition. If S is a subgraph of G, and $\mathcal{R} \subset \mathcal{P}(\mathbb{R}^2)^{E(G)}$, then let $\mathcal{R}|_{E(S)} = \{R|_{E(S)} : R \in \mathcal{R}\}$.

Corollary 2.3 (Recoverability of Subgraphs)**.** *Suppose S is a subgraph of G and $\mathcal{R} \subset \mathcal{P}(\mathbb{R}^2)^{E(G)}$. If $R \in \mathcal{R}$ and (G, R) is recoverable over $\mathcal{R}$, then $(S, R|_{E(S)})$ is recoverable over $\mathcal{R}|_{E(S)}$. If the graph G is recoverable over $\mathcal{R}$, then S is recoverable over $\mathcal{R}|_{E(S)}$*

Proof. Assume $\mathcal{R} \neq \emptyset$. Suppose $\Sigma = (S, R_S)$ and $\Sigma' = (S, R'_S)$ with $R_S \neq R'_S \in \mathcal{R}|_{E(S)}$, and that Σ and Σ' are electrically equivalent. We can extend R_S and R'_S to functions R and R' on $E(G)$ such that $R_e = R'_e$ for $e \in E(G) \setminus E(S)$. Let $S^* = G \setminus S$, and Σ^* be the subnetwork on S^* with PCR's given by R_e. Then Σ and Σ^* are a partition of (G, R), and Σ' and Σ^* are a partition of (G, R'), so (G, R) and (G, R') are electrically equivalent. □

3 Signed Linear Conductances

3.1 The Kirchhoff Matrix

Let Γ be an electrical network where

$$R_e = \{(x, y) : y = a_e x\},$$

for some $a_e \in \mathbb{R}$, $a_e \neq 0$. Since $R_{\overline{e}} = -R_e$, we must have $a_{\overline{e}} = a_e$. This R_e is given by conductance function $\gamma_e(x) = a_e x$ or equivalently, resistance function $\rho_e(y) = y/a_e$. Assume the vertices have been indexed by integers $1, 2, \ldots, |V|$. Define the *Kirchhoff matrix* $K \in \mathbb{R}^{V \times V}$ by setting

$$\kappa_{pq} = -\sum_{\substack{e:\iota(e)=p,\\ \tau(p)=q}} a_e, \text{ for } p \neq q.$$

and

$$\kappa_{pp} = -\sum_{q \neq p} \kappa_{pq}.$$

Then K is symmetric and has row sums zero. For $u \in \mathbb{R}^V$, $p \in V$,

$$(Ku)_p = u_p \sum_{q \neq p} \kappa_{pq} - \sum_{q \neq p} u_q \sum_{\substack{e:\iota(e)=p,\\ \tau(p)=q}} a_e = \sum_{e:\iota(e)=p} a_e(u_p - u_{\tau(e)}).$$

Thus, if u has a compatible current function, $(Ku)_p$ gives the net current on vertex p. Thus, u is a harmonic potential if and only if $(Ku)_p = 0$ for $p \in I$. By dividing the vertices into boundary and interior, we can write K in block form as

$$K = \begin{pmatrix} K_{B,V} \\ K_{I,V} \end{pmatrix} = \begin{pmatrix} K_{B,B} & K_{B,I} \\ K_{I,B} & K_{I,I} \end{pmatrix}$$

So u is a harmonic potential if and only if $u \in \ker K_{I,V}$.

The Dirichlet and Neumann problems have an interpretation in terms of linear algebra. In the following, we will assume G is connected. There is no real loss of generality, since a harmonic function on G restricts to a harmonic function on any connected component, and harmonic functions on the connected components combine to form a harmonic function on G. And components with no boundary vertices are of little interest. If G has multiple connected components $G_1, \ldots, G_N$ and we reorder the vertices of G so that the vertices of G_1 are first, then $V(G_2)$, and so on, then the Kirchhoff matrix will decompose into blocks

$$K = \begin{pmatrix} K_1 & 0 & \cdots & 0 \\ 0 & K_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_N \end{pmatrix},$$

so the behavior of the whole can easily be understood from the behavior of the smaller blocks.

Consider the Dirichlet problem. For $\phi \in \mathbb{R}^B$, we want to find a harmonic potential u with $u|_B = \phi$. This is the same as letting $u = (\phi, w)$, where w satisfies

$$K_{I,B}\phi + K_{I,I}w = 0.$$

This will have a unique solution if and only if $K_{I,I}$ is invertible. As we will see, this does not always happen. But suppose $K_{I,I}$ is invertible. Then $w = -K_{I,I}^{-1}K_{I,B}\phi$. The current on each edge can be computed from the conductance functions. The net current on the boundary vertices is

$$\psi = K_{B,B}\phi + K_{B,I}w = (K_{B,B} - K_{B,I}K_{I,I}^{-1}K_{I,B})\phi.$$

The matrix $\Lambda = K_{B,B} - K_{B,I}K_{I,I}^{-1}K_{I,B}$ is the *Schur complement* $K/K_{I,I}$. Λ is called the *response matrix* and it acts as a *Dirichlet-to-Neumann map* $\mathbb{R}^B \to \mathbb{R}^B$ sending boundary potentials to the boundary net currents of the corresponding harmonic function. Then $L = \{(\phi, \Lambda\phi) : \phi \in \mathbb{R}^B\}$.

The Neumann problem has a similar interpretation. For $\psi \in \mathbb{R}^B$, we want to find a potential u such that

$$Ku = \begin{pmatrix} \psi \\ 0 \end{pmatrix}.$$

Of course, if ψ came from a valid current function, its entries must sum to zero as mentioned in the Introduction. We cannot expect the solution to the Neumann problem to be unique either. Indeed, if we take a harmonic function and raise the potentials on all the vertices by some constant, then the new function will be harmonic and have the same boundary currents.

So we revise the Neumann problem as follows: Let $A \subset \mathbb{R}^V$ be the set of functions whose entries sum to zero. For $(\psi, 0)^T \in A$, does there exist a unique harmonic (u, c) with $u \in A$ and $Ku = (\psi, 0)^T$? The answer is yes if and only if $K|_A$ is invertible. Since the image of K is contained in A, this happens if and only if $\operatorname{rank} K = \dim A$, which is $|V| - 1$.

3.2 Spanning Forests

Our main tool to determine when certain submatrices of K are invertible is the following combinatorial result, which generalizes the matrix-tree theorem attributed to Kirchhoff. A more general version of this formula is found in [2].

Let G be a graph. A *spanning tree* T is a subgraph (without boundary) such that T is connected, every vertex is in T, and T has no cycles. A *spanning forest* F is a subgraph such that every vertex is in T and T has no cycles; the components of F have no cycles, and are therefore *trees*.

Let P and Q be disjoint subsets of B with $|P| = |Q| = n$. Let $\mathcal{F}(P,Q)$ be the set of forests F such that each connected component either contains exactly one vertex from P and one from Q or it contains exactly one vertex from $B \setminus (P \cup Q)$. Let $K_{P\cup I, Q\cup I}$ be the submatrix of K with rows indexed by $P \cup I$ and columns by $Q \cup I$, ordered according to a given indexing of vertices by the integers $1, \ldots, |V|$. Let $p_1, \ldots, p_n$ be the vertices of P and $q_1, \ldots, q_n$ the vertices of Q ordered according to the same indexing. For any $F \in \mathcal{F}(P,Q)$, there is a permutation $\tau \in S_n$ such that p_j and $q_{\tau(j)}$ are in the same component of F; call this permutation τ_F.

Theorem 3.1. *Let P and Q be disjoint subsets of B with $|P| = |Q| = n$. Then*

$$\det K_{P\cup I, Q\cup I} = (-1)^n \sum_{F\in\mathcal{F}(P,Q)} \operatorname{sgn} \tau_F \prod_{e\in E'(F)} a_e.$$

Proof. Let $m = |I|$. Let $p_1, \ldots, p_{n+m}$ be the vertices of $P \cup I$ and $q_1, \ldots, q_{n+m}$ be the vertices of $Q \cup I$, so that $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$ and for $j > n$, $p_j = q_j \in I$. Suppose $\sigma \in S_{n+m}$; if $p_j = q_{\sigma(j)}$, then p_j must be interior. Let m_σ be the number of indices with $p_j = q_{\sigma(j)}$. By definition, $\det K_{P\cup I, Q\cup I}$ is

$$\begin{aligned}
&\sum_{\sigma\in S_{n+m}} \operatorname{sgn}\sigma \prod_{j=1}^{n+m} \kappa_{p_j, q_{\sigma(j)}} \\
&= \sum_{\sigma\in S_{n+m}} \operatorname{sgn}\sigma \left(\prod_{p_j\neq q_{\sigma(j)}} \sum_{\substack{e:\iota(e)=p_j \\ \tau(e)=q_{\sigma(j)}}} (-a_e) \right) \left(\prod_{p_j=q_{\sigma(j)}} \sum_{e:\iota(e)=p_j} a_e \right) \\
&= \sum_{\sigma\in S_{n+m}} (-1)^{n+m-m_\sigma} \operatorname{sgn}\sigma \left(\prod_{p_j\neq q_{\sigma(j)}} \sum_{\substack{e:\iota(e)=p_j \\ \tau(e)=q_{\sigma(j)}}} a_e \right) \left(\prod_{p_j=q_{\sigma(j)}} \sum_{e:\iota(e)=p_j} a_e \right)
\end{aligned}$$

Our goal is to expand each of the sums inside the product. Fix σ; choosing one term from each of the inner sums amounts to choosing for each j an edge e_j such that (1) $\iota(e_j) = p_j$ and (2) if $p_j \neq q_{\sigma(j)}$, then $\tau(e) = q_{\sigma(j)}$. Let $\mathcal{Y}$ be the collection of all sets $Y = \{e_1, \ldots, e_{n+m}\}$ such that $\iota(e_j) = p_j$. We say $\sigma \in S_{n+m}$

and $Y \in \mathcal{Y}$ are compatible if (1) and (2) are satisfied for every $e_j \in Y$. Then

$$\det K_{P\cup I,Q\cup I} = \sum_{\sigma\in S_{n+m}} (-1)^{n+m-m_\sigma} \operatorname{sgn}\sigma \sum_{\substack{\text{compatible}\\ Y\in\mathcal{Y}}} \prod_{e\in Y} a_e$$
$$= \sum_{Y\in\mathcal{Y}} \sum_{\substack{\text{compatible}\\ \sigma\in S_{n+m}}} (-1)^{n+m-m_\sigma} \operatorname{sgn}\sigma \prod_{e\in Y} a_e$$

Suppose that Y contains a sequence of edges $e_{j_1}, \dots, e_{j_k}$ with $\tau(j_\ell) = \iota(j_{\ell+1})$ for $\ell = 1, \dots, k-1$ and $\tau(e_{j_k}) = \iota(e_{j_1})$. (Either such a sequence forms a cycle or $k = 2$ and it is a pair $e, \overline{e}$.) If σ is compatible with Y, there are two possibilities: Either (1) $\sigma(j_\ell) = j_\ell$ for all ℓ or (2) $j_1 \mapsto j_2 \mapsto \dots \mapsto j_k \mapsto j_1$ is a cycle of σ. In fact, there is a one-to-one correspondence between compatible permutations satisfying (1) and those satisfying (2), and we can partition the compatible permutations into pairs $\{\sigma, \xi\sigma\}$, where $\xi \in S_{n+m}$ is the cycle $j_1 \mapsto j_2 \mapsto \dots \mapsto j_k \mapsto j_1$, such that σ satisfies (1) and $\xi\sigma$ satisfies (2). Then $m_{\xi\sigma} = m_\sigma - k$ and $\operatorname{sgn}\xi = (-1)^{k+1}$, so

$$(-1)^{n+m-m_{\xi\sigma}} \operatorname{sgn}(\xi\sigma) = (-1)^{n+m-m_\sigma-k}(-1)^{k+1} \operatorname{sgn}\sigma = -(-1)^{n+m-m_\sigma} \operatorname{sgn}\sigma.$$

Thus,

$$\sum_{\substack{\text{compatible}\\ \sigma\in S_{n+m}}} (-1)^{n+m-m_\sigma} \operatorname{sgn}\sigma = 0$$

because the terms for σ and $\xi\sigma$ cancel.

Therefore, it suffices to consider elements $Y \in \mathcal{Y}$ which do not contain cycles or pairs $\{e, \overline{e}\}$. For any such Y, there is a unique spanning forest F with $E(F) = Y \cup \overline{Y}$. I claim that

1. If Y is compatible with σ, then the corresponding F is in $\mathcal{F}(P,Q)$,

2. There is a one-to-one correspondence between compatible (Y,σ) pairs and forests F, and

3. For each (Y,σ), we have $(-1)^{n+m-m_\sigma} \operatorname{sgn}\sigma = (-1)^n \operatorname{sgn}\tau_F$.

To prove (1), it suffices to show that every component of F includes exactly one vertex from $B \setminus P$, that is, one vertex from Q or one from $B \setminus \{P \cup Q\}$. For each p_j, there is a unique outgoing $e_j \in Y$ with $\iota(e_j) = p_j$. We start at p_j and construct a path following the oriented edges of Y. As long as the last vertex is in $P \cup I$, we can continue the path. Since Y has no cycles or conjugate pairs, we cannot repeat vertices, so eventually we will reach a vertex in $B \setminus P$, so every component has one vertex from $B \setminus P$. Suppose for the sake of contradiction that it had more than one. Then there would be $r, r' \in B \setminus P$ and a path from r to r' using oriented edges $\epsilon_1, \dots, \epsilon_K \in Y \cup \overline{Y}$. We can assume without loss of generality that r and r' are the only vertices in $B \setminus P$ in the path. If $e \in Y$, then $\iota(e) \in P \cup Q$. Thus, $\epsilon_1 \notin Y$, $\epsilon_K \in Y$. Let k be the first index such that

$\epsilon_k \in Y$. Then $\epsilon_{k-1} \in \overline{Y}$, so ϵ_k and $\overline{\epsilon}_{k-1}$ are two edges in Y with the same initial vertex, which contradicts our definition of $\mathcal{Y}$.

(2) Choose F, and we will show there is a unique (Y, σ) which corresponds to F. We obtain Y from $E(F)$ by choosing one orientation for each edge. For an $e \in E(F)$, there is an embedded path from $\iota(e)$ to some $r \in B \backslash P$; this embedded path must be unique because F is a forest. There is also an embedded path from $\tau(e)$ to r, and one of the two paths must use e or $\overline{e}$. We choose the orientation which matches the orientation of the path. These orientations are uniquely determined: If we assume $e \in Y$ for some Y but that the orientation of e does not match the orientation of the path, then we reach a contradiction by the same argument as above.

To construct σ, we decompose τ_F into disjoint cycles $\eta_1, \ldots, \eta_K$. For each η_k, we define a cycle $\sigma_k \in S_{n+m}$ as follows: Let η_k be given by $i_1 \mapsto i_2 \mapsto i_R \mapsto i_1$ (the dependence on k has been suppressed in the notation). There is a unique embedded path in F from p_{i_r} to $q_{i_{r+1}}$ and the other vertices in the path are interior, so the vertices in all the paths have the form

$$\begin{array}{l} p_{i_1},\, p_{j_{1,1}} = q_{j_{1,1}},\, p_{j_{1,2}} = q_{j_{1,2}},\, \ldots,\, p_{j_{1,k_1}} = q_{j_{1,n_1}},\, q_{i_2} \\ p_{i_2},\, p_{j_{2,1}} = q_{j_{2,1}},\, p_{j_{2,2}} = q_{j_{2,2}},\, \ldots,\, p_{j_{2,k_2}} = q_{j_{2,n_2}},\, q_{i_3} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ p_{i_R},\, p_{j_{R,1}} = q_{j_{R,1}},\, p_{j_{R,2}} = q_{j_{R,2}},\, \ldots,\, p_{j_{R,k_2}} = q_{j_{R,n_R}},\, q_{i_1}. \end{array}$$

We define the cycle ξ_k by

$$\begin{array}{l} i_1 \mapsto j_{1,1} \mapsto j_{1,1} \mapsto j_{1,2} \mapsto \ldots \mapsto j_{1,n_1} \mapsto i_2 \\ i_2 \mapsto j_{2,1} \mapsto j_{2,1} \mapsto j_{2,2} \mapsto \ldots \mapsto j_{2,n_2} \mapsto i_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ i_R \mapsto j_{R,1} \mapsto j_{R,1} \mapsto j_{R,2} \mapsto \ldots \mapsto j_{R,n_R} \mapsto i_1. \end{array}$$

Then let $\sigma = \xi_1 \xi_2 \ldots \xi_n$. To show σ is uniquely determined, suppose σ is product of cycles $\xi_1, \ldots, x_k$. Suppose ξ_k given by $j_1 \mapsto j_2 \mapsto \ldots \mapsto j_L \mapsto j_1$. If each j_ℓ was greater than n (corresponding to an interior vertex), then we would have $p_{j_\ell} = q_{j_\ell} \in I$ and the edges $e_{j_1}, \ldots, e_{j_L} \in Y$ would form a cycle or pair $\{e, \overline{e}\}$, contradicting our assumptions about Y. Thus, some of the indices in the cycle are $\leq n$; it follows that ξ_k must represent boundary-to-boundary paths just as in our original construction, and the paths are uniquely determined by F

(3) Consider a cycle η_k which maps $i_1 \mapsto i_2 \mapsto i_R \mapsto i_1$, and let $j_{i,1}, \ldots, j_{i,n_r}$ be as above. Let $z_k = \sum_{r=1}^R n_r$, which is the number of interior vertices in the paths corresponding to η_k. Then $\operatorname{sgn} \xi_k = (-1)^{z_k} \operatorname{sgn} \eta_k$. The total number of interior vertices in the paths is $\sum_{k=1}^K z_k$. The interior vertices not in the paths are exactly the vertices p_j for which $\sigma(j) = j$. Hence, $\sum_{k=1}^K z_k = m - m_\sigma$. Therefore,

$$\operatorname{sgn} \sigma = \operatorname{sgn}(\xi_1 \ldots \xi_n) = (-1)^{\sum_k z_k} \operatorname{sgn}(\eta_1 \ldots \eta_n) = (-1)^{m - m_\sigma} \operatorname{sgn} \tau_F.$$

Thus, $(-1)^{n+m-m_\sigma} \operatorname{sgn}\sigma = (-1)^n \operatorname{sgn}\tau_F$. Therefore,

$$\sum_{Y\in\mathcal{Y}} \sum_{\substack{\text{compatible} \\ \sigma\in S_{n+m}}} (-1)^{n+m-m_\sigma} \operatorname{sgn}\sigma \prod_{e\in Y} a_e = (-1)^n \sum_{F\in\mathcal{F}(P,Q)} \operatorname{sgn}\tau_F \prod_{e\in E'(F)} a_e.$$

□

Corollary 3.2. *Let* $\mathcal{F} = \mathcal{F}(\varnothing, \varnothing)$. *Then* $\det K_{I,I} = \sum_{F\in\mathcal{F}} \prod_{e\in E'(F)} a_e$.

Proof. The proof is the same except that $n = 0$ and there is no τ_F. □

Corollary 3.3 (Matrix-Tree Theorem)**.** *Let G be a connected graph (without boundary). Let K be the Kirchhoff matrix of the electrical network where each edge has conductance $a_e = 1$. For $p, q \in V$, $(-1)^{p-q} \det K_{V\setminus\{p\},V\setminus\{q\}}$ is the number of spanning trees of G.*

Proof. If $p = q$, then make G into a graph with boundary by setting $B = \{p\}$. Reindex the vertices so that p occurs first; this does not change the determinant. Then by the previous theorem,

$$\det K_{V\setminus\{p\},V\setminus\{p\}} = \det K_{I,I} = \sum_{F\in\mathcal{F}} \operatorname{sgn}\tau_F.$$

Since p is the only boundary vertex, each forest is a spanning tree, so the result is the number of spanning trees. If $p \neq q$, set $B = \{p, q\}$. Reindex the vertices so that p and q occur first; this does not change the determinant, but it does change $(-1)^{p-q}$ to -1. Compute

$$\det K_{V\setminus\{p\},V\setminus\{q\}} = \det K_{I\cup\{q\},I\cup\{p\}} = -\sum_{F\in\mathcal{F}(\{q\},\{p\})} \operatorname{sgn}\tau_F.$$

Again, since p and q are the only boundary vertices, each spanning forest is a spanning tree, and τ_F is the identity. □

3.3 Singular Networks

A network for which the Dirichlet problem does not have a unique solution is called *Dirichlet-singular*; if the (revised) Neumann problem does not have a unique solution, it is *Neumann-singular*. Using the spanning forest formula, we will show that the Dirichlet and Neumann problems have a unique solution for reasonable graphs when $a_e > 0$, but if we allow a_e to be positive or negative, one can generally find values of a_e which create a Dirichlet-singular or Neumann-singular network. We assume throughout that G is connected and has some boundary vertices.

As the reader can verify, this implies that there is at least one spanning forest in $\mathcal{F}$. Hence, if $a_e > 0$,

$$\det K_{I,I} = \sum_{F\in\mathcal{F}} \prod_{e\in E'(F)} a_e > 0.$$

Therefore, the Dirichlet problem has a unique solution.

However, for most graphs there will be (possibly negative) values of $a_e \neq 0$ for which $K_{I,I}$ is not invertible. Suppose every interior vertex has valence ≥ 2. Then there are at least two forests F_1 and F_2 in $\mathcal{F}$. It is clear from the proof of Theorem 3.1 that each forest has the same number of edges. So there is an edge $e_0 \in E(F_1) \setminus E(F_2)$. Let W be the set of $a = \{a_e\}_{e \in E}$ with $a_{\overline{e}} = a_e$ and $a_e > 0$ for $e \neq e_0$, $a_{e_0} < 0$. In the spanning forest formula, the term for F_1 is negative for $a \in W$ and the term for F_2 is positive. Let $\epsilon < 1/|\mathcal{F}|$. If we choose $a \in W$ with $|a_e| = 1$ for $e \in F_1$ and $|a_e| = \epsilon$ for all other e, then $\det K_{I,I} < 0$ because the F_1 term dominates. If we set $|a_e| = 1$ for $e \in F_2$ and $|a_e| = \epsilon$ for other e, then $\det K_{I,I} > 0$. Since W is connected, the intermediate value theorem implies that there is an $a \in W$ with $\det K_{I,I} = 0$.

So most graphs have signed conductances which make them Dirichlet-singular. A more delicate question is, what are the possible values of $\dim \ker K_{I,I}$? This depends on the graph, but in some cases, it is easy to find a lower bound: Suppose $G_1, \ldots, G_N$ form a subgraph partition of G and $B(G_k) \subset B(G)$ for all k. Suppose there are Dirichlet-singular conductances for each G_k, and let the conductances on G be the same as the conductances on the G_k's. Since $\ker K_{I,I}$ is nontrivial for each G_k, there is a nonzero harmonic potential u_k on G_k, and we can extend it to G by setting it to zero on the other vertices. The potentials thus defined are linearly independent because u_k is nonzero on G_k, but u_j for $j \neq k$ is zero on G_k. Thus, $\dim \ker K_{I,I} \geq N$.

If $a_e > 0$, the Neumann problem has a unique solution. By similar reasoning as in Corollary 3.3, for any p, q,

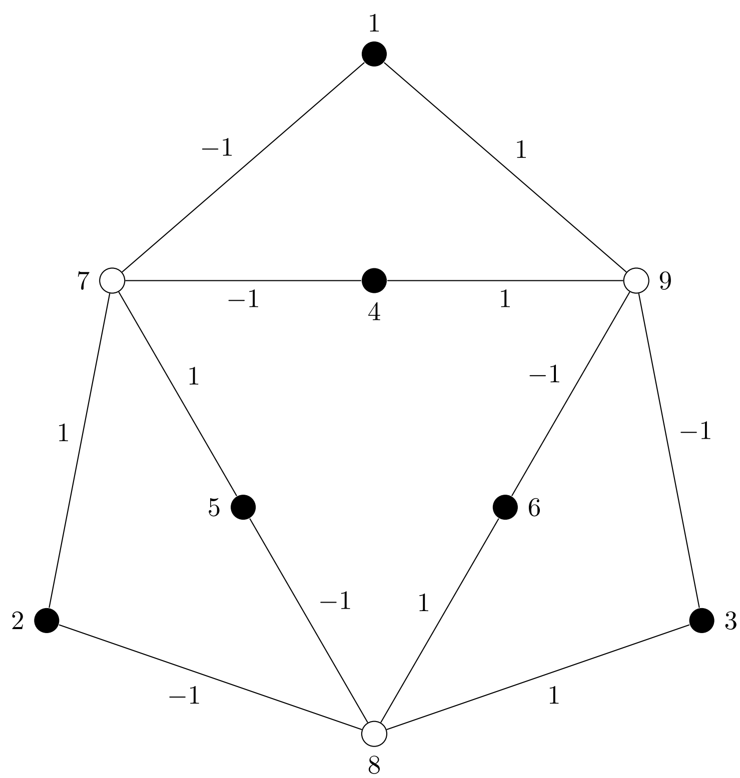
$$(-1)^{p-q} \det K_{V \setminus \{p\}, V \setminus \{q\}} = \sum_{\substack{\text{spanning} \\ \text{trees } T}} \prod_{e \in E'(T)} a_e.$$

Since G is connected, it has a spanning tree, so the right hand side is positive if $a_e > 0$. So K has rank $|V| - 1$ and the Neumann problem has a unique solution. This also shows that the determinant of any $|V| - 1$ by $|V| - 1$ submatrix of K is the same up to sign, so to see whether the Neumann problem has a unique solution, it suffices to check one of them.

If G is a tree (that is, it has no cycles), then there is only one spanning tree of G, which is all of G, so the Neumann problem has a unique solution. However, if G has a cycle, there is more than one spanning tree, so by the same argument as before, there exist signed a_e's which produce a Neumann-singular network.

What are the possible values of $\dim \ker K$? It must be ≥ 1. Now suppose $G_1, \ldots, G_N$ form a subgraph partition of G, such that each G_k is connected and any cycle of G is contained in some G_k. Suppose there exist Neumann-singular conductances on each G_k, and use them to define conductances on G. Then for each G_k, there exists a non-constant harmonic potential u_k on G_k with net current zero on every vertex. We can extend u_k to G by defining it to be constant on each G_k; this will be consistent because every cycle is contained in some G_k. Then the u_k's are linearly independent, so $\dim \ker K \geq N + 1$.

Figure 1: Singular conductances on the triangle-in-triangle network. Boundary vertices are colored in.



For some networks, it is possible for a nonzero harmonic function to have potential and current zero on the boundary, even if there are no components without boundary vertices. Consider the "triangle-in-triangle" network with boundary vertices $\{1, \ldots, 6\}$ and interior vertices $\{7, 8, 9\}$ and edges with coefficients a_e shown in the figure. The Kirchhoff matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

Let χ_p be the vector with 1 on vertex p and zero elsewhere. Then $\chi_7 + \chi_8 + \chi_9$ is a harmonic potential which is zero on the boundary and the corresponding current function has net current zero on the boundary.

3.4 Properties of L

For linear conductances, the space of harmonic functions $\mathcal{H}$ is a linear subspace of $\mathbb{R}^V \times \mathbb{R}^E$, and L is a linear subspace of $\mathbb{R}^B \times \mathbb{R}^B$. The harmonic potentials are the kernel of $K_{I,V}$, which has dimension at least $|V| - |I| = |B|$. If (u, c) is harmonic, then the boundary potentials and currents are given by $u|_B$ and $(Ku)|_B$. Let $\Phi : \ker K_{I,V} \to \mathbb{R}^{2n} : u \mapsto (u|_B, (Ku)|_B)$. Then $L = \Phi(\ker K_{I,V})$. Hence, $\dim L \leq \dim \ker K_{I,V}$. If there is a harmonic function with zero potential and current on the boundary, as in the last example, then $\ker \Phi$ is nontrivial, so this inequality is strict.

In general, we would expect $\mathcal{H}$ and L to have dimension $|B|$; this is the case if either the Dirichlet problem or the Neumann problem has a unique solution. Sometimes $\dim \mathcal{H} > |B|$; however, in all cases,

Proposition 3.4. $\dim L = |B|$.

Proof. The kernel of Φ consists of harmonic potentials which are zero on the boundary have zero current on the boundary, that is, $\ker \Phi$ consists of elements of $\ker K$ whose boundary entries are zero. Hence, $\ker \Phi$ is isomorphic to $\ker K_{V,I}$. By the rank-nullity theorem and symmetry of K,

$$\begin{aligned} \operatorname{rank} \Phi + \dim \ker \Phi &= \dim \ker K_{I,V} \\ &= |V| - \operatorname{rank} K_{I,V} \\ &= |V| - \operatorname{rank} K_{V,I} \\ &= |V| - |I| + \dim \ker K_{V,I} \\ &= |B| + \dim \ker \Phi. \end{aligned}$$

Thus, $\dim L = \operatorname{rank} \Phi = |B|$. □

If the Dirichlet problem has a unique solution, then the Dirichlet-to-Neumann map $\Lambda = K_{B,B} - K_{B,I}K_{I,I}^{-1}K_{I,B}$ is symmetric. So if (ϕ_1, ψ_1) and (ϕ_2, ψ_2) are the boundary data of harmonic functions, then

$$\phi_1 \cdot \psi_2 = \phi_1^T \Lambda \phi_2 = \phi_2^T \Lambda \phi_1 = \phi_2 \cdot \psi_1.$$

Actually, this holds even for Dirichlet-singular networks:

Proposition 3.5. *$\phi_1 \cdot \psi_2 = \phi_2 \cdot \psi_1$ for $(\phi_1, \psi_1), (\phi_2, \psi_2) \in L$.*

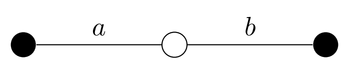
Proof. Suppose (ϕ_1, ψ_1) and (ϕ_2, ψ_2) are in L, and let u_1 and u_2 be the corresponding harmonic potentials. Let $w_1 = u_1|_I$ and $w_2 = u_2|_I$. Then $\psi_j = K_{B,B}\phi_j + K_{I,B}w_j$. Since $u_j \in \ker K_{I,V}$, we have $0 = K_{I,V}u_j = K_{I,B}\phi_j + K_{I,I}w_j$, which implies $K_{I,B}\phi_j = -K_{I,I}w_j$. Hence, applying the symmetry of K,

$$\begin{aligned}
\phi_1 \cdot \psi_2 = \phi_1^T \psi_2 &= \phi_1^T (K_{BB}\phi_2 + K_{BI}w_2) \\
&= \phi_1^T K_{B,B}\phi_2 + (K_{I,B}\phi_1)^T w_2 \\
&= \phi_1^T K_{B,B}\phi_2 - (K_{I,I}w_1)^T w_2 \\
&= \phi_1^T K_{B,B}\phi_2 - w_1^T K_{I,I} w_2 \\
&= \phi_2^T K_{B,B}\phi_1 - w_2^T K_{I,I} w_1 \\
&= \phi_2 \cdot \psi_1.
\end{aligned}$$

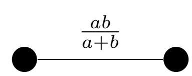
□

3.5 Local Electrical Equivalences

A *series* is the following configuration:



If $a + b \neq 0$, then it is electrically equivalent to



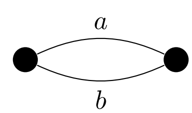
In other words, a series can be reduced to a single edge, and the resistances add: The original resistances were $1/a$ and $1/b$, and the new resistance is $1/a + 1/b$. This shows that the series is not recoverable; in fact, there is a one-parameter family of conductances on the series graph which produce the same boundary behavior.

If $a + b = 0$, then the series is Dirichlet-singular. The two boundary vertices must have the same potential. The potential of the interior vertex is independent of the boundary potentials, but depends on the current flowing from one boundary vertex to the other. In this case, changing the conductances to ca and cb for some $c \neq 0$ will produce an electrically equivalent network.

Any network which has a series as a subnetwork is not recoverable over the signed linear conductances. If $a+b \neq 0$, we can produce an electrically equivalent network by replacing the series subnetwork with a single-edge subnetwork, as follows from Corollary 2.2. This transformation is called a *series reduction* and we call it one type of *local electrical equivalence*. We also call the inverse operation is also a local electrical equivalence.

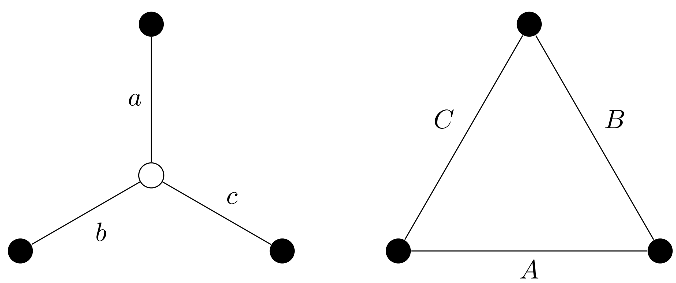
Suppose $a + b = 0$ and p and q are the endpoints of the series, and r is the middle vertex. If the series is a subnetwork of a larger network in which p is an interior vertex, then we can produce an electrically equivalent network by "collapsing" the series–identifying p and q and removing r and the edges in the series. This is because any harmonic function must have the same potential on p and q, and the amount of current flowing from p to q is independent of the potentials. This is another type of local electrical equivalence.

A parallel circuit is the following configuration:



If $a + b \neq 0$, then this is equivalent to a single edge with conductance $a + b$. If $a + b = 0$, then it is equivalent to a network with no edges. Substituting a parallel edge for a single edge or no edge is another local electrical equivalence.

A Y (left) and a Δ (right) are the following types of networks:



For any Y with $a + b + c \neq 0$, there is a unique equivalent Δ with

$$A = \frac{bc}{a+b+c}, \quad B = \frac{ac}{a+b+c}, \quad C = \frac{ab}{a+b+c}.$$

This can be proved by computing the response matrix Λ for each network. If $a + b + c = 0$, then in the Y the Dirichlet problem does not always have a solution; however, this is impossible in a Δ, so there is no equivalent Δ. For any Δ with $1/A + 1/B + 1/C \neq 0$, there is a unique equivalent Y with

$$a = \frac{AB + BC + CA}{A}, \quad b = \frac{AB + BC + CA}{B}, \quad c = \frac{AB + BC + CA}{C}.$$

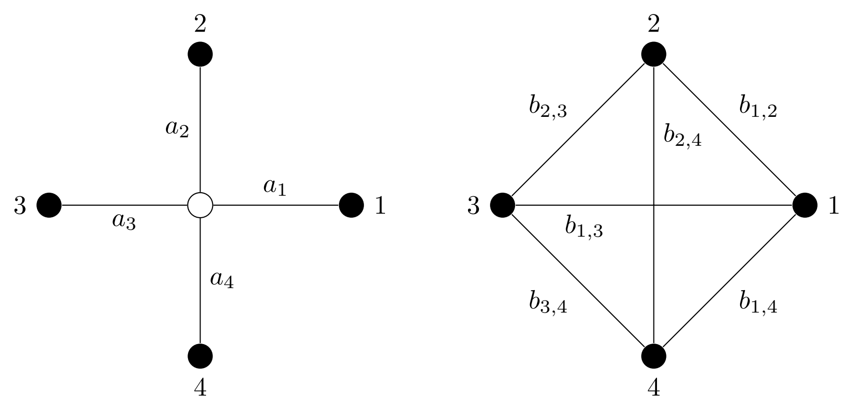
However, if $1/A + 1/B + 1/C = 0$, then the Δ is Neumann-singular because it is a tree, so there is no equivalent Y. A Y-Δ *transformation* is the transformation that replaces a Y subnetwork with an equivalent Δ subnetwork or vice versa.

Y-Δ transformations preserve recoverability over the positive linear conductances. For suppose G' is obtained from G by a Y-Δ transformation and G'

is recoverable over the positive linear conductances. For any positive linear conductances on G, we can find equivalent conductances on G'. These conductances are uniquely determined by L over the positive linear conductances. In particular, the conductances on the Y or Δ in G' are determined, but then we can find the conductances on the corresponding Δ or Y in G, so G is also recoverable.

We say two graphs are Y-Δ *equivalent* if there is a sequence of Y-Δ transformations which will change one into the other. This is an equivalence relation. If G is Y-Δ equivalent to G' and G' has a series or parallel configuration, then G' is not recoverable, and hence G is not recoverable over the positive linear conductances. This is one of the best methods for showing a graph is not recoverable, and it is applied in [1] to circular planar networks.

The final type of local electrical equivalence is the $\bigstar$-$\mathcal{K}$ *transformation* described in [6] and [3]. An n-star is a graph with n boundary vertices and one interior vertex, and one edge from the interior vertex to each boundary vertex. The *complete graph* $\mathcal{K}_n$ is a graph with n boundary vertices and one edge between each pair of distinct boundary vertices. For example, here are networks on 4-star and $\mathcal{K}_4$ graphs:



Index the vertices of the n-star and $\mathcal{K}_n$ by $1, \ldots, n$. Let a_j be the conductance of the star edge incident to j and $b_{i,j}$ the conductance of the edge in the $\mathcal{K}_n$ between vertices i and j. Let $\sigma = a_1 + \cdots + a_n$. For any star with $\sigma \neq 0$, there is an equivalent $\mathcal{K}_n$ with conductances $b_{i,j} = a_i a_j / \sigma$. If $\sigma = 0$, then the star is Dirichlet-singular and hence not equivalent to a $\mathcal{K}_n$. If $n \geq 4$, most $\mathcal{K}_n$'s are not equivalent to a star, unlike the $n = 3$ case of Y-Δ transformations:

Lemma 3.6. *Let $n \geq 4$. A network on a $\mathcal{K}_n$ is equivalent to a star if and only if*

- *It satisfies the quadrilateral rule: $b_{i,j} b_{k,\ell} = b_{i,k} b_{j,\ell}$ for distinct i, j, k, ℓ.*
- *It is not Neumann-singular.*

Proof. If the network is equivalent to a star, then for distinct i, j, k, ℓ,

$$b_{i,j} b_{k,\ell} = \frac{a_i a_j a_k a_\ell}{\sigma^2} = b_{i,k} b_{j,\ell}.$$

A star is a tree and is therefore not Neumann-singular.

Suppose conversely that a $\mathcal{K}_n$ network satisfies these three conditions. Fix i and choose distinct $k, \ell \neq i$, and let

$$a_i = \sum_{j \neq i} b_{i,j} + \frac{b_{i,k} b_{i,\ell}}{b_{k,\ell}}.$$

The quadrilateral rule guarantees that the right hand side is independent of k and ℓ. This is the current on vertex i of the potential $\chi_i - (b_{i,\ell}/b_{k,\ell})\chi_k$ on the $\mathcal{K}_n$ network. This function has net current 0 on vertex ℓ, but since $b_{i,\ell}/b_{k,\ell}$ is independent of ℓ, it has current 0 on all $\ell \neq k$. Since the potential is not constant, there must be nonzero net current on i and k, so a_i must be nonzero.

Observe $\operatorname{sgn}(b_{i,k} b_{k,\ell} b_{i,\ell}) = \operatorname{sgn}(b_{i,k} b_{i,\ell}/b_{i,k})$ is independent of k and ℓ. However, since it is symmetric in i, k, and ℓ, it is also independent of i. Suppose $\operatorname{sgn}(b_{i,k} b_{k,\ell} b_{i,\ell}) = 1$. For each i, choose c_i such that

- $|c_i| = \sqrt{b_{i,k} b_{i,\ell}/b_{k,\ell}}$ for distinct $k, \ell \neq i$.
- $\operatorname{sgn} c_1 = 1$.
- For $i \neq 1$, $\operatorname{sgn} c_i = \operatorname{sgn} b_{1,i}$.

Then for $i \neq j$, we can choose k distinct from i, j and

$$|c_i c_j| = \sqrt{\frac{b_{i,j} b_{i,k}}{b_{j,k}}} \sqrt{\frac{b_{i,j} b_{j,k}}{b_{i,k}}} = |b_{i,j}|.$$

Also, $\operatorname{sgn}(c_i c_j) = \operatorname{sgn} b_{i,j}$; this is clear if i or j equals 1, and otherwise,

$$\operatorname{sgn}(c_i c_j) = \operatorname{sgn} b_{1,i} \operatorname{sgn} b_{1,j} = \operatorname{sgn} b_{i,j}.$$

Then

$$a_i = \sum_{j \neq i} b_{i,j} + \frac{b_{i,k} b_{i,\ell}}{b_{k,\ell}} = \sum_{j \neq i} c_i c_j + c_i^2 = c_i \sum_{j=1}^{n} c_j.$$

Since $a_i \neq 0$, the sum is nonzero; hence,

$$\sigma = \sum_{i=1}^{n} c_i \sum_{j=1}^{n} c_j = \left(\sum_{i=1}^{n} c_i \right)^2 \neq 0.$$

The $\mathcal{K}_n$ is equivalent to the star because

$$\frac{a_i a_j}{\sigma} = \frac{\left(c_i \sum_{k=1}^{n} c_k\right)\left(c_j \sum_{k=1}^{n} c_k\right)}{\left(\sum_{k=1}^{n} c_k\right)^2} = c_i c_j = b_{i,j}.$$

The case where $\operatorname{sgn}(b_{i,k} b_{k,\ell} b_{i,\ell}) = -1$ follows from recognizing that a star with conductances $-a_i$ will produce a complete graph with conductances $-b_{i,j}$. □

For any finite graph G, there is a sequence of $\bigstar$-$\mathcal{K}$ moves and parallel circuit reductions that will transform it into a graph with no interior vertices. Let Γ be a signed linear network on G, and suppose that at each step, the star is nonsingular, so an equivalent $\mathcal{K}$ can be found. After the final step, the response matrix is exactly the Kirchhoff matrix because there are no interior vertices. So the $\bigstar$-$\mathcal{K}$ transformation provides a way to compute the response matrix from the Kirchhoff matrix in small steps, and in some cases, this is a useful technique for determining recoverability over positive linear conductances.

4 The Dirichlet Problem

4.1 Solutions to the Dirichlet Problem

We consider the Dirichlet problem on the following type of network: For each edge e of a graph G, let $\gamma_e : \mathbb{R} \to \mathbb{R}$ be an increasing function with $\gamma_e(0) = 0$ and $\gamma_{\overline{e}}(x) = -\gamma_e(-x)$. Let $\gamma_e(x^-) = \lim_{x' \to x^-} \gamma_e(x')$ and $\gamma(x^+) = \lim_{x' \to x} \gamma_e(x')$; these limits exist and $\gamma_e(x^-) \leq \gamma_e(x^+)$. Let

$$R_e = \{(x, y) \in \mathbb{R}^2 : \gamma_e(x^-) \leq y \leq \gamma_e(x^+)\}.$$

For $\phi \in \mathbb{R}^B$, $\mathcal{H}_\phi$ be the set of solutions to the Dirichlet problem, that is, the set of harmonic (u, c) with $u|_B = \phi$. Let $\mathcal{U}_\phi = \pi_1(\mathcal{H}_\phi)$ be the set of potentials of functions $(u, c) \in \mathcal{H}_\phi$ and let $\mathcal{C}_\phi = \pi_2(\mathcal{H}_\phi)$ be the set of current functions.

The following theorem was proved by Will Johnson [4] in the case where γ_e is continuous:

Theorem 4.1.

i. There exists a $(u, c) \in \mathcal{H}_\phi$ *satisfying* $\min_{p \in B} \phi_p \leq \min_{q \in I} u_q \leq \max_{q \in I} u_q \leq \max_{p \in B} \phi_p$.

ii. Every $u \in \mathcal{U}_\phi$ *is compatible with every* $c \in \mathcal{C}_\phi$.

iii. $\mathcal{U}_\phi$ *and* $\mathcal{C}_\phi$ *are convex sets.*

iv. For each edge e, either the potential drop $u_{\iota(e)} - u_{\tau(e)}$ *or the current* c_e *is uniquely determined. If* γ_e *is continuous, the current is uniquely determined. If* γ_e *is strictly increasing, the potential drop is uniquely determined.*

For convenience, I will say that any function u *satisfies the maximum principle* if $\min_{p \in B} \phi_p \leq \min_{q \in I} u_q \leq \max_{q \in I} u_q \leq \max_{p \in B} \phi_p$. To prove the theorem, we need the following definitions and results from convex analysis:

Definition. $S \subset \mathbb{R}^d$ is convex if for all $x, y \in S$ and $t \in [0, 1]$, $(1 - t)x + ty \in S$.

Definition. A function $f : \mathbb{R}^d \to \mathbb{R}$ is convex if for any $x, y \in \mathbb{R}^d$ and $t \in [0, 1]$,

$$f((1 - t)x + ty) \leq (1 - t)f(x) + tf(y).$$

Definition. Let $f : \mathbb{R}^d \to \mathbb{R}$. A vector $v \in \mathbb{R}^d$ is a called a *subgradient* of f at x if

$$f(y) - f(x) \leq v \cdot (y - x) \text{ for all } x \in \mathbb{R}^d.$$

The *subdifferential* $\partial f(x)$ is the set of all subgradients of f at x.

Lemma 4.2. *If f is convex, then for any x, $\partial f(x)$ is nonempty and convex.*

Lemma 4.3. *If $f : \mathbb{R} \to \mathbb{R}$ is increasing, then $g(x) = \int_0^x f(t)\,dt$ is convex and $\partial g(x) = [f(x^-), f(x^+)]$.*

Lemma 4.4. *If $f_1, \ldots, f_n$ are convex, then $f = f_1 + \cdots + f_n$ is convex, and*

$$\partial f(x) = \partial f_1(x) + \cdots + \partial f_n(x),$$

where addition denotes addition of sets.

Proof of Theorem 4.1. For $e \in E$, define $Q_e : \mathbb{R} \to \mathbb{R}$ by

$$Q_e(x) = \int_0^x \gamma_e(t)\,dt.$$

Then Q_e is nonnegative convex function with $Q_{\overline{e}}(x) = Q_e(-x)$ and $Q_e(0) = 0$. Define the total pseudopower $Q : \mathbb{R}^V \to \mathbb{R}$ by

$$Q(u) = \frac{1}{2} \sum_{e \in E} Q_e(u_{\iota(e)} - u_{\tau(e)}) = \sum_{e \in E'} Q_e(u_{\iota(e)} - u_{\tau(e)}).$$

The last expression makes sense because $Q_{\overline{e}}(u_{\iota(\overline{e})} - u_{\tau(\overline{e})}) = Q_e(u_{\iota(e)} - u_{\tau(e)})$.

For $\phi \in \mathbb{R}^B$ and $w \in \mathbb{R}^I$, I will write $u = (\phi, w)$ for $u|_B = \phi$ and $u|_I = w$. Fix ϕ and let $Q_\phi(w) = Q(u)$, where $u = (\phi, w)$. Q_ϕ is also convex. We can write

$$Q_\phi(w) = \sum_{e \in E'} F_{\phi,e}(w), \text{ where } F_{\phi,e}(w) = Q_e(u_{\iota(e)} - u_{\tau(e)}).$$

Let χ_p be the vector in $\mathbb{R}^I$ with a 1 on vertex p and 0 elsewhere, and let

$$\chi_e = \begin{cases} \chi_{\iota(e)} - \chi_{\tau(e)}, & \text{if } \iota(e) \in I, \tau(e) \in I \\ \chi_{\iota(e)}, & \text{if } \iota(e) \in I, \tau(e) \in B \\ -\chi_{\tau(e)}, & \text{if } \iota(e) \in B, \tau(e) \in I \\ 0, & \text{if } \iota(e) \in B, \tau(e) \in B. \end{cases}$$

Then it is not too hard to show

$$\begin{aligned} \partial F_{\phi,e}(w) &= \chi_e \cdot \partial Q_e(u_{\iota(e)} - u_{\tau(e)}) \\ &= \chi_e \cdot [\gamma_e((u_{\iota(e)} - u_{\tau(e)})^-), \gamma_e((u_{\iota(e)} - u_{\tau(e)})^+)] \end{aligned}$$

Thus,

$$\partial Q_\phi(w) = \sum_{e \in E'} \partial F_{\phi,e}(w) = \sum_{e \in E'} \chi_e \cdot [\gamma_e((u_{\iota(e)} - u_{\tau(e)})^-), \gamma_e((u_{\iota(e)} - u_{\tau(e)})^+)].$$

I claim that (ϕ, w) has a compatible current function if and only if $0 \in \partial Q_\phi(w)$. Indeed, if $0 \in Q_\phi(w)$, then for each $e \in E'$, we can choose $c_e \in [\gamma_e((u_{\iota(e)} - u_{\tau(e)})^-), \gamma_e((u_{\iota(e)} - u_{\tau(e)})^+)]$ such that

$$\sum_{e \in E'} c_e \chi_e = 0.$$

For each $p \in I$, examining the p-entry of this equation yields

$$\sum_{\substack{e \in E' \\ \iota(e)=p}} c_e - \sum_{\substack{e \in E' \\ \tau(e)=p}} c_e = 0,$$

which means the net current on p is 0. Hence c defines a current function which is compatible with u. By reversing this reasoning, we see that if c is a current function compatible with u, then $0 \in \partial Q_\phi(w)$.

Observe that $0 \in \partial Q_\phi(w)$ if and only if w is a global minimum of Q_ϕ, so our goal is show that a minimum is achieved. Let $m = \min_{p \in B} \phi_p$ and $M = \max_{p \in B} \phi_p$. Since $[m, M]^I$ is compact, Q_ϕ achieves a minimum on $[m, M]^I$ at some point w^*. I claim w^* is a global minimum of Q_ϕ. Suppose $w \in \mathbb{R}^I$. Let $\tilde{w} \in \mathbb{R}^I$ be given by

$$\tilde{w}_p = \begin{cases} m, & w_p < m, \\ w_p, & m \le w_p \le M, \\ M, & w_p \ge M. \end{cases}$$

Let $u = (\phi, w)$ and $\tilde{u} = (\phi, \tilde{w})$. Then for each e,

$$|\tilde{u}_{\iota(e)} - \tilde{u}_{\tau(e)}| \le |u_{\iota(e)} - u_{\tau(e)}|, \qquad \operatorname{sgn}(\tilde{u}_{\iota(e)} - \tilde{u}_{\tau(e)}) = \operatorname{sgn}(u_{\iota(e)} - u_{\tau(e)}).$$

Now Q_e is increasing for $x \ge 0$ and decreasing for $x \le 0$; therefore,

$$Q_e(\tilde{u}_{\iota(e)} - \tilde{u}_{\tau(e)}) \le Q_e(u_{\iota(e)} - u_{\tau(e)}).$$

Hence, $Q(\tilde{u}) \le Q(u)$. Since $\tilde{w} \in [m, M]^I$, we have

$$Q_\phi(w) \ge Q_\phi(\tilde{w}) \ge Q_\phi(w^*),$$

so w^* is indeed a global minimum. Thus, $u^* = (\phi, w^*)$ has a compatible current function c^*. By construction, u^* satisfies $m \le \min_{q \in I} u_q^* \le \max_{q \in I} u_q^* \le M$, so (i) is proved.

To prove (ii), it suffices to show that if u and $\tilde{u}$ are in $\mathcal{U}_\phi$ and u is compatible with c, then $\tilde{u}$ is also compatible with c. Because c_e is a subderivative of Q_e at $u_{\iota(e)} - u_{\tau(e)}$, we have

$$Q_e(\tilde{u}_{\iota(e)} - \tilde{u}_{\tau(e)}) - Q_e(u_{\iota(e)} - u_{\tau(e)}) - c_e\Big((\tilde{u}_{\iota(e)} - \tilde{u}_{\tau(e)}) - (u_{\iota(e)} - u_{\tau(e)})\Big) \ge 0.$$

Summing the left hand side over $e \in E'$ yields

$$Q_\phi(\tilde{w}) - Q_\phi(w) - \sum_{e \in E'} c_e\Big((\tilde{u}_{\iota(e)} - \tilde{u}_{\tau(e)}) - (u_{\iota(e)} - u_{\tau(e)})\Big),$$

and the first two terms cancel because $\tilde{w}$ and w must both achieve the global minimum of Q_ϕ. The other sum is

$$\begin{aligned}\sum_{e\in E'} c_e\Big((\tilde{u}_{\iota(e)}-\tilde{u}_{\tau(e)})-(u_{\iota(e)}-u_{\tau(e)})\Big) &= \sum_{e\in E'} c_e\Big((\tilde{u}_{\iota(e)}-u_{\iota(e)})-(\tilde{u}_{\tau(e)}-u_{\tau(e)})\Big)\\ &= \sum_{e\in E} c_e(\tilde{u}_{\iota(e)}-u_{\iota(e)})\\ &= \sum_{p\in V}\sum_{\substack{e\in E\\ \iota(e)=p}} c_e(\tilde{u}_p-u_p).\end{aligned}$$

This is zero because If $p\in I$, then $\sum_{\iota(e)=p} c_e = 0$, but if $p\in B$, then $\tilde{u}_p - u_p = \phi_p - \phi_p = 0$. Hence,

$$\sum_{e\in E'}\bigg(Q_e(\tilde{u}_{\iota(e)}-\tilde{u}_{\tau(e)})-Q_e(u_{\iota(e)}-u_{\tau(e)})-c_e\Big((\tilde{u}_{\iota(e)}-\tilde{u}_{\tau(e)})-(u_{\iota(e)}-u_{\tau(e)})\Big)\bigg)=0,$$

but each term is nonnegative, so each term must be zero. Since $c_e \in \partial Q_e(u_{\iota(e)} - u_{\tau(e)})$, we have for any $x\in\mathbb{R}$,

$$\begin{aligned}&Q_e(x)-Q_e(\tilde{u}_{\iota(e)}-\tilde{u}_{\tau(e)})-c_e\Big(x-(\tilde{u}_{\iota(e)}-\tilde{u}_{\tau(e)})\Big)\\ &\quad= Q_e(x)-Q_e(u_{\iota(e)}-u_{\tau(e)})-c_e\Big(x-(u_{\iota(e)}-u_{\tau(e)})\Big)\\ &\quad\geq 0.\end{aligned}$$

Therefore, c_e is a subderivative of Q_e at $\tilde{u}_{\iota(e)} - \tilde{u}_{\tau(e)}$, and hence

$$\gamma_e((\tilde{u}_{\iota(e)}-\tilde{u}_{\tau(e)})^-) \leq c_e \leq \gamma_e((\tilde{u}_{\iota(e)}-\tilde{u}_{\tau(e)})^+),$$

and $\tilde{u}$ is compatible with c.

For (iii), note that the set of minimizers of a convex function is convex, so the set of w's which minimize Q_ϕ is convex. Thus, if u and $\tilde{u}$ are in $\mathcal{U}_\phi$, then so is $(1-t)u + t\tilde{u} = (\phi, (1-t)w + t\tilde{w})$. Thus, $\mathcal{U}_\phi$ is convex. Next, suppose c and $\tilde{c}$ are in $\mathcal{C}_\phi$. Then by (ii), there is a u which is compatible with both c and $\tilde{c}$. Then $(1-t)c + t\tilde{c}$ will be a valid current function because it has net current zero on the interior vertices, and it will be compatible with u, because if $c_e, \tilde{c}_e \in [\gamma_e((u_{\iota(e)} - u_{\tau(e)})^-, \gamma_e((u_{\iota(e)} - u_{\tau(e)})^+]$, then so is $(1-t)c_e + t\tilde{c}_e$. Thus, $\mathcal{C}_\phi$ is convex.

For (iv), choose an edge e. Suppose the current on e is not uniquely determined, so that there exist $c, \tilde{c} \in \mathcal{C}_\phi$ with $c_e < \tilde{c}_e$. Any $u \in \mathcal{U}_\phi$ must be compatible with both c and $\tilde{c}$, so

$$c_e, \tilde{c}_e \in [\gamma_e((u_{\iota(e)}-u_{\tau(e)})^-), \gamma_e((u_{\iota(e)}-u_{\tau(e)})^+)].$$

Since γ_e is increasing, this can only happen for one value of $u_{\iota(e)} - u_{\tau(e)}$, and it is impossible if γ_e is continuous. If γ_e is strictly increasing, then different potential drops on e cannot produce the same current, so (ii) implies that any $u \in \mathcal{U}_\phi$ has the same potential drop on e. □

Now that we have existence and something like uniqueness of a solution, a natural question to ask is whether u depends continuously on ϕ in some sense. The maximum principle asserts that we can make u depend continuously on ϕ at 0, and indeed, we can find a value of u that depends continuously on ϕ. For $u \in \mathbb{R}^V$, let $\|u\|_\infty$ be the uniform norm $\max_{p\in V} |u_p|$, and make the same definition for $\phi \in \mathbb{R}^B$. Then

Proposition 4.5. *There exists a continuous $U : \mathbb{R}^B \to \mathbb{R}^V$ such that $U(\phi) \in \mathcal{U}_\phi$ and*

$$\|U(\phi_1) - U(\phi_2)\|_\infty = \|\phi_1 - \phi_2\|_\infty .$$

To prove this, we need a few results from analysis:

Definition. A sequence of functions $\{f_n\}$ from $\mathbb{R}^d \to \mathbb{R}^{d'}$ is *equicontinuous* if for any $x \in \mathbb{R}^d$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$|y - x| < \delta \text{ implies } |f_n(y) - f_n(x)| < \epsilon \text{ for all } n.$$

Definition. A sequence of functions $\{f_n\}$ is *pointwise bounded* if for any $x \in \mathbb{R}^d$, $\{f_n(x)\}$ is a bounded set.

Lemma 4.6 (Arzela-Ascoli Theorem)**.** *Suppose $f_n : \mathbb{R}^d \to \mathbb{R}^{d'}$ is a sequence which is equicontinuous and pointwise bounded. Then there is a subsequence which converges uniformly on compact sets to a continuous function f.*

Proof of Proposition 4.5. We can assume without loss of generality that every component of the graph has a boundary vertex. Indeed, on a component with no boundary vertex, we can always set $U(\phi)$ to be identically zero.

First consider the case where γ_e is strictly increasing. Then by (iv), the potential drop on each edge is uniquely determined, and if every component has a boundary vertex, the potentials themselves are uniquely determined. Let (u^*, c^*) be a harmonic function on Γ. Define

$$\widehat{\gamma}_e(x) = \gamma_e(u^*_{\iota(e)} - u^*_{\tau(e)} + x) - c^*_e$$

Then $\widehat{\gamma}_e$ is strictly increasing and because c^*_e is between the right and left-hand limits of $\gamma_e(u^*_{\iota(e)} - u^*_{\tau(e)})$, we can make it zero-preserving by changing the value at 0 if necessary. Let $\widehat{\Gamma}$ be the corresponding electrical network. If (u, c) is a harmonic potential on Γ, then $(u - u^*, c - c^*)$ is a harmonic potential on $\widehat{\Gamma}$. Since the potential of the solution to Dirichlet problem on $\widehat{\Gamma}$ is unique and it must satisfy the maximum principle, we have

$$\|u - u^*\|_\infty \leq \|\phi - \phi^*\|_\infty , \text{ where } \phi = u|_B, \quad \phi^* = u^*|_B.$$

Thus, if $U(\phi)$ is the harmonic potential with boundary potentials ϕ, then U is continuous and satisfies the desired estimate.

Now suppose γ_e is weakly increasing. Let $\gamma_{n,e}(x) = \gamma_e(x) + x/n$, so that $\gamma_{n,e}$ is strictly increasing and $Q_{n,e}(x) = Q_e(x) + x^2/2n$. Note $Q_{n,e} \to Q_e$ and

$Q_n \to Q$ uniformly on compact sets. Let $U_n(\phi)$ be the unique harmonic potential for $\gamma_{n,e}$. Because $\|U_n(\phi_1) - U_n(\phi_2)\|_\infty = \|\phi_1 - \phi_2\|_\infty$ and $\|U_n(\phi)\|_\infty = \|\phi\|_\infty$, the sequence $\{U_n\}$ is equicontinuous and pointwise bounded. Therefore, by the Arzela-Ascoli theorem, there is a subsequence $\{U_{n_k}\}$ converging uniformly on compact sets to a continuous function U.

Suppose u is any potential with $u|_B = \phi$. Since $Q_{n_k} \geq Q$ and $U_{n_k}(\phi)$ minimizes Q over potential functions with boundary values ϕ,

$$Q_{n_k}(u) \geq Q_{n_k}(U_{n_k}(\phi)) = (Q_{n_k}(U_{n_k}(\phi)) - Q(U_{n_k}(\phi))) + Q(U_{n_k}(\phi))$$

By uniform convergence on compact sets, $Q_{n_k}(U_{n_k}(\phi)) - Q(U_{n_k}(\phi)) \to 0$ and by continuity of Q, $Q(U_{n_k}(\phi)) \to Q(U(\phi))$. Thus, taking $k \to \infty$ yields $Q(u) \geq Q(U(\phi))$. Hence, $U(\phi)$ minimizes Q over potential functions with boundary values ϕ, so it is a harmonic potential for conductances γ. Also,

$$\|U(\phi_1) - U(\phi_2)\|_\infty = \lim_{k\to\infty} \|U_{n_k}(\phi_1) - U_{n_k}(\phi_2)\|_\infty = \|\phi_1 - \phi_2\|_\infty . \qquad \square$$

4.2 The Dirichlet-to-Neumann Map Λ

Let Γ be as in the previous section, and in addition assume that γ_e is continuous. Then there exists a solution (u, c) to the Dirichlet problem and c is uniquely determined. In particular, the net current ψ on the boundary vertices is uniquely determined by the boundary potentials ϕ. Hence, there a well-defined Dirichlet-to-Neumann map $\Lambda : \mathbb{R}^B \to \mathbb{R}^B : \phi \mapsto \psi$.

Since γ_e is continuous, Q_e is C^1 and

$$\nabla Q(u) = \nabla_u \sum_{e\in E'} Q_e(u_{\iota(e)} - u_{\tau(e)}) = \sum_{e\in E'} \chi_e \gamma_e(u_{\iota(e)} - u_{\tau(e)}).$$

So

$$\partial_p Q(u) = \frac{\partial}{\partial u_p} Q(u) = \sum_{\substack{e\in E \\ \iota(e)=p}} \gamma_e(u_{\iota(e)} - u_{\tau(e)}),$$

and in particular, if (u, c) is a solution to the Dirichlet problem,

$$\partial_p Q(u) = \begin{cases} 0, & \text{if } p \in I \\ \psi_p, & \text{if } p \in B. \end{cases}$$

Hence, if $\pi_B : \mathbb{R}^V \to \mathbb{R}^B$ is projection onto the boundary vertices, then

$$\Lambda(\phi) = \pi_B \circ \nabla Q(u) \text{ for any } u \in \mathcal{U}_\phi.$$

And if $U(\phi)$ is a harmonic potential depending continuously on ϕ, as in Proposition 4.5, then

$$\Lambda(\phi) = \pi_B \circ \nabla Q \circ U(\phi),$$

which shows that Λ is continuous. It also depends "continuously" on γ:

Proposition 4.7. *Suppose that γ_n and γ_0 are continuous, increasing conductances on a graph G and Λ_n and Λ_0 are the corresponding Dirichlet-to-Neumann maps. If $\gamma_{n,e} \to \gamma_{0,e}$, then $\Lambda_n \to \Lambda_0$ uniformly on compact sets.*

We need the following lemmas, whose proofs are left as exercises:

Lemma 4.8. *Suppose g_n and g are increasing functions $\mathbb{R} \to \mathbb{R}$ and $g_n \to g$. If g is continuous, then the convergence is uniform on compact sets.*

Lemma 4.9. *Let $f_n : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ and $g_n : \mathbb{R}^{d_2} \to \mathbb{R}^{d_3}$ be continuous. If $f_n \to f$ uniformly on compact sets and $g_n \to g$ uniformly on compact sets, then $g_n \circ f_n \to g \circ f$ uniformly on compact sets.*

Lemma 4.10. *Let $f_n : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$. If every subsequence of $\{f_n\}$ has in turn a subsequence converging uniformly on compact sets to f, then $f_n \to f$ uniformly on compact sets.*

Proof of Proposition 4.7. Observe that if $\pi_I : \mathbb{R}^V \to \mathbb{R}^I$ is the projection onto the interior vertices, then $u = (\phi, w)$ is in $\mathcal{U}_\phi$ if and only if w minimizes Q_ϕ if and only if $\pi_I \circ \nabla Q(u) = \nabla Q_\phi(w) = 0$.

Let Q_n and Q_0 be the pseudopower corresponding to γ_n and γ_0, and let $U_n(\phi)$ and $U_0(\phi)$ be harmonic potentials as in Proposition 4.5. Since $\gamma_{n,e} \to \gamma_{0,e}$ uniformly on compact sets, the same is true for ∇Q_n and ∇Q_0. Let $\{\Lambda_{n_k}\}$ be any subsequence of $\{\Lambda_n\}$. Since $\{U_{n_k}\}$ is equicontinuous and pointwise bounded, there is a subsequence $\{U_{n_{k_j}}\}$ converging uniformly on compact sets to a function U_0. Since $Q_n \to Q_0$ uniformly on compact sets, we see that U_0 is a harmonic potential by the same argument as in Proposition 4.5. By Lemma 4.9, $\nabla Q_{n_{k_j}} \circ U_{n_{k_j}} \to \nabla Q_0 \circ U_0$ on compact sets; hence,

$$\Lambda_{n_{k_j}} = \pi_B \circ \nabla Q_{n_{k_j}} \circ U_{n_{k_j}} \to \pi_B \circ \nabla Q_0 \circ U_0 = \Lambda_0$$

uniformly on compact sets. □

It is actually not necessary to assume that all the γ_e's are continuous. If we assume instead that γ_e is continuous for every edge incident to a boundary vertex, then by Theorem 4.1 (iv), the boundary currents are uniquely determined. They also depend continuously on ϕ.

Proposition 4.7 also generalizes: If $\gamma_{n,e}$ is increasing but not necessarily continuous, then pointwise convergence of $\gamma_{n,e} \to \gamma_{0,e}$ implies $Q_{n,e} \to Q_{0,e}$ uniformly on compact sets. Since $Q_n \to Q_0$ uniformly on compact sets, U_0 will still be a harmonic potential. If we assume $\gamma_{n,e}$ and γ_e are continuous when e is incident to a boundary vertex, and hence $\gamma_{n,e} \to \gamma_e$ uniformly on compact sets, then we still obtain $\Lambda_{n_{k_j}} \to \Lambda_0$ uniformly on compact sets.

4.3 Differentiation of Λ and U

The goal of this section is to differentiate (or linearly approximate) the Dirichlet-to-Neumann map Λ. Assume each γ_e is differentiable. For $u \in \mathbb{R}^V$, define a set of linear conductance functions $d_u\gamma$ by

$$(d_u\gamma)_e(x) = \gamma_e'(u_{\iota(e)} - u_{\tau(e)})x.$$

These conductances satisfy

$$(d_u\gamma)_{\overline{e}}(x) = \gamma_{\overline{e}}'(u_{\tau(e)} - u_{\iota(e)})x = \gamma_e'(u_{\iota(e)} - u_{\tau(e)})x = -(d_u\gamma)_e(-x)$$

because

$$\gamma_{\overline{e}}'(x) = \frac{d}{dx}\gamma_{\overline{e}}(x) = -\frac{d}{dx}\gamma_e(-x) = \gamma_e'(-x).$$

Let Λ_γ be the Dirichlet-to-Neumann map for the network on G with conductances γ, and let $\Lambda_{d_u\gamma}$ be the Dirichlet-to-Neumann map for the network on G with conductances $d_u\gamma$. Then

Theorem 4.11. *Λ_γ is differentiable with respect to ϕ. For a given ϕ, the differential $D_\phi\Lambda_\gamma : \mathbb{R}^B \to \mathbb{R}^B$ is given by $D_\phi\Lambda_\gamma = \Lambda_{d_u\gamma}$, where u is any harmonic potential with $u|_B = \phi$.*

We need the following lemma on linear conductances:

Lemma 4.12. *Let $a_e \geq 0$, $a_{\overline{e}} = a_e$, and $a = \{a_e\}_{e\in E}$. For linear conductances $\gamma_e(x) = a_e x$ with $a_e \geq 0$, the Dirichlet-to-Neumann map Λ_γ is a linear transformation given by a response matrix M_a, which depends continuously on a.*

Proof. If the conductances are linear, then a linear combination of harmonic functions is a harmonic function, so the Dirichlet-to-Neumann map is linear. Hence, it is given by a matrix M_a.

To show continuity, suppose $a_n = \{a_{n,e}\}_{e\in E}$, and that $a_{n,e} \to a_e$. If $\gamma_{n,e}(x) = a_{n,e}x$ and $\gamma_e(x) = a_e x$, then $\gamma_{n,e} \to \gamma_e$. Thus, by Proposition 4.7, $\Lambda_{\gamma_n} \to \Lambda_\gamma$ uniformly on compact sets. This implies that the matrices M_{a_n} converge to M_a entry-wise. □

Proof of Theorem 4.11. Using a similar translation argument as in Proposition 4.5, we can reduce to the case where $\phi = 0$ and $u = 0$. For $u \in \mathbb{R}^V$, define coefficients

$$a_{u,e} = \begin{cases} \frac{\gamma_e(u_{\iota(e)} - u_{\tau(e)})}{u_{\iota(e)} - u_{\tau(e)}}, & \text{if } u_{\iota(e)} \neq u_{\tau(e)} \\ \gamma_e'(0), & \text{if } u_{\iota(e)} = u_{\tau(e)}. \end{cases}$$

Let $a_u = \{a_{u,e}\}_{e\in E}$ and $(\Delta_u\gamma)_e(x) = a_{u,e}x$ and $\Delta_u\gamma = \{(\Delta_u\gamma)_e\}_{e\in E}$. Then $\Delta_u\gamma$ defines a set of a linear conductances and $\Lambda_{\Delta_u\gamma}(\phi) = M_{a_u}\phi$. Also, at $u = 0$, $\Delta_0\gamma = d_0\gamma$. Since $a_{u,e}$ depends continuously on u, we know M_{a_u} depends continuously on u. For each edge e,

$$(\Delta_u\gamma)_e(u_{\iota(e)} - u_{\tau(e)}) = \frac{\gamma_e(u_{\iota(e)} - u_{\tau(e)})}{u_{\iota(e)} - u_{\tau(e)}}(u_{\iota(e)} - u_{\tau(e)}) = \gamma_e(u_{\iota(e)} - u_{\tau(e)}).$$

if $u_{\iota(e)} = u_{\tau(e)}$, and if $u_{\iota(e)} = u_{\tau(e)}$, then both sides are zero. In particular, if u is compatible with a current function c on the network with conductances γ, then it is compatible with c on the network with conductances $\Delta_u\gamma$. So if ϕ and ψ represent the boundary potentials and net currents for (u, c), we have

$$\Lambda_\gamma(\phi) = \psi = \Lambda_{\Delta_u\gamma}(\phi) = M_{a_u}\phi.$$

Let $U(\phi)$ be a solution of the Dirichlet problem satisfying the maximum principle. Then

$$\Lambda_\gamma(\phi) = M_{a_{U(\phi)}}\phi,$$

and since M_{a_u} depends continuously on u, $\lim_{\phi\to 0} M_{a_{U(\phi)}} = M_{a_0}$. Therefore, Λ_γ is differentiable at 0, and the differential $D_0\Lambda_\gamma$ is the linear transformation given by the matrix M_{a_0}, which is exactly $\Lambda_{d_0\gamma}$. □

In the case where $\gamma_e' > 0$, we can say more. In the following, we identify the linear transformation $D_\phi\Lambda$ with its matrix.

Proposition 4.13. *Suppose every component of G has a boundary vertex. Let γ_e be differentiable with $\gamma_e' > 0$. Let H_uQ be the Hessian matrix of the total pseudopower at $u \in \mathbb{R}^V$. Then*

i. The Dirichlet problem has a unique solution.

ii. For any $u \in \mathbb{R}^V$, $H_uQ_{I,I}$ is invertible.

iii. Let $U(\phi)$ be the potential for the solution to the Dirichlet problem and $W(\phi) = \pi_I \circ U(\phi)$. Then $D_\phi W = -(H_{U(\phi)}Q)_{I,I}^{-1}(H_{U(\phi)}Q)_{I,B}$.

iv. $D_\phi\Lambda$ is the Schur complement $H_{U(\phi)}Q/(H_{U(\phi)}Q)_{I,I}$.

Proof. The solution to the Dirichlet problem is unique because γ_e is strictly increasing. By computation, the mixed partial

$$\partial_p\partial_q Q(u) = \begin{cases} \sum_{\substack{e:\iota(e)=p\\ \tau(e)=q}} \gamma_e'(u_{\iota(e)} - u_{\tau(e)}), & p \neq q \\ \sum_{e:\iota(e)=p} \gamma_e'(u_{\iota(e)} - u_{\tau(e)}), & p = q. \end{cases}$$

Thus, H_uQ is exactly the Kirchhoff matrix of the network with linear conductances $d_u\gamma$. Invertibility of $(H_uQ)_{I,I}$ follows from our discussion of positive linear conductances. For (iii), it suffices to consider the case $\phi = 0$. Since $u = U(\phi)$ is a harmonic potential with respect to $\Delta_u\gamma$,

$$W(\phi) = -(K_{\Delta_u\gamma})_{I,I}^{-1}(K_{\Delta_u\gamma})_{I,B}\phi,$$

where $K_{\Delta_u\gamma}$ is the Kirchhoff matrix for the linear conductances $\Delta_u\gamma$. Thus,

$$D_0W = -(K_{d_0\gamma})_{I,I}(K_{d_0\gamma})_{I,B} = -(H_0Q)_{I,I}^{-1}(H_0Q)_{I,B}.$$

(iv) follows from the chain rule:

$$\begin{aligned} D_\phi\Lambda &= D_\phi\pi_B \circ \nabla Q(U(\phi)) \\ &= (H_uQ)_{B,B} + (H_uQ)_{B,I} \circ D_\phi W \\ &= (H_uQ)_{B,B} - (H_uQ)_{B,I}(H_uQ)_{I,I}^{-1}(H_uQ)_{B,I}. \end{aligned}$$ □

4.4 Linearizing the Inverse Problem

Proposition 4.13 allows us to "linearize" the inverse problem for differentiable conductances. Suppose that G is recoverable over positive linear conductances, and that γ_e is differentiable with positive derivative. Let $(U(\phi), C(\phi))$ be the solution to the Dirichlet problem. Linear recoverability guarantees that $H_{U(\phi)}Q$ is uniquely determined by $D_\phi \Lambda$ (which is uniquely determined by L). From $H_{U(\phi)}Q$, we can find $D_\phi W$ and $D_\phi U$, and hence $D_\phi C$. A function is uniquely determined by its derivative and the value at one point, and we know $U(0) = 0$ and $C(0) = 0$; thus, U and C are uniquely determined by Λ. Therefore, $\gamma_e(U_{\iota(e)} - U_{\tau(e)})$ is determined by Λ.

Suppose that for any $t \in \mathbb{R}$, there exists a harmonic (u, c) with $u_{\iota(e)} - u_{\tau(e)} = t$. Then for each t, $\gamma_e(t)$ is determined by Λ, so γ is determined by Λ. Thus, Γ is recoverable over differentiable conductances with positive derivatives.

However, for a given t, there may not be a harmonic (u, c) with $u_{\iota(e)} - u_{\tau(e)} = t$. For example, consider a Y with boundary vertices $\{1, 2, 3\}$ and interior vertex 4, with oriented edge e_j from 4 to j. Suppose $|\gamma_{e_1}(t)| \leq M$ and $|\gamma_{e_2}(t)| \leq M$ are bounded, but γ_{e_3} is unbounded. If $\gamma_{e_3}(t) > 2M$, then there is no way to make

$$\gamma_{e_1}(u_4 - u_1) + \gamma_{e_2}(u_4 - u_2) + \gamma_{e_3}(u_4 - u_3) = 0, \quad u_4 - u_3 = t.$$

Thus, the network is not recoverable over differentiable conductances with positive derivatives.

We will return in §6.5 to the question of when there exists a harmonic function with a specified potential drop on a specified edge. But in a sense, knowing U and C is almost as good as recovering the conductances, since it completely describes the behavior of the network. Thus, we will say a network is *weakly recoverable* over a class of PCR's if the space of harmonic functions is uniquely determined by L. Then

Proposition 4.14. *If G is recoverable over the positive linear conductances, then it is weakly recoverable over differentiable conductances with $\gamma_e' > 0$ and $\gamma_e(0) = 0$.*

Future research could apply this approach to the inverse problem to graphs which are not recoverable over the positive linear conductances.

5 The Neumann Problem

5.1 Solutions to the Neumann Problem

We approach the Neumann problem in a similar way to the Dirichlet problem. For each edge e of a graph G, let $\rho_e : \mathbb{R} \to \mathbb{R}$ be an increasing function with $\rho_e(0) = 0$ and $\rho_{\overline{e}}(y) = -\rho_e(-y)$. Let

$$R_e = \{(x, y) \in \mathbb{R}^2 : \rho_e(y^-) \leq x \leq \rho_e(y^+)\}.$$

For $\psi \in \mathbb{R}^B$, let $\mathcal{H}_\psi$, $\mathcal{U}_\psi$, and $\mathcal{C}_\psi$ be as in the previous section.

This theorem was proved by [4] in the case where ρ_e is continuous.

Theorem 5.1. *Suppose $\psi \in \mathbb{R}^B$ and its entries sum to zero on every connected component of G.*

i. There exists a $(u, c) \in \mathcal{H}_\phi$ satisfying $\max_{e\in E} |c_e| \leq \frac{1}{2}\sum_{p\in B} |\psi_p|$.

ii. Every $u \in \mathcal{U}_\psi$ is compatible with every $c \in \mathcal{C}_\psi$.

iii. $\mathcal{U}_\psi$ and $\mathcal{C}_\psi$ are convex sets.

iv. For each edge e, either the potential drop $u_{\iota(e)} - u_{\tau(e)}$ or the current c_e is uniquely determined. If ρ_e is continuous, the potential drop is uniquely determined. If ρ_e is strictly increasing, the current is uniquely determined.

Proof. Let $\mathcal{X} \subset \mathbb{R}^E$ be the space of current functions and $\mathcal{Y}$ the space of current functions with net current zero on each boundary vertex. For $e \in E$, define $Q_e : \mathbb{R} \to \mathbb{R}$ by

$$Q_e(y) = \int_0^y \rho_e(t)\,dt.$$

Then Q_e is nonnegative convex function with $Q_{\overline{e}}(y) = Q_e(-y)$ and $Q_e(0) = 0$. Define the total pseudopower $Q : \mathcal{X} \to \mathbb{R}$ by

$$Q(u) = \frac{1}{2}\sum_{e\in E} Q_e(c_e) = \sum_{e\in E^*} Q_e(c_e),$$

where $E^* \subset E$ be a set with one oriented edge for each edge in E'.

Fix $\psi \in \mathbb{R}^B$. As the reader can verify, there exists a current function c_0 whose net currents are given by ψ. Let Q^* be the restriction of Q to $c_0 + \mathcal{Y}$, the space of current functions with boundary net current ψ. Define $F_e : c_0 + \mathcal{Y} \to \mathbb{R}$ by $F_e(c) = Q_e(c_e)$. Let

$$\partial F_e(c) = \{h \in \mathcal{Y} : F_e(c') - F_e(c) \geq h \cdot (c' - c) \text{ for } c' \in c_0 + \mathcal{Y}\}.$$

If $\chi_e \in \mathbb{R}^E$ is the vector which is 1 on e and 0 on the other edges, then

$$\partial F_e(c) + \mathcal{Y}^\perp = \chi_e[\rho_e(c_e^-), \rho_e(c_e^+)] + \mathcal{Y}^\perp,$$

Since $\mathcal{Y}$ is a finite-dimensional real inner product space, Lemma 4.4 applies and

$$\partial Q^*(c) = \sum_{e\in E^*} \partial F_e(c).$$

Hence,

$$\begin{aligned}\partial Q^*(c) + \mathcal{Y}^\perp &= \sum_{e\in E^*} \chi_e[\rho_e(c_e^-), \rho_e(c_e^+)] + \mathcal{Y}^\perp \\ &= \sum_{e\in E^*} \frac{1}{2}(\chi_e - \chi_{\overline{e}})[\rho_e(c_e^-), \rho_e(c_e^+)] + \mathcal{Y}^\perp\end{aligned}$$

because $\chi_e + \chi_{\overline{e}} \in \mathcal{Y}^\perp$.

I claim that $c \in c_0 + \mathcal{Y}$ has a compatible potential function if and only if $0 \in \partial Q^*(c)$. Indeed, if $0 \in Q_\phi(w)$, then for each $e \in E^*$, we can choose $h_e \in [\rho_e(c_e^-), \rho_e(c_e^+)]$ such that $g = \sum_{e\in E^*}(\chi_e - \chi_{\overline{e}}) \in \mathcal{Y}_\perp$. Note $g_{\overline{e}} = -g_e$, so for all e, $g_e \in [\rho_e(c_e^-), \rho_e(c_e^+)]$. Suppose $e_1, \dots, e_n$ form a cycle. Then $\sum_{j=1}^n (\chi_{e_j} - \chi_{\overline{e}_j})$ is in $\mathcal{Y}$. Since $g \in \mathcal{Y}^\perp$,

$$0 = g \cdot \sum_{j=1}^n (\chi_{e_j} - \chi_{\overline{e}_j}) = 2 \sum_{j=1}^n g_{e_j}.$$

Since g sums to zero over every cycle, we can find $u \in \mathbb{R}^V$ such that $g_e = u_{\iota(e)} - u_{\tau(e)}$, and u is a potential compatible with c. Conversely, suppose u is a potential compatible with c. Let $g_e = u_{\iota(e)} - u_{\tau(e)}$. Any $c' \in \mathcal{Y}$ can be written as a linear combination of functions of the form $\sum_{j=1}^n (\chi_{e_j} - \chi_{\overline{e}_j})$ for a cycle $e_1, \dots, e_n$. Since g sums to zero over every cycle, $g \in \mathcal{Y}^\perp$. Also, $g \in \partial Q^*(c) + \mathcal{Y}^\perp$, so $0 \in \partial Q^*(c)$.

Now $0 \in \partial Q^*(c)$ if and only if c is a global minimum of Q^*, so our goal is show a minimum is achieved. Let $\mathcal{Z}$ be the set of current functions $c \in c_0 + \mathcal{Y}$ such that there is no cycle of oriented edges $e_1, \dots, e_n$ with $c_{e_j} > 0$ for all j. Then $\mathcal{Z}$ is closed. I claim it is also bounded, and in fact, that every $c \in \mathcal{Z}$ satisfies the maximum principle $\max_{e\in E} |c_e| \leq \frac{1}{2}\sum_{p\in B} |\psi_p|$. Fix $c \in \mathcal{Z}$ and $e_0 \in E$, and we will prove $|c_e| \leq \frac{1}{2}\sum_{p\in B}|\psi_p|$. If $c_{e_0} = 0$, we are done, so assume $c_{e_0} \neq 0$, and assume without loss of generality $c_{e_0} > 0$. Let P be the set of vertices p such that there exists a path from p to $\iota(e_0)$ along oriented edges with strictly positive current (including $\iota(e_0)$), and let R be the set of edges along these paths (including e_0). If $p \in P$ and $\tau(e) = p$ and $c_e > 0$, then $e \in R$. Thus,

$$\sum_{\substack{e\in R \\ \iota(e)=p}} c_e - \sum_{\substack{e\in R \\ \tau(e)=p}} c_e = \sum_{\substack{e\in R \\ \iota(e)=p}} c_e - \sum_{\substack{e\in E \\ c_e>0 \\ \tau(e)=p}} c_e \leq \sum_{\substack{e\in E \\ c_e>0 \\ \iota(e)=p}} c_e - \sum_{\substack{e\in E \\ c_e>0 \\ \tau(e)=p}} c_e = \sum_{e:\iota(e)=p} c_e.$$

Summing over $p \in P$ gives

$$\sum_{\substack{e\in R \\ \iota(e)=p}} c_e - \sum_{\substack{e\in R \\ \tau(e)=p}} c_e \leq \sum_{p\in P} \sum_{e:\iota(e)=p} c_e = \sum_{p\in P\cap B} \psi_p.$$

All edges in R except e_0 have both endpoints in P, and e_0 has $\iota(e_0) \in P$, $\tau(e_0) \notin P$. Thus, all the terms on the left hand side cancel except c_{e_0}, and hence,

$$c_{e_0} \leq \sum_{p\in P\cap B} \psi_p \leq \sum_{p\in P\cap B} \max(0, \psi_p) \leq \sum_{\substack{p\in B \\ \psi_p>0}} \psi_p.$$

Since $\sum_{p\in B} \psi_p = 0$,

$$\sum_{\substack{p\in B \\ \psi_p>0}} |\psi_p| = \sum_{\substack{p\in B \\ \psi_p<0}} |\psi_p| = \frac{1}{2}\sum_{p\in B} |\psi_p|,$$

and hence $|c_e| \leq \frac{1}{2}\sum_{p\in B}|\psi_p|$.

This shows $\mathcal{Z}$ is bounded and hence compact. Thus, Q^* attains a minimum at some $c^* \in \mathcal{Z}$. I claim c^* is a global minimum. Suppose $c \in c_0 + \mathcal{Y}$ and $c \notin \mathcal{Z}$. Then there is some cycle with edges $e_1, \dots, e_n$ such that $c_{e_j} > 0$. Let m be the minimum over j of c_{e_j}. Define c' by letting $c'_{e_j} = c_{e_j} - m$, $c'_{\overline{e}_j} = c_{\overline{e}_j} + m$, and $c'_e = c_e$ for all other e. Then $|c'_e| \leq |c_e|$ and $\operatorname{sgn} c'_e = \operatorname{sgn} c_e$; hence, $Q_e(c') \leq Q_e(c)$, and $Q^*(c') \leq Q^*(c)$. If $c' \notin \mathcal{Z}$, then we can repeat the process; at each step, we decrease the number of edges on which current is flowing, so the process must end after finitely many steps, and we have a $c'' \in \mathcal{Z}$ with $Q^*(c'') \leq Q^*(c)$. So the global minimum is achieved in $\mathcal{Z}$, at c^*. Therefore, c^* has a compatible potential function, and we already showed it satisfies the maximum principle, so (i) is proved.

To prove (ii), it suffices to show that if c and $\tilde{c}$ are in $\mathcal{C}_\phi$ and c is compatible with u, then $\tilde{c}$ is also compatible with u. Because $u_{\iota(e)} - u_{\tau(e)}$ is a subderivative of Q_e at c_e, we have

$$Q_e(\tilde{c}_e) - Q_e(c_e) - (u_{\iota(e)} - u_{\tau(e)})(\tilde{c}_e - c_e) \geq 0.$$

Summing the left hand side over $e \in E^*$ yields

$$Q^*(\tilde{c}) - Q^*(c) - \sum_{e\in E^*}(u_{\iota(e)} - u_{\tau(e)})(\tilde{c}_e - c_e),$$

and the first two terms cancel because $\tilde{c}$ and c must both achieve the global minimum of Q^*. The other sum is

$$\begin{aligned}\sum_{e\in E^*}(u_{\iota(e)} - u_{\tau(e)})(\tilde{c}_e - c_e) &= \sum_{e\in E} u_{\iota(e)}(\tilde{c}_e - c_e) \\ &= \sum_{p\in V}\sum_{e:\iota(e)=p} u_p(\tilde{c}_e - c_e) \\ &= \sum_{p\in V} u_p\left(\sum_{e:\iota(e)=p}\tilde{c}_e - \sum_{e:\iota(e)=p} c_e\right) = 0\end{aligned}$$

because c and $\tilde{e}$ have the same net current on each vertex. Hence,

$$\sum_{e\in E'}\Big(Q_e(\tilde{c}_e) - Q_e(c_e) - (u_{\iota(e)} - u_{\tau(e)})(\tilde{c}_e - c_e)\Big) = 0,$$

but each term is nonnegative, so each term must be zero. Since $u_{\iota(e)} - u_{\tau(e)} \in \partial Q_e(c_e)$, the same argument as in the Dirichlet problem shows that $u_{\iota(e)} - u_{\tau(e)} \in \partial Q_e(\tilde{c}_e)$, and hence $\tilde{c}$ is compatible with u.

The arguments for (iii) and (iv) are the same as before, and the details are left to the reader. □

Proposition 5.2. *Let A be the set of $\psi \in \mathbb{R}^B$ whose entries sum to zero on each connected component of G. There exists a continuous $C : \mathbb{R}^B \to \mathbb{R}^E$ such* *that* $C(\psi) \in \mathcal{C}_\psi$ *and*

$$\max_{e\in E} |C(\psi_1) - C(\psi_2)| \leq \frac{1}{2} \sum_{p\in B} |(\psi_1)_p - (\psi_2)_p|.$$

Proof. The argument is the same as for Proposition 4.5. □

5.2 The Neumann-to-Dirichlet Map Ω

Let Γ be as in the previous section, and in addition assume that ρ_e is continuous and each component of G has a boundary vertex. For any $\psi \in A$, there is solution (u, c) to the Neumann problem and the potential drops are uniquely determined. Thus, there is unique potential function u such that the boundary potentials sum to zero on each connected component. Hence, there a well-defined Neumann-to-Dirichlet map $\Omega : A \to A : \psi \mapsto \phi$. We also have the following results; the proofs are straightforward adaptations of the analogous proofs for the Dirichlet problem, and are left to the reader:

Proposition 5.3. Ω *is continuous.*

Proposition 5.4. *Suppose that* ρ_n *and* ρ_0 *are continuous, increasing resistance functions on a graph* G *and* Ω_n *and* Ω_0 *are the corresponding Neumann-to-Dirichlet map. If* $\rho_{n,e} \to \rho_{0,e}$*, then* $\Omega_n \to \Omega_0$ *uniformly on compact sets.*

Theorem 5.5. Ω_ρ *is differentiable with respect to* ψ*. The differential* $d_\psi\Omega_\rho : A \to A$ *is given by* $d_\phi\Omega_\rho = \Omega_{d_c\rho}$*, where* c *is any element of* $\mathcal{C}_\psi$*.*

6 Reduction Operations

6.1 Definition

A *boundary spike* is an edge $\{e, \bar{e}\}$ such that $\iota(e) \in B$, $\tau(e) \in I$, and $\iota(e)$ has valence 1. If G has a boundary spike e and G' satisfies

$$V(G') = V(G) \setminus \{\iota(e)\}, E(G') = E(G) \setminus \{e, \bar{e}\}, I(G') = I(G) \setminus \{\tau(e)\},$$

then the transformation $G \mapsto G'$ is a called a *boundary spike contraction.* The reverse transformation is called a *boundary spike expansion.*

A *boundary edge* is an edge $\{e, \bar{e}\}$ with $\iota(e) \in B$ and $\tau(e) \in B$. If e is a boundary edge and G' satisfies

$$V(G') = V(G), E(G') = E(G) \setminus \{e, \bar{e}\}, I(G') = I(G),$$

then the transformation $G \mapsto G'$ is a *boundary edge deletion.* The reverse transformation is called a *boundary edge addition.*

A *disconnected boundary vertex* is a boundary vertex with valence 0. If p is such a vertex, and

$$V(G') = V(G) \setminus \{p\}, E(G') = E(G), I(G') = I(G),$$

then the transformation $G \mapsto G'$ is a *disconnected boundary vertex deletion*. The opposite is *disconnected boundary vertex addition*.

Boundary spike contraction, boundary edge deletion, and disconnected boundary vertex deletion are called *reduction operations*. We say G is *reducible* to H if there is a sequence of reduction operations that will transform G into H. In this case, H must be a subgraph of G. We say G and H are *reduction-equivalent* if there is a sequence of reduction operations and their inverses which transforms G into H. This is an equivalence relation.

The motivation for considering reduction operations is the "layer-stripping" approach to the inverse problem. The idea is to determine the PCR's on boundary spikes and boundary edges, then to contract the spikes or delete the edges, and then to repeat this process on the reduced graph. If G is reducible to the empty graph, then we will eventually recover all the PCR's of all edges of G, assuming that at each step we can determine the set of boundary data of the reduced graph.

6.2 Reduction to Embedded Flowers

Not all graphs are reducible to the empty graph. In particular, a *flower* is graph with no boundary spikes, boundary edges, or disconnected boundary vertices. A flower cannot be reduced the empty graph unless it is already the empty graph.

Every (finite) graph can be reduced to a flower. Indeed, if it is not a flower, we can perform a reduction operation, which will either decrease the number of vertices or decrease the number of edges. If we keep performing reduction operations we will eventually either reach the empty graph or some subgraph of G which cannot be reduced, which is a flower. It turns out that the flower we reach is independent of the sequence of reduction operations:

Theorem 6.1.

i. Every graph G is reducible to a unique flower $G^{✿}$.

ii. G and H are reduction-equivalent if and only if $G^{✿} = H^{✿}$.

iii. If H is a subgraph of G, then $H^{✿}$ is a subgraph of $G^{✿}$.

We start with a few lemmas:

Lemma 6.2. *If G is reducible to H and S is a subgraph of G, then S is reducible to $S \cap H$, where $S \cap H$ is defined by*

$$V(S \cap H) = V(S) \cap V(H),$$
$$E(S \cap H) = E(S) \cap E(H),$$
$$I(S \cap H) = I(S) \cap I(H).$$

Proof. Suppose S is a subgraph of G. Let $G = G_0, G_1, \ldots, G_N = H$ be a sequence of graphs where G_{n+1} is obtained from G_n by a single decomposition operation. Let $S_n = G_n \cap S$. We want to show that S_n is reducible to S_{n+1}. There are several cases:

1. Suppose G_{n+1} is obtained from G_n by deleting a disconnected boundary vertex p. If $p \notin V(S_n)$, then $S_n = S_{n+1}$, so we are done. If $p \in V(S_n)$, then it is a disconnected boundary vertex as a consequence of the definition of subgraph. Thus, S_n is reducible to S_{n+1}.

2. Suppose G_{n+1} is obtained from G_n by deleting a boundary edge e. If $e \notin E(S_n)$, then $S_n = S_{n+1}$, and we are done. Otherwise, e must be a boundary edge of S_n, so S_n is reducible to S_{n+1}.

3. Suppose G_{n+1} is obtained from G_n by a contracting a boundary spike e. If $\iota(e) \notin V(S_n)$, then $e \notin E(S_n)$ and $\tau(e)$ is either a boundary vertex of S_n or is not in $V(S_n)$; thus, $S_n = S_{n+1}$, and we are done. If $\iota(e) \in V(S_n)$, but $e \notin E(S_n)$, then $\tau(e)$ is either a boundary vertex of S_n or is not in $V(S_n)$; also, $\iota(e)$ is a disconnected boundary vertex, so we can delete it to obtain S_{n+1}. If $e \in E(S_n)$, then $\iota(e)$ must be a boundary vertex of S_n. If $\tau(e)$ is interior in S_n, then e is a spike in S_n, which we can contract. If $\iota(e)$ is a boundary vertex in S_n, then e is a boundary edge and $\iota(e)$ has degree 1. Thus, we can obtain S_{n+1} by deleting the boundary edge e, then deleting the disconnected boundary vertex $\iota(e)$. □

Corollary 6.3. *If G is reducible to the empty graph, then so is every subgraph of G.*

Lemma 6.4. *If G is a flower and G is reduction-equivalent to H, then G is a subgraph of H. In particular, if two flowers are reduction-equivalent, they are equal.*

Proof. There is a sequence of graphs $G = G_0, G_1, \dots, G_N = H$, where G_{n+1} is obtained from G_n by a single operation. We prove the lemma for each G_n by induction. We already know it is true for G_0. Suppose it is true for G_n. Then either G_n is a subgraph of G_{n+1} or G_{n+1} is a subgraph of G. If G_n is a subgraph of G_{n+1}, we are done because G is a subgraph of G_n. Otherwise, G_{n+1} is obtained from G_n by contracting a boundary spike, deleting a boundary edge, or deleting a disconnected boundary vertex. The boundary spike or boundary edge or disconnected boundary vertex in question cannot be part of G, because then by similar reasoning as in the previous proposition, it would be a boundary spike or boundary edge or disconnected boundary vertex of G, which is impossible because G is a flower. Therefore, G must be a subgraph of G_{n+1}. □

Proof of Theorem. We already showed G can be reduced to a flower, and uniqueness follows from Lemma 6.4. Clearly, G is reduction-equivalent to $G^{✿}$ and H to $H^{✿}$. Thus, G is equivalent to H if and only if $G^{✿}$ is equivalent to $H^{✿}$ if and only if $G^{✿} = H^{✿}$.

For (iii), suppose H is a subgraph of G. Since G is reducible to G^*, we know H is reducible to $G^{✿} \cap H$. Then $G^{✿} \cap H$ must be reduction-equivalent to $H^{✿}$. By Lemma 6.4, $H^{✿}$ must be a subgraph of $G^{✿} \cap H$, which is a subgraph of $G^{✿}$. □

6.3 Electrical Properties

If we want to solve the inverse problem by layer-stripping, we need to know that when we perform a reduction operation, the set of boundary data of the reduced network is uniquely determined by the boundary data of the original network and the PCR of the edge removed. This is the purpose of the following lemmas:

Lemma 6.5. *Let Γ' be the subnetwork of Γ obtained by contracting a spike e, and let L and L' be the corresponding sets of boundary data. Suppose R_e is given by a resistance function ρ_e. Then L' is uniquely determined by L and ρ_e.*

Proof. Define $\Xi : \mathbb{R}^{B(G')} \times \mathbb{R}^{B(G')} \to \mathbb{R}^{B(G)} \times \mathbb{R}^{B(G)}$ by $(\phi', \psi') \mapsto (\phi, \psi)$, where

- For $p \in B(G) = B(G') = B(G) \setminus \{\iota(e)\}$, we have $\phi_p = \phi'_p$ and $\psi_p = \psi'_p$.
- $\phi_{\iota(e)} = \phi'_{\tau(e)} + \rho_e(\psi_p)$.
- $\psi_{\iota(e)} = \psi'_{\tau(e)}$.

I claim $L' = \Xi^{-1}(L)$. Suppose $(\phi', \psi') \in L'$ and it is the boundary data of a harmonic (u', c') on Γ'. We can extend (u, c) to a harmonic function (u, c) on Γ by setting $c_e = \psi'_{\tau(e)}$ and $u_{\iota(e)} = u_{\tau(e)} + \rho_e(\psi'_{\iota(e)})$. This harmonic function has boundary data $(\phi, \psi) = \Xi(\phi', \psi')$, so $(\phi', \psi') \in \Xi^{-1}(L)$. Conversely, suppose $(\phi, \psi) \in L$ is the boundary data of a harmonic (u, c) on Γ. Since e is a spike, c_e must equal $\psi_{\iota(e)}$. Hence, $u_{\iota(e)} - u_{\tau(e)} = \rho_e(\psi_{\iota(e)})$. Thus, when we restrict (u, c) to Γ', the boundary data becomes $\Xi^{-1}(\phi, \psi)$, so $\Xi^{-1}(\phi, \psi) \in L'$. ☐

Lemma 6.6. *Let Γ' be the subnetwork of Γ obtained by deleting a boundary edge e, and let L and L' be the corresponding sets of boundary data. Suppose R_e is given by a conductance function γ_e. Then L' is uniquely determined by L and γ_e.*

Proof. Observe $B(G) = B(G')$. Define $\Xi : \mathbb{R}^{B(G')} \times \mathbb{R}^{B(G')} \to \mathbb{R}^{B(G)} \times \mathbb{R}^{B(G)}$ by $(\phi', \psi') \mapsto (\phi, \psi)$, where

- $\phi = \phi'$.
- For $p \in B(G) \setminus \{\iota(e), \tau(e)\}$, we have $\psi_p = \psi'_p$.
- $\psi_{\iota(e)} = \psi'_{\iota(e)} + \gamma_e(\phi_{\iota(e)} - \phi_{\tau(e)})$.
- $\psi_{\tau(e)} = \psi'_{\tau(e)} - \gamma_e(\phi_{\iota(e)} - \phi_{\tau(e)})$.

Then $L' = \Xi^{-1}(L)$. The proof is similar to the previous one and is left to the reader. ☐

Clearly, if Γ' is obtained from Γ by deleting a disconnected boundary vertex, L' is determined by L. Thus, we have the following corollary: If Γ is reducible to Γ' and each PCR is given by a bijective conductance function, then L' is uniquely determined by L and the γ_e's of the edges removed in the reduction.

6.4 Regularity of L

Suppose G is reducible to the empty graph through a sequence of reduction operations. Then we can assume that the disconnected boundary vertex deletions occur last, since leaving a disconnected boundary vertex in the graph longer does not prevent boundary spike contractions or boundary edge deletions. Thus, there is a sequence of boundary spike contractions and boundary edge deletions that reduce G to a graph with no edges and only boundary vertices.

Suppose Γ is a network on G with bijective conductance functions. Let $G = G_0, G_1, \ldots, G_N$ be the sequence of graphs obtained by reduction operations such that G_N has no edges or interior vertices, and let $L = L_0, \ldots, L_N$ their sets of boundary data. Each graph has the same number of boundary vertices as G. On G_N, any potentials are possible, but the net currents must all be zero, so the boundary relationship $L_N = \mathbb{R}^{B(G_N)} \times \{0\}^{B(G_N)}$.

If G_{n-1} is obtained from G_n, then there is a function Ξ_n mapping boundary data on G_n to boundary data on G_{n-1}, as seen in the proof of Lemmas 6.5 and 6.6. Thus, $L_{n-1} = \Xi_n(L_n)$, and

$$L = \Xi_1 \circ \Xi_2 \circ \cdots \circ \Xi_N(\mathbb{R}^{B(G_N)} \times \{0\}^{B(G_N)}).$$

Thus, we have the following result:

Proposition 6.7. *Suppose G is reducible to the empty graph. Let Γ be a network on G with bijective conductances.*

1. *If each γ_e is continuous, then L is homeomorphic to $\mathbb{R}^{|B|}$.*

2. *If each γ_e is C^k with $\gamma_e' \neq 0$, then the homeomorphism is a C^k. Hence, L is a C^k manifold of dimension $n = |B|$ embedded in $\mathbb{R}^B \times \mathbb{R}^B$*

3. *In this case, if L_γ is the set of boundary data for conductances γ, and if (ϕ, ψ) is the boundary data of a harmonic function (u, c), then the tangent space is $T_{(\phi,\psi)}(L_\gamma) = L_{d_u \gamma}$.*

4. *Let $\mathcal{H} \subset \mathbb{R}^V \times \mathbb{R}^E$ be the space of harmonic functions on Γ. If γ_e and ρ_e are C^k, then $\mathcal{H}$ is a C^k manifold of dimension n, and the tangent space is $T_{(u,c)}(\mathcal{H}_\gamma) = \mathcal{H}_{d_u \gamma}$.*

5. *If two harmonic functions have the same boundary data, they are equal.*

Proof.

1. If γ_e is continuous and bijective, then so is $\rho_e = \gamma_e^{-1}$. Also, each Ξ_n is continuous. For adding a boundary spike, the inverse of Ξ_n is the same as Ξ_n but with the $-\rho_e$ substituted for ρ_e. For adding a boundary edge, the inverse is obtained by changing the sign of the conductance γ_e. So each Ξ_n is a homeomorphism, so restricting $\Xi = \Xi_1 \circ \cdots \circ \Xi_N$ to $\mathbb{R}^{B(G_N)} \times \{0\}^{B(G_N)}$ provides a homeomorphism onto L, and $\mathbb{R}^{B(G_N)} \times \{0\}^{B(G_N)}$ is naturally homeomorphic to $\mathbb{R}^{|B|}$.

2. If each γ_e and ρ_e is C^n, then so are Ξ_n and Ξ_n^{-1}.

3. Let $x \in \mathbb{R}^{B(G_N)}$ be the restriction of u to $B(G_N)$. By direct computation, the differential $d\Xi_n$ for γ at a point $\Xi_{n+1} \circ \cdots \circ \Xi_N(x_u, 0)$ is the same as the Ξ_n for the linear conductances $d_u\gamma$. Thus, $D\Xi = D\Xi_1 \circ \cdots \circ D\Xi_N$ is the same as the Ξ-map for the linear conductances. So if Ξ_γ represents the map for conductances γ, we have $D_{(x,0)}\Xi_\gamma = \Xi_{d_u\gamma}$. Since the potentials on $B(G_N)$ provide a parametrization of L_γ,

$$T_{(\phi,\psi)}(L_\gamma) = D_{(x,0)}\Xi_\gamma(\mathbb{R}^{B(G_N)} \times 0) = \Xi_{d_u\gamma}(\mathbb{R}^{B(G_N)} \times 0) = L_{d_u\gamma}.$$

4. We can also parametrize $\mathcal{H}_\gamma$ in terms of the potentials on $B(G_N)$. This is because each vertex is in $B(G_n)$ for some n, so at some step of the above argument, it was given as an entry of $\Xi_n \circ \cdots \circ \Xi_N(x, 0)$. Similarly, each edge was a boundary edge or spike at some step.

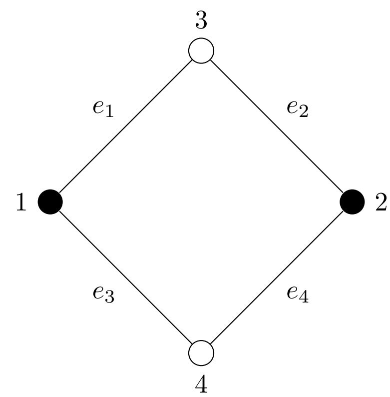
5. Each Ξ_n was bijective, so the boundary data on G determines the boundary data on each G_n, and hence the potentials and currents on the whole network. □

Part (3) is an analogue of the formulas for differentiating the Dirichlet-to-Neumann and Neumann-to-Dirichlet maps. The analogous result for the inverse problem is:

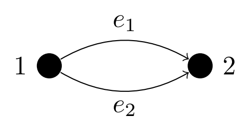
Corollary 6.8. *If G is layerable recoverable over signed linear conductances, then it is weakly recoverable over bijective, zero-preserving C^1 conductances with $\gamma_e' \neq 0$.*

Proof. Since L is a C^1 manifold, we can compute the tangent space at each point (ϕ, ψ). From this, $L_{d\gamma_u}$ is determined, and by recoverability, $d\gamma_u$ is uniquely determined, and hence we can find the derivative of (u, c) with respect to (ϕ, ψ). This, together with the fact that $(u, c) = 0$ when $(\phi, \psi) = 0$, uniquely determines (u, c) as a function of (ϕ, ψ). □

As we saw, (4) and (5) of the Proposition do not hold for all graphs, not even for linear conductances. Actually, (1), (2), and (3) can fail for some graphs with bijective C^∞ conductances with nonzero derivatives. Consider the following graph:



Let $\rho_{e_1}(t) = \rho_{e_3}(t) = t + \frac{1}{2}\sin t$ (the orientation of the edge does not matter since the function is odd), and let $\rho_{e_2}(t) = \rho_{e_3}(t) = -t$. These are bijective C^∞ resistance functions with a C^∞ inverse. The series with resistance functions ρ_{e_1} and ρ_{e_2} is equivalent to a single-edge with resistance $\rho_{e_1} + \rho_{e_2}$. Thus, the network is equivalent to a parallel connection



in which each edge has resistance function $\rho(t) = \frac{1}{2}\sin t$. Let e_1 and e_2 be the oriented edges shown in the picture. Thus, (u, c) is harmonic if and only if

$$u_1 - u_2 = \tfrac{1}{2}\sin c_{e_1} = \tfrac{1}{2}\sin c_{e_2}.$$

Now $\sin c_{e_1} = \sin c_{e_2}$ is equivalent to $c_{e_2} = c_{e_1} + 2\pi n$ or $c_{e_2} = \pi - c_{e_1} + 2\pi n$. If $c_{e_1} = c_{e_2} + 2\pi n$, then the net current $\psi_1 = c_{e_1} + c_{e_2} = 2c_{e_1} + 2\pi n$ and $\psi_2 = -\psi_1$ and $u_1 - u_2$ must be $\frac{1}{2}\sin\psi_1/2$. If $c_{e_2} = \pi - c_{e_1} + 2\pi n$, then $\psi_1 = (2n+1)\pi$ and $\psi_2 = -\psi_1$ and $u_1 - u_2$ could be any number in $[-1, 1]$. Thus,

$$\begin{aligned} L =&\{(\phi, \psi) : \phi_1 - \phi_2 = \tfrac{1}{2}\sin\psi_1/2,\ \psi_1 = -\psi_2\} \\ &\cup \{(\phi, \psi) : \phi_1 - \phi_2 \in [-1, 1],\ \psi_1 = (2n+1)\pi,\ \psi_2 = -\psi_1\}. \end{aligned}$$

This is not a smooth manifold because there is neighborhood of the points with $\phi_1 - \phi_2 = \pm 1$ and $\psi_1 = (2n+1)\pi$ which is homeomorphic to $\mathbb{R}^2$.

6.5 Faithful Networks

In §4.4, we wanted to guarantee that for some $t \in \mathbb{R}$ and $e \in E$, there was a harmonic (u, c) with $u_{\iota(e)} - u_{\tau(e)} = t$. We can now answer that question for many graphs. We say a network is *faithful* if for any $e \in E$, for any $(x, y) \in R_e$, there exists a harmonic (u, c) with $u_{\iota(e)} - u_{\tau(e)} = x$ and $c_e = y$. If a network is faithful and is weakly recoverable over $\mathcal{R}$, then it is recoverable over $\mathcal{R}$.

Proposition 6.9. *Let Γ be a network on a graph G which is reducible to the empty graph, and suppose every vertex is contained a boundary-to-boundary path. Suppose R_e satisfies*

- $(0,0) \in R_e$.
- *If $(x,y),(x',y') \in R_e$, then $x \leq x'$ if and only if $y \leq y'$.*
- *For any x, there exists y with $(x,y) \in R_e$ and for any y, there exists x with $(x,y) \in R_e$.*

Then Γ is faithful.

Proof. As the reader may verify, for any such R_e we can find an increasing, zero-preserving γ_e and ρ_e such that

$$R_e = \{(x,y) : \gamma_e(x^-) \leq y \leq \gamma_e(x^+)\} = \{(x,y) : \rho_e(y^-) \leq x \leq \rho_e(y^+)\}.$$

Let $G_0, G_1, \ldots, G_N$ be a sequence of graphs such that $G_0 = G$, G_N has no edges or interior vertices, and G_n is obtained from G_{n-1} by deleting a boundary edge e_n or contracting a boundary spike e_n. Choose n and $(x,y) \in R_{e_n}$. I claim there is a harmonic (u,c) on G_{n-1} with $u_{\iota(e)} - u_{\tau(e)} = x$ and $c_e = y$. There are two cases:

- Suppose e_n is a boundary edge in G_{n-1}. Since the Dirichlet problem has a solution, we can find a harmonic (u,c) with $u_{\iota(e_n)} - u_{\tau(e_n)} = x$. If $c_e \neq y$, we can change it to y without affecting the net current on the interior vertices.
- Suppose e_n is a boundary spike in G_{n-1}. Since every vertex in G is contained in a boundary-to-boundary path, this is also true of any subgraph of G and in particular G_{n-1}. Hence, any component of G_{n-1} with an interior vertex has at least two boundary vertices. Since $\tau(e)$ is interior, the component with e has at least two boundary veritices. So we can choose ψ with $\psi_{\iota(e)} = y$ such that the entries of ψ sum to zero on each component of G_{n-1}. Let (u,c) be a solution to the Neumann problem for ψ. If $u_{\iota(e)} - u_{\tau(e)} \neq x$, we can change $u_{\iota(e)}$ to make it so without affecting the net currents.

It is easy to verify that a harmonic function on G_k extends to G_{k-1}. Thus, by induction, we can extend (u,c) to G. Since every oriented edge is either some e_n or some $\overline{e}_n$, we are done. ∎

Corollary 6.10. *Let R_e be as above. Suppose $G_1, \ldots, G_N$ are a subgraph partition of G with $B(G) = \bigcup_{k=1}^N B(G_k)$. If the networks $\Gamma_1, \ldots, \Gamma_N$ are faithful, so is Γ. The same holds if $G_1, \ldots, G_N$ are a subgraph partition such that every cycle of G is contained in some G_k, and each G_k is connected and has at least two boundary vertices.*

Proof. Suppose $G_1, \ldots, G_N$ are a subgraph partition of G with $B(G) = \bigcup_{k=1}^N B(G_k)$. Suppose e is an edge in G_n and $(x, y) \in R_e$. There exists a harmonic (u_n, c_n) on Γ_n with $(u_n)_{\iota(e)} - (u_n)_{\tau(e)} = x$ and $(c_n)_e = y$. Since the Dirichlet problem has a solution, for $k \neq n$, we can find a harmonic (u_k, c_k) on Γ_k with $(u_k)_p = (u_n)_p$ for $p \in B(G_n) \cap B(G_k)$ and $(u_k)_p = 0$ for $p \in B(G_k) \setminus B(G_n)$. Since $B(G) = \bigcup_{k=1}^N B(G_k)$, these join to form a harmonic function on G.

Suppose $G_1, \ldots, G_N$ are a subgraph partition of G such that every cycle of G is contained in some G_k. For $e \in E(G_n)$ and $(x, y) \in R_e$, we can find a harmonic (u_n, c_n) on Γ_n with $(u_n)_{\iota(e)} - (u_n)_{\tau(e)} = x$ and $(c_n)_e = y$. Using the existence of a solution to the Neumann problem, we can extend (u_n, c_n) to a harmonic (u, c) on Γ. □

Corollary 6.11. *If G is reducible to the empty graph and recoverable over the positive linear conductances, then it is recoverable over bijective, differentiable, zero-preserving conductances with $\gamma_e' > 0$.*

Proof. Every vertex must be contained in a boundary-to-boundary path. If this were not the case, then there would be a nontrivial connected subgraph of G with only one boundary vertex; every harmonic function on this subgraph must be constant. Hence, changing the conductances on this subgraph would not affect L; thus, G would not be recoverable over the positive linear conductances. It follows by Proposition 6.9 that Γ is faithful. By Proposition 4.14, it is weakly recoverable over bijective, differentiable, zero-preserving conductances with $\gamma_e' > 0$; therefore, it is recoverable. □

7 Layerings and the Inverse Problem

7.1 Two-Boundary Graphs and Layerings

We now describe a construction which will allow us to recover bijective zero-preserving conductances on a large class of graphs, and it has several other useful consequences.

A *two-boundary graph* is a graph together with two sets of vertices B_{upper} and B_{lower}. Every two-boundary graph corresponds to a graph with boundary with $B = B_{\text{upper}} \cup B_{\text{lower}}$. But a graph with boundary may correspond to many different two-boundary graphs. We do *not* assume B_{upper} and B_{lower} are disjoint.

Suppose G, G_1, and G_2 are two-boundary graphs. Then $G = G_1 \bowtie G_2$ means that

- $V(G) = V(G_1) \cup V(G_2)$.
- $E(G) = E(G_1) \cup E(G_2)$.
- $E(G_1) \cap E(G_2) = \varnothing$.
- $V(G_1) \cap V(G_2) = B_{\text{lower}}(G_1) = B_{\text{upper}}(G_2)$.

- $B_{\text{upper}}(G) = B_{\text{upper}}(G_1)$.
- $B_{\text{lower}}(G) = B_{\text{lower}}(G_2)$.

A consequence is that, when considered as graphs with boundary, G_1 and G_2 are subgraphs of G, and in fact, a subgraph partition of G.

Let G be a two-boundary graph and Γ be an electrical network on (the graph with boundary corresponding to) G. We define the *two-boundary relationship* $X \subset (\mathbb{R}^{B_{\text{upper}}} \times \mathbb{R}^{B_{\text{upper}}}) \times (\mathbb{R}^{B_{\text{lower}}} \times \mathbb{R}^{B_{\text{lower}}})$ as follows. Suppose $x = (x_u, x_c) \in \mathbb{R}^{B_{\text{upper}}} \times \mathbb{R}^{B_{\text{upper}}}$ and $y = (y_u, y_c) \in \mathbb{R}^{B_{\text{lower}}} \times \mathbb{R}^{B_{\text{lower}}}$. Then $(x, y) \in X$ if and only if there exists a $(\phi, \psi) \in L$ such that

$$\begin{aligned}
\phi|_{B_{\text{upper}}} &= x_u \\
\phi|_{B_{\text{lower}}} &= y_u \\
\psi|_{B_{\text{upper}} \setminus B_{\text{lower}}} &= -x_c|_{B_{\text{upper}} \setminus B_{\text{lower}}} \\
\psi|_{B_{\text{upper}} \cap B_{\text{lower}}} &= y_c|_{B_{\text{upper}} \cap B_{\text{lower}}} - x_c|_{B_{\text{upper}} \cap B_{\text{lower}}} \\
\psi|_{B_{\text{lower}} \setminus B_{\text{upper}}} &= y_c|_{B_{\text{lower}} \setminus B_{\text{upper}}}.
\end{aligned}$$

Here x represents voltage and current data on B_{upper} (with the sign of the net current changed) and y represents voltage and net current data on B_{lower}. If a vertex p is in $B_{\text{upper}} \cap B_{\text{lower}}$, then $(x, y) \in \Xi$ implies that $(x_v)_p = (y_v)_p$, and that the net current on p is $(y_c)_p - (x_c)_p$. We think of y_c as representing current flowing into the network on the lower boundary and x_c as representing current flowing out of the network on the upper boundary. If p is in both boundary sets, then current can flow directly from one boundary to the other through p.

If $\Gamma = \Gamma_1 \bowtie \Gamma_2$, then, by similar reasoning as in the section about subnetworks, X is

$$X_2 \odot X_1 = \{(x, y) : \text{ there exists } z \text{ such that } (x, z) \in X_1 \text{ and } (z, y) \in X_2\}.$$

Using "$\bowtie$," we can express complicated networks as combinations of simpler ones. Our building blocks are networks on the following four types of two-boundary graphs, called *elementary layers*:

1. A *horizontal-edge layer* is a two-boundary graph with $V = B_{\text{upper}} = B_{\text{lower}}$ and $E \neq \varnothing$. Its edges are called *horizontal edges.*

2. A *vertical-edge layer* is a two-boundary graph such that each connected component is either a single valence-0 vertex $p \in B_{\text{upper}} \cap B_{\text{lower}}$, or it consists of two vertices $p \in B_{\text{upper}} \setminus B_{\text{lower}}$ and $q \in B_{\text{lower}} \setminus B_{\text{upper}}$ with a single edge $\{e, \overline{e}\}$ with $\iota(e) = p$, $\tau(e) = q$.

3. A *upper-stub layer* is a two-boundary graph with $V = B_{\text{upper}} \supsetneq B_{\text{lower}}$ and $E = \varnothing$. The vertices in $B_{\text{upper}} \setminus B_{\text{lower}}$ are called *upper stubs.*

4. A *lower-stub layer* is a two-boundary graph with $V = B_{\text{lower}} \supsetneq B_{\text{upper}}$ and $E = \varnothing$. The vertices in $B_{\text{lower}} \setminus B_{\text{upper}}$ are called *lower stubs.*

The advantage of elementary layers is that their X relationships are simple to describe.

If G is a graph with boundary, then a *layering of* G is a sequence of elementary layers $G_1, \ldots, G_N$ such that

- $G_1 \bowtie \cdots \bowtie G_K$ is a two-boundary graph corresponding to G;
- If G_j is a upper-stub layer and G_k is a lower-stub layer, then $j < k$.

If G has no edges and only boundary vertices, then we say that there is a layering of G with $B_{\text{upper}}(G) = B_{\text{lower}}(G) = V(G)$ such that there are zero elementary layers in the layering. This is a trivial case, but the definition will make the statements and proofs of results simpler to write.

For example, see Figure 2. For a nontrivial layering, the following properties are consequences of the definition:

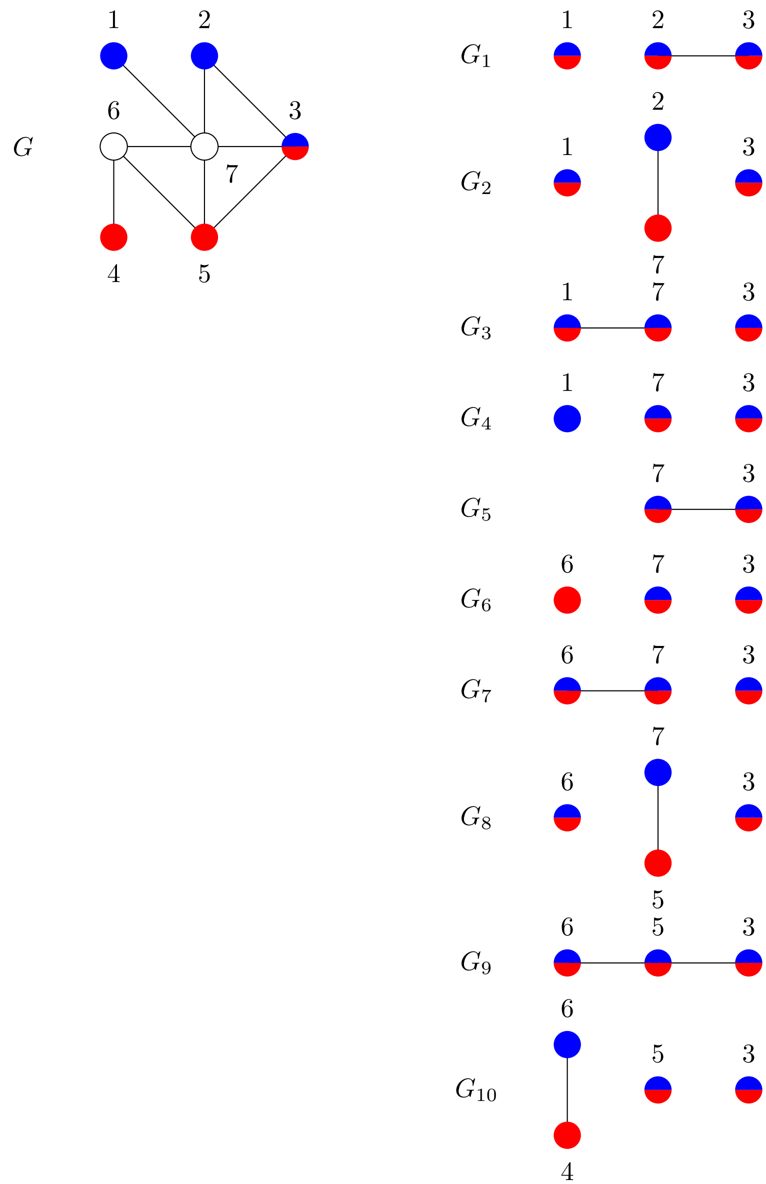
- $B(G) = B_{\text{upper}}(G_1) \cup B_{\text{lower}}(G_K)$.
- For each n, $B_{\text{upper}}(G_n) = B_{\text{lower}}(G_{n-1})$.
- If a vertex p is in G_i and G_k and $i < j < k$, then it is in G_j,
- If $p \in I(G)$, then p must be incident to two vertical edges.

For finite graphs, any vertical-edge layer can be written as $G_1 \bowtie \cdots \bowtie G_K$, where each G_k is a vertical edge layer with only one edge, and the same applies to horizontal-edge layers. Thus, we can assume if we wish that each vertical- or horizontal-edge layer in a layering has only one edge, and similarly each upper- or lower-stub layer has only one stub.

An *upper-boundary spike* is an oriented edge e with $\iota(e) \in B_{\text{upper}}$ and $\tau(e) \notin B_{\text{upper}}$. An *upper-boundary edge* is an edge with $\iota(e)$ and $\tau(e) \in B_{\text{upper}}$, and an *upper-boundary stub* is a disconnected boundary vertex in $B_{\text{upper}} \setminus B_{\text{lower}}$. If $G = G_1 \bowtie G_2$ where G_1 is a horizontal-edge layer with one edge, then G_2 is obtained from G by an upper-boundary edge deletion; conversely, if G has an upper boundary edge, then it can be expressed as $G_1 \bowtie G_2$, where G_1 is a horizontal-edge layer. The same holds when G_1 is an upper-stub layer with one stub, and G_2 is obtained from G by deleting an upper-boundary stub; and when G_1 is a vertical-edge layer, and G_2 is obtained from G by removing an upper-boundary spike e. For the vertical-edge layer, removing the upper-boundary spike may not actually be a spike contraction when we consider G as a graph with boundary: If $\tau(e) \in I$, then upper boundary spike is removed by a spike contraction, but if $\tau(e) \in B_{\text{lower}}$, then it is removed by a boundary edge deletion followed by a disconnected boundary vertex deletion of $\iota(e)$. We make the same definitions and observations for the lower boundary.

Joining elementary layers thus provides an interpretation of reduction operations in terms of subgraph partitions. In particular, a graph G is reducible to the empty graph if and only if there exists a layering of G.

Figure 2: A layering of a graph. The upper boundary is shown in blue and the lower boundary in red. The horizontal-edge layers are G_1, G_3, G_7, and G_9; the vertical-edge layers are G_2, G_8, and G_{10}; the upper-stub layer is G_4 and the lower-stub is G_6.



7.2 Recovering Boundary Spikes and Boundary Edges

If $e \in E$, then an *e-horizontal layering* is a layering $G_1, \dots, G_K$ such that

- $e \in E(G_j)$ for some horizontal-edge layer G_j.
- G_j comes between the upper- and lower-stub layers. That is, if G_k is an upper-stub layer, then $k < j$, and if G_j is a lower-stub layer, then $k > j$.

Similarly, an *e-vertical layering* is a layering $G_1, \dots, G_K$ such that $e \in E(G_j)$ for some vertical edge layer G_j which comes between the upper- and lower-stub layers. Assume $\iota(e)$ is in the upper boundary of G_j and $\tau(e)$ is in the lower boundary.

Lemma 7.1. *Let $\Gamma = (G, R)$ be an electrical network with bijective, zero-preserving conductances. Suppose e is a boundary spike and there is an e-horizontal layering of G. Then γ_e is uniquely determined by L.*

Proof. Let $G_1, \dots, G_K$ be an e-horizontal layering. With some abuse of notation, I will use G to mean the two-boundary graph with $B_{\text{upper}}(G) = B_{\text{upper}}(G_1)$ and $B_{\text{lower}}(G) = B_{\text{lower}}(G_K)$. Note $\iota(e)$, the boundary vertex of the spike, must be in $B_{\text{upper}}(G) \cap B_{\text{lower}}(G)$.

Let $t \in \mathbb{R}$. I claim that there exists a harmonic (u, c) with

- $u_{\iota(e)} = t$.
- For $p \in B_{\text{upper}}(G) \setminus \{\iota(e)\}$, $u_p = 0$.
- For $p \in B_{\text{upper}}(G) \setminus B_{\text{lower}}(G)$, the net current on p is zero.

Further, I claim that for any such (u, c), $u_{\tau(e)}$ must be zero. Hence, the net current on $\iota(e)$ is $\gamma_e(u_{\iota(e)} - u_{\tau(e)}) = \gamma_e(t)$. This implies that $\gamma_e(t)$ is uniquely determined by L; indeed, we only have to find some $(\phi, \psi) \in L$ which satisfies the boundary conditions listed above; we know that such a (ϕ, ψ) exists, and that whichever we choose, it will have $\psi_{\iota(e)} = \gamma_e(t)$. Since t was arbitrary, γ_e will be uniquely determined by L.

To prove the claims, let $x \in \mathbb{R}^{B_{\text{upper}}} \times \mathbb{R}^{B_{\text{upper}}}$ be the upper boundary data which has potential t on $u_{\iota(e)}$ and 0 elsewhere and current 0 on B_{upper}. Solving the above boundary value problem is equivalent to finding a (u, c) with upper boundary data x. (For $p \in B_{\text{upper}} \cap B_{\text{lower}}$ specifying the upper boundary current does not determine the net current since we did not specify the lower boundary current.) We construct (u, c) inductively, first defining it on G_1, then on $G_1 \bowtie G_2$, and so on until we reach G_K. We rely on the fact that the relationship X of G equals $X_K \odot X_{K-1} \odot \cdots \odot X_1$, where X_k is the relationship of G_k.

For $n = 1, \dots, j - 1$, we claim that there is a unique harmonic (u_n, c_n) on $G_1 \bowtie \cdots \bowtie G_n$ with upper boundary data x, and all potentials and currents are zero except that $u_{\iota(e)} = t$. Note that G_1 is either a upper-stub, horizontal-edge, or vertical-edge network. In each case, any harmonic function on G_1 with upper boundary data zero must have lower boundary data zero, since we are dealing with bijective, zero-preserving conductances (and clearly, setting the potential and current to zero *does* define a harmonic function). However, the same holds if we change the potential of $\iota(e)$ to t. Indeed, if $1 < j$, there are no edges incident to $\iota(e)$ in G_1, and $\iota(e)$ is on both boundaries of G_1. So the lower boundary data on G_1 must be zero except on $\iota(e)$. The same argument applies for each G_n, $n = 1, \ldots, j-1$. Each vertex and edge of $G_1 \bowtie \cdots \bowtie G_{j-1}$ was in one of the layers, so it has potential or current zero, except for $\iota(e)$.

Note $\tau(e) \in B_{\text{lower}}(G_j) = B_{\text{upper}}(G_1 \bowtie \cdots \bowtie G_{j-1})$. The lower boundary data of $G_1 \bowtie \cdots \bowtie G_{j-1}$ has potential 0 except on $\iota(e)$, and in particular, the potential on $\tau(e)$ must be zero.

Finally, we extend our harmonic function to $G_1 \bowtie \cdots \bowtie G_n$ for $n \geq j$. If there is a harmonic (u_n, c_n) on $G_1 \bowtie \cdots \bowtie G_n$, we let its lower boundary data be y_n. G_{n+1} is a lower-stub, horizontal-edge, or vertical edge layer with bijective conductances. For any such network, we can find a harmonic function on G_{n+1} with upper boundary data y_n (this function is not unique for a lower-stub layer), and joining it with (u_n, c_n) produces a harmonic function on $G_1 \bowtie \cdots \bowtie G_{n+1}$. □

Lemma 7.2. *Let $\Gamma = (G, R)$ be an electrical network with bijective, zero-preserving conductances. Suppose e is a boundary edge and there is an e-vertical layering of G. Then γ_e is uniquely determined by L.*

Proof. Let $G_1, \ldots, G_K$ be an e-vertical layering. Observe that $\iota(e) \in B_{\text{upper}}(G)$ and $\tau(e) \in B_{\text{lower}}(G)$. Let $t \in \mathbb{R}$. I claim that there exists a harmonic (u, c) with

- A net current of t on $\iota(e)$.
- For $p \in B_{\text{upper}}(G)$, $u_p = 0$.
- For $p \in B_{\text{upper}}(G) \setminus B_{\text{lower}}(G) \setminus \{\iota(e)\}$, the net current on p is zero.

Further, I claim that for any such (u, c), the potential is zero on all neighbors of $\iota(e)$ except $\tau(e)$. This will imply that

$$u_{\iota(e)} - u_{\tau(e)} = \gamma_e^{-1}(t) = \rho_e(t).$$

Thus, $\rho_e(t)$ and hence γ_e are uniquely determined by L.

Let x be the upper boundary data with potentials and net currents zero except that the net current on $\iota(e)$ is $-t$. The boundary value problem above is equivalent to finding a harmonic (u, c) with upper boundary data x. As before, we define (u, c) inductively on $G_1 \bowtie \cdots \bowtie G_n$, for $n = 1, \ldots, K$. The potential and current must be zero on $G_1 \bowtie \cdots \bowtie G_{j-1}$, except that the net current for $\iota(e)$ on the upper and lower boundary is $-t$. Since e is a vertical edge in G_j, $\iota(e)$ is not in $V(G_n)$ for any $n > j$. Thus, any edges incident to e must have been in the layers $G_1, \ldots, G_{j-1}$, so the potential on all neighbors of $\iota(e)$ except $\tau(e)$ is zero. □

7.3 Sufficient Conditions for Recoverability

We can now state precisely what conditions we need in order to recover bijective, zero-preserving conductances using layerings. We say a graph G is *recoverably layerable* if there is a sequence of subgraphs $G = G_0, G_1, \ldots G_K$ such that

- G_{k+1} is obtained from G_k by a boundary spike contraction or boundary edge deletion.
- G_K has no edges, and all vertices are boundary vertices.
- If it G_{k+1} obtained by contracting a spike e, then there is an e-horizontal layering of G_k.
- If it G_{k+1} is obtained by deleting a boundary edge e, then there is an e-vertical layering of G_k.

Let e_n be the edge removed from G_n to obtained G_{n+1}, and let L_n be the set of boundary data for G_n. By Lemmas 7.1 and 7.2, the conductance of e_n is uniquely determined by L_n; by Lemmas 6.5 and 6.6, L_{n+1} is uniquely determined by L_n and γ_{e_n}. Thus, by induction γ_{e_n} is uniquely determined by L for all n, which means G is recoverable over the bijective, zero-preserving conductances.

A more symmetric and (it turns out) stronger condition is *total layerability.* We say G is *totally layerable* if for each edge e, there exists an e-horizontal layering, and an e-vertical layering. We will prove later that all critical circular planar graphs are totally layerable, as well as many others.

Proposition 7.3. *If G is totally layerable, then it is recoverably layerable.*

To prove this, note that a totally layerable graph must be layerable. Thus, there is a spike or boundary edge e. We can also find an e-horizontal or e-vertical layering. It only remains to show that after contracting the spike or deleting the boundary edge, the reduced graph is also totally layerable. To do this, we use the following lemma:

Lemma 7.4. *Suppose S is a subgraph of G and $e \in E(S)$. If there is an e-horizontal (respectively e-vertical) layering of G, then there is an e-horizontal (respectively e-vertical) layering of S.*

Proof. S can be obtained from G in three steps:

1. Change all vertices of $I(G) \setminus I(S)$ to boundary.
2. Delete all edges in $E(G) \setminus E(S)$, which must be boundary edges.
3. Delete all vertices in $V(G) \setminus V(S)$, which must be disconnected boundary vertices.

Thus, it suffices to show that if G' is obtained from G by changing a single interior vertex to boundary, deleting a single boundary edge, or deleting a disconnected boundary vertex, and if $G_1, \dots, G_K$ is an e-horizontal / e vertical layering of G, then we can find such a layering for G'.

Suppose G' is obtained by changing an interior vertex p to boundary. Let $G_1, \dots, G_K$ be an e-horiztonal or e-vertical layering of G, and assume each layer has one edge or one stub. Let G_j be the layer which includes e. Then p is contained in some layer G_n, and either $n \leq j$ or $n \geq j$. Assume $n \leq j$; the other case is symmetrical. Since p is interior, there must be some vertical edge layer G_m with $p \in B_{\text{lower}}(G_m) \setminus B_{\text{upper}}(G_m)$. Then $m \leq n \leq j$. Let q be the adjacent vertex in $B_{\text{upper}}(G_m)$, and let e' be the edge from q to p. We define elementary layers G'_k as follows:

- For $k < m$, we obtain G'_k from G_k by adding p to both the upper and lower boundary.
- Let G'_m be the horizontal edge layer with $V(G'_m) = V(G_m)$ and the single edge e'. Let G''_m be the upper-stub layer with $B_{\text{upper}}(G''_m) = V(G_m)$ and $B_{\text{lower}}(G''_m) = B_{\text{lower}}(G_m) = V(G_m) \setminus \{q\}$.

Then $G'_1, \dots, G'_{m-1}, G'_m, G''_m, G_{m+1}, \dots, G_K$ form a layering of G'. Each G'_k for $k < m$ is the same type of elementary layer as G_k. The only upper-stub layer we added was G''_m which comes before G_j, and e is in the same type of layer it was before (horizontal or vertical). Thus, the new layering is an e-horizontal or e-vertical layering as desired.

Suppose G' is obtained from G by deleting a boundary edge e'. Let $G_1, \dots, G_K$ and G_j be as before. Then e' is in some G_n. Then we modify the layering by removing G_n to obtain a layering for G'. If e' is a vertical edge, assume $\iota(e')$ is on the upper boundary and $\tau(e')$ is on the lower boundary. For each $k < n$, G_k must have $\iota(e')$ on both the upper and lower boundary. Let G'_k be obtained from G_k by adding $\tau(e')$ to both boundaries. For $k > n$, G_k must have $\tau(e')$ on both boundaries. Let G'_k be obtained from G_k by adding $\iota(e')$ to both boundaries. Then $G'_1, G'_2, \dots, G'_{n-1}, G'_{n+1}, \dots, G'_K$ form a layering of G', and if the original layering was an e-vertical / e-horiztonal layering, so is the new one.

The case where G' is obtained from G by deleting a disconnected boundary vertex is easy and is left to the reader. □

Corollary 7.5. *A subgraph of a totally layerable graph is totally layerable. A subgraph of a recoverably layerable graph is recoverably layerable.*

Proof. The first statement follows easily from the lemma and definition of total layerability. For the second, use a similar technique to Lemma 6.2; the details are left to the reader. □

8 Layerings, Connections, and Mixed Problems

8.1 Columns and Connections

Let P and Q be sets of boundary vertices. A *connection from P to Q* is a collection of disjoint boundary-to-boundary paths such that each path has its intial vertex in P and its terminal vertex in Q; each vertex in P is in exactly one of the paths, and each vertex in Q is in exactly one of the paths. There may be a vertex $p \in P \cap Q$; in this case, any connection from P to Q must include the length-0 path from p to itself. Thus, there is a one-to-one correspondence between connections from P to Q and connections from $P \setminus Q$ to $Q \setminus P$. If there is a connection from P to Q, then P and Q must have the same cardinality.

Let $M(P, Q)$ be the maximum number of paths in any connection that exists from some $P' \subset P$ to some $Q' \subset Q$. [1] shows that if G is a circular planar graph with positive linear conductances, then $M(P, Q) = \operatorname{rank} \Lambda_{P,Q}$, where Λ is the response matrix. This section generalizes their results by using layerings to relate $M(P, Q)$ to certain properties of L.

If G is a graph with boundary such that $B = P \cup Q$, then a *layering from P to Q* is a layering with $P = B_{\text{upper}}(G_1)$ and $Q = B_{\text{lower}}(G_K)$. For a layering from P to Q, the vertical edges can be joined together into embedded paths, which we call *columns*. For a given column, there are three possibilities:

- The column forms a boundary-to-boundary path with one endpoint in P and the other in Q.
- The column has one endpoint in P and the other is an upper stub.
- The column has one endpoint in Q and the other is a lower stub.

The *width* of the layering is the number of columns which form boundary-to-boundary paths.

Proposition 8.1. *If there is a layering from P to Q, then the width of the layering is $M(P, Q)$.*

Proof. Let w be the width. The columns which form boundary-to-boundary paths furnish a connection of size w from some subset of P to some subset of Q, so $M(P, Q) \geq w$. To show the reverse inequality, note that since the upper-stub layers come before the lower-stub layers, there must be some G_n such that $B_{\text{lower}}(G_n) = B_{\text{upper}}(G_{n+1})$ has w elements, one from each of the columns which form boundary-to-boundary paths. Any path from a vertex in P to a vertex in Q must pass through every layer of the layering, so it must have a vertex in $B_{\text{lower}}(G_n)$. Thus, there can be at most w disjoint paths from a vertex in P to a vertex in Q, and $M(P, Q) \leq w$. □

8.2 The Two-Boundary Relationship

Let P and Q be sets of boundary vertices with $B = P \cup Q$. Consider G as a two-boundary graph with $B_{\text{upper}}(G) = P$ and $B_{\text{lower}}(G) = Q$. Let $X \subset (\mathbb{R}^{B_{\text{upper}}} \times$

$\mathbb{R}^{B_{\text{upper}}}) \times (\mathbb{R}^{B_{\text{lower}}} \times \mathbb{R}^{B_{\text{lower}}})$ be the two-boundary relationship defined in 7.1. Let π_1 and π_2 be the projections onto $\mathbb{R}^{B_{\text{upper}}} \times \mathbb{R}^{B_{\text{upper}}}$ and $\mathbb{R}^{B_{\text{lower}}} \times \mathbb{R}^{B_{\text{lower}}}$. For $x \in \mathbb{R}^{B_{\text{upper}}} \times \mathbb{R}^{B_{\text{upper}}}$, let $X^x = \{y : (x,y) \in X\}$, and for $y \in \mathbb{R}^{B_{\text{lower}}} \times \mathbb{R}^{B_{\text{lower}}}$, let $X_y = \{x : (x,y) \in X\}$. Then

Proposition 8.2. *Let Γ be an eletrical network with bijective continuous conductances. Suppose there exists a layering from P to Q, and let X be as above. Let w be the width of the layering, s_u the number of upper stubs, and s_ℓ the number of lower stubs. Then*

i. $\pi_1(X)$ *is homeomorphic to* $\mathbb{R}^{2w+s_u}$*.*

ii. $\pi_2(X)$ *is homeomorphic to* $\mathbb{R}^{2w+s_\ell}$*.*

iii. For any $x \in \pi_1(X)$*,* X^x *is homeomorphic to* $\mathbb{R}^{n_\ell}$*.*

iv. For any $y \in \pi_2(X)$*,* X_y *is homeomorphic to* $\mathbb{R}^{n_u}$*.*

Proof. Let $G_1, \ldots, G_K$ be a layering from P to Q, such that each layer has one edge or one stub. Choose n such that for any upper-stub layer G_k, $k \leq n$, for any lower-stub layer G_k, $k > n$. Let X_k be the two-boundary relationship for G_k, so that $X = X_K \odot X_{K-1} \odot \cdots \odot X_1$. We parametrize X in terms of three things: $\xi \in \mathbb{R}^{B_{\text{lower}}(G_n)} \times \mathbb{R}^{B_{\text{lower}}(G_n)}$, $\eta \in \mathbb{R}^{S_u}$, and $\zeta \in \mathbb{R}^{S_\ell}$.

Choose ξ, η, ζ. Let $\xi_k = \xi$. If G_n is a horizontal-edge or vertical-edge layer, then there is a unique ξ_{k-1} with $(\xi_{n-1}, \xi_n) \in X_n$. If G_n is an upper-stub layer and p is the stub, then there is a unique ξ_{n-1} with $(\xi_{n-1}, \xi_n) \in X_n$, and the potential on p equal to η_p. We apply the same reasoning to G_{n-1}, G_{n-2}, and so on. Then we let $x = \xi_0$.

Similarly, if G_{n+1} is a horizontal-edge or vertical-edge layer, then there is a unique ξ_{n+1} with $(\xi_n, \xi_{n+1}) \in X_{n+1}$. If G_{n+1} is a lower-stub layer with stub p, then there is a unique ξ_{n+1} with $(\xi_n, \xi_{n+1}) \in X_{n+1}$ and potential ζ_p on p. Apply the same reasoning to $G_{n+2}, \ldots, G_K$, and let $y = \xi_K$.

The (x, y) thus constructed depends continuously on ξ, η, ζ. Actually, x only depends on ξ and η, and y depends on ξ and ζ. Conversely, for $k = 1, \ldots, n$, ξ_k depends continuously on ξ_{k-1}, so ξ and η depend continuously on x, and similarly, ξ and ζ depend continuously on y. We have parametrized all of X. Since $\pi_1(X)$ is the set of x's and $\pi_2(X)$ is the set of y's, we have proven (i) and (ii).

For (iii), fix $x \in \pi_1(X)$. Then ξ and η are uniquely determined by x; however, ζ does not depend on x, so the set of y's with $(x, y) \in X$ can be parametrized by ζ, and is thus homeomorphic to $\mathbb{R}^{n_\ell}$. The proof of (iv) is symmetrical. □

Corollary 8.3. *Under the above conditions, $M(P, Q)$ can be computed from L, and*

$$2M(P,Q) = \dim \pi_1(X) - \dim X_y = \dim \pi_2(X) - \dim X^x.$$

Proof. Dimension here means the dimension of a topological manifold: If S is locally homeomorphic to $\mathbb{R}^k$, then we say $\dim S = k$. This is well-defined by the "invariance of domain" theorem from topology. The corollary follows immediately, since $w = M(P, Q)$ and since X can be computed from L and vice versa. ☐

Corollary 8.4. *Let Γ be an electrical network with signed linear conductances, and suppose the Dirichlet problem has a unique solution. Let Λ be the response matrix, and let $P' = P \setminus Q$, $Q' = Q \setminus P$. If there is a layering from P to Q, then* $\operatorname{rank} \Lambda_{P',Q'} = M(P', Q')$.

8.3 Stubless Layerings

A *stubless layering* of a graph G is a layering with no upper-stub or lower-stub networks. For stubless layerings, the relationship between layerings, mixed problems, and connections is much stronger:

Theorem 8.5. *Let G be a graph in which every interior vertex has degree ≥ 2. Let $B = P \cup Q$ and $P' = P \setminus Q = B \setminus Q$ and $Q' = Q \setminus P = B \setminus P$. The following are equivalent:*

1. *There is a stubless layering from P to Q.*
2. *For all bijective conductances, potentials on P and net currents on P' determine a unique harmonic function.*
3. *For all (nonzero) signed linear conductances, potentials on P and net currents on P' determine a unique harmonic function.*
4. *There is a unique connection from P to Q, and this connection uses all the interior vertices.*

Remark. Let $(*)$ be the condition that potentials on P and net currents on P' determine a unique harmonic function. In (2) and (3) it is important that $(*)$ holds for *all* conductances. Even if it holds for *most* signed linear conductances, the stubless layering may not exist.

Proof. (1) $\implies$ (2). Let X be the two-boundary relationship corresponding to P and Q. Since there are no stub layers, the reasoning in the previous Proposition implies that for bijective conductances, $\pi_1(X)$ is all of $\mathbb{R}^P \times \mathbb{R}^P$ and any $x \in \mathbb{R}^P \times \mathbb{R}^P$ determines a unique harmonic function on the network.

(2) $\implies$ (3) by definition.

(3) $\implies$ (4). For signed linear conductances $\{a_e\}$, $(*)$ is equivalent to the submatrix $K_{P' \cup I, Q' \cup I}$ being invertible. Recall that

$$\det K_{P' \cup I, Q' \cup I} = (-1)^n \sum_{F \in \mathcal{F}(P,Q)} \operatorname{sgn} \tau_F \prod_{e \in F} a_e.$$

I claim that if $(*)$ holds for all signed linear conductances, then $\mathcal{F}(P, Q)$ has exactly one element. Clearly, it has at least one element, since otherwise $\det K_{P' \cup I, Q' \cup I}$ is always zero. Suppose it has two elements F_1 and F_2. There is some edge $e_0 \in F_1 \setminus F_2$ or vice versa. Thus, we can assign a sign ± 1 to each edge, such that

$$(-1)^n \operatorname{sgn} \tau_{F_1} \prod_{e \in F_1} \operatorname{sgn} e = 1, \qquad (-1)^n \operatorname{sgn} \tau_{F_2} \prod_{e \in F_2} \operatorname{sgn} e = -1.$$

Thus, by the same argument as in §3.3, there exist signed conductances with $\det K_{P' \cup I, Q' \cup I} = 0$. So $\mathcal{F}(P, Q)$ has only one element.

Let F be this element. Each component contains either one vertex in $P \cap Q$, or it contains one vertex in P' and one in Q'. Each component is a tree, but I claim that each component is actually a path. If some component were not a path, then there would be an interior vertex p with only one edge e in F incident to it. By assumption, there is another edge e' incident to p. The other endpoint of e' is in some component of F, so $F \setminus \{e\} \cup \{e'\}$ is another spanning forest. The components of F thus provide a connection from P to Q. The connection is unique because if there were another connection, then we could add edges to complete it to a different spanning forest. Thus, (3) implies (4).

(4) $\implies$ (1). Suppose there is a unique connection from P to Q, and that this connection uses all the interior vertices. Our goal is to produce a stubless layering whose columns are the paths in the connection. Call the edges in the paths "vertical" and the other edges "horizontal." Let E^* be the set of oriented edges e such that either (a) e is horizontal or (b) e is vertical and oriented in the same direction as the paths, from P to Q. For $e, e' \in E^*$ define $e \prec e'$ if $\tau(e) = \iota(e')$ and at least one of the oriented edges is vertical.

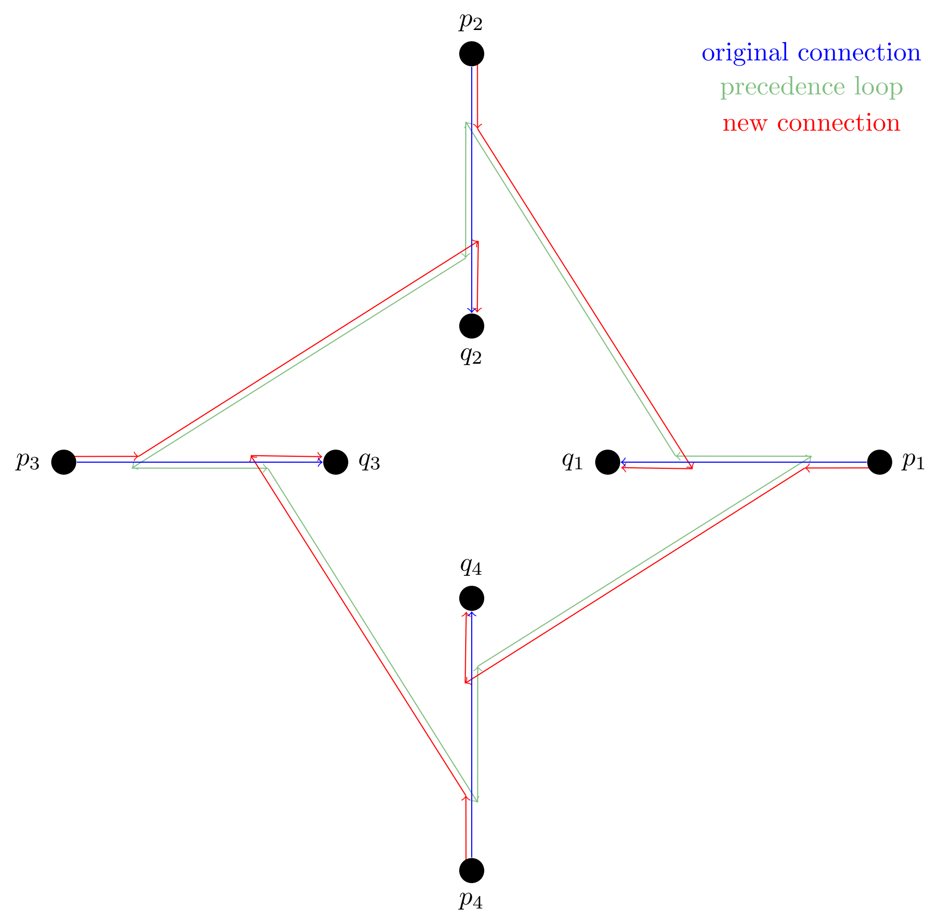
I claim that there does not exist a sequence $e_1, \dots, e_K \in E^*$ with

$$e_1 \prec e_2 \prec \cdots \prec e_K \prec e_1.$$

Call such a sequence a *precedence loop*. Suppose for the sake of contradiction a precedence loop exists, and that $e_1, \dots, e_K \in E^*$ is the precedence loop with the minimum number of horizontal edges. The idea is to use the precedence loop to construct a different connection from P to Q, as shown in Figure 3. However, we have many details to attend to first. Observe:

- Every precedence loop must have horizontal edges.
- If there were $j < k$ with $e_j = e_k$, then $e_1, \dots, e_j, e_{k+1}, \dots, e_K$ and $e_{j+1}, \dots, e_k$ would both be precedence loops. So $e_{j+1}, \dots, e_k$ would be a precedence loop with fewer horizontal edges. Thus, the oriented edges in our precedence loop must be distinct.
- Call a subset of $\{1, \dots, K\}$ an interval if it has the form $\{j, \dots, k\}$ or $\{1, \dots, j, k, \dots, K\}$ for some $1 \leq j < k \leq K$. If α is a path of the connection, let $I_\alpha = \{k : e_k \text{ is in } \alpha\}$. Then I_α is an interval. For suppose not. By reindexing if necessary, we can assume e_K is not in α. Then there exist $j < k < \ell < m$ such that e_{j-1} is a horizontal edge, $e_j, \dots, e_k$ are in α, e_{k+1} is a horizontal edge; $e_{\ell-1}$ is a horizontal edge, $e_\ell, \dots, e_m$ are in α, and e_{k+1} is a horizontal edge. We can assume that $e_j, \dots, e_k$ occur earlier

Figure 3: Proof of Theorem 8.5: (4) $\implies$ (1)



in α than $e_\ell, \ldots, e_m$ (earlier meaning closer to P). So there is a sequence of edges $e'_1, \ldots, e'_n$ in α with $e'_1 \prec e'_2 \prec \cdots \prec e'_n$, $e'_1 = e_j$, and $e'_n = e_m$. Thus, we obtain a precedence loop

$$e_1, \ldots, e_{j-1}, e_j = e'_1, e'_2, \ldots, e'_n = e_m, e_{m+1}, \ldots, e_K,$$

which has fewer horizontal edges than the original, since we removed at least one horizontal edge e_{k+1} and added only vertical edges.

- Let $v_j = \tau(e_j)$. Each v_j is in exactly one α. Since each v_j must be the endpoint of some vertical edge, it follows from the previous observation that for a given α, the set of j with v_j in α is an interval. Thus, none of the v_j's in the same path can be equal to each other.

Now we use the loop to create a different connection from P to Q. Let $p_1, \ldots, p_N$ be the vertices in P and $q_1, \ldots, q_N$ be the vertices in Q. Let α_n be the path from p_n to q_n. If the precedence loop does not contain any edges of α_n, let $\alpha'_n = \alpha_n$. If the loop contains some edges $e_j, \ldots, e_k$ from α_n, let α'_n start at p_n, follow α_n until it reaches the first vertex in the loop (v_{j-1}), then follow the horizontal edge $\overline{e}_{j-1}$ to a vertex (v_{j-2}) in a different path α_m, and finally follow α_m until it reaches q_m. It follows from the properties listed above that the paths α'_n are disjoint.

Therefore, (4) implies that there is no precedence loop. Thus, there must be some $e_1 \in E^*$ such that $e' \not\prec e$ for all $e' \in E^*$.

- Suppose e_1 is vertical. Then there are no vertical or horizontal edges incident to $\iota(e)$. Therefore, e_1 is an upper boundary spike. We define a vertical-edge layer G_1 with $B_{\text{upper}}(G_1) = P$ and $B_{\text{lower}}(G_1) = P \setminus \{\iota(e_1)\} \cup \{\tau(e_1)\}$ and $E(G_1) = \{e_1, \overline{e}_1\}$.
- Suppose e_1 is horizontal. Then there is no vertical edge $e \in E^*$ with $\tau(e)$ equal to one of the endpoints of e_1. Thus, e_1 must be an upper-boundary edge. We define a horizontal-edge layer G_1 with $B_{\text{upper}}(G_1) = B_{\text{lower}}(G_1) = P$ and $E(G_1) = \{e_1, \overline{e}_1\}$.

Next, we find an edge e_2 such that for all $e \in E^*$, $e \neq e_1$, we have $e \not\prec e_2$. In a similar way, we define an elementary layer with edge e_2, and continue inductively until all edges of G have been exhausted. The resulting layers $G_1, \ldots, G_K$ form a stubless layering of G from P to Q. □

This result is rather surprising. Two purely geometric conditions (1) and (4) are equivalent to the algebraic conditions (2) and (3). And it is not at all obvious that (2) and (3) should be equivalent since bijective conductances can behave much worse than signed linear conductances. Nor is it immediate that (1) and (4) are equivalent, and one consequence of this is that a unique connection in a flower cannot use all the interior vertices. However, it is relatively easy to show directly that (1) $\implies$ (4), and I leave this as an exercise.

9 Critical Circular Planar Networks

9.1 Medial Graphs

For any graph G, there is a corresponding topological space $\mathcal{G}$, the quotient space obtained from $E \times [0,1]$ by identifying (e,t) with $(\overline{e}, 1-t)$ and identifying $(e,0)$ and $(e',0)$ if $\iota(e) = \iota(e')$. An *embedding* of a graph on a surface with boundary S is a function $f : \mathcal{G} \to \overline{S}$ which is a homeomorphism onto its image, such that

- $f(x) \in \partial S$ if and only if x corresponds to a boundary vertex.
- Each component of $S \setminus f(\mathcal{G})$ is homeomorphic to an open disc.

In the future, we will identify G, $\mathcal{G}$, and $f(\mathcal{G})$.

The components of $S \setminus G$ are called *cells*. The boundary of each cell is a union of edges of G and pieces of ∂S. Two cells $\mathcal{X}$ and $\mathcal{Y}$ are *adjacent* if they share an edge (that is, there is an edge contained in $\overline{\mathcal{X}} \cap \overline{\mathcal{Y}}$). A *two-coloring* of the cells is an assignment to each cell of a color "white" or "black," such that adjacent cells have opposite colors. Not all graphs admit such a coloring.

If S is a surface with boundary, then a *medial graph on S* is a graph embedded on S such that each interior vertex has valence 4 and each boundary vertex has valence 1, together with a two-coloring of the cells. If C is a curve on $\overline{S}$ or any subset of $\overline{S}$, we say a medial cell $\mathcal{X}$ *touches* C if $\overline{\mathcal{X}} \cap C \neq \varnothing$. If G is embedded on S, then we say a medial graph $\mathcal{M}$ is *compatible* with G if the following conditions are satisfied:

- Each edge of G contains exactly one interior vertex of $\mathcal{M}$, not at either of the endpoints of the edge, and each interior vertex of $\mathcal{M}$ is contained in one edge of G.
- The edges of $\mathcal{M}$ only intersect G at their endpoints.
- Each black cell of $\mathcal{M}$ contains exactly one vertex of G, and each vertex of G is contained in a black cell.
- Each white cell of $\mathcal{M}$ is contained a cell of G, and each cell of G contains exactly one white cell of $\mathcal{M}$.
- A vertex of G is a boundary vertex if and only if its medial cell touches ∂S.

According to this definition, there may be more than one medial graph for a given G with a given embedding.

If e and e' are medial edges incident to an interior vertex v of a medial graph on S, we say e and e' are *adjacent* (at v) if there is a cell $\mathcal{X}$ with $e, e' \subset \partial \mathcal{X}$. Otherwise, e and e' are *opposite* (at v). A *geodesic arc* is a path in $\mathcal{M}$ with vertices and oriented edges $v_0, e_1, v_1, e_2, \dots, e_K, v_K$ such that e_k and e_{k+1} are opposite at v_k and $e_1, \dots, e_K$ are distinct. A *geodesic* is a geodesic arc such that either $v_0 = v_K$ or v_0 and v_K are both boundary vertices.

If G is embedded on S with a medial graph $\mathcal{M}$, then subgraph partitions of G naturally arise from partitions of the surface S. Suppose C is a simple curve on S which divides it into two regions S_1 and S_2 which are themselves surfaces with boundary. Assume no vertices of $\mathcal{M}$ are on C, that C intersects each edge of $\mathcal{M}$ in finitely many points, and that for each medial cell $\mathcal{X}$, $\mathcal{X} \cap S_1$ and $\mathcal{X} \cap S_2$ are homeomorphic to D. For $j = 1, 2$, define a graph G_j by letting $V(G_j)$ be the set of vertices whose medial cells intersect S_j, $E(G_j)$ the set of oriented edges whose medial vertices are in S_j, and $B(G_j)$ the set of vertices whose medial cells touch ∂S_1. Then G_1 and G_2 form a subgraph partition of G. Similarly, we can divide S into surfaces $S_1, \dots, S_K$ and find a corresponding subgraph partition of G.

We can embed G_j into S_j by restricting the embedding of G and altering it slightly. Then there is compatible medial graph $\mathcal{M}_j$ whose cells are the intersections of the cells of $\mathcal{M}$ with S_j. However, the embeddings for G_1 and G_2 thus constructed may or may not be consistent with each other. For example, if G is divided into three subgraphs G_1, G_2, G_3, one medial black cell may be cut into three regions with no common boundary points, and in this case it is impossible to find a position for the vertex which will work for all three subgraphs. In this case, it is best not to worry about the position of the vertices, but instead focus on the medial cells.

For a graph G and surface S and $C \subset \partial S$, let B_C be the set of vertices whose medial cells touch C. If G, G_1, and G_2 are as above, define two-boundary graphs by letting

$$\begin{aligned}
B_{\text{upper}}(G_1) &= B_{\partial S_1 \cap \partial S}(G_1) \\
B_{\text{lower}}(G_1) &= B_C(G_1) \\
B_{\text{upper}}(G_2) &= B_C(G_2) \\
B_{\text{lower}}(G_2) &= B_{\partial S_2 \cap \partial S}(G_2).
\end{aligned}$$

Then $G = G_1 \bowtie G_2$. In particular, we may be able to create layerings of a graph embedded on S by constructing curves which divide S into surfaces $S_1, \dots, S_K$ such that each G_k is an elementary layer.

9.2 Total Layerability

A graph is *circular planar* if it can be embedded in the unit disc D, where ∂D is the unit circle. Equivalently, it is circular planar if it can be embedded in some surface with boundary S with $\overline{S}$ homeomorphic to $\overline{D}$. If G is circular planar and every component has a boundary vertex, then there exists a compatible medial graph on D; the case where G is connected is proven in [1] and we leave the rest to the reader.

A *lens* in a medial graph is a closed path in $\mathcal{M}$ formed by one or two geodesic arcs with distinct edges. A medial graph is *lensless* if it has no lenses; equivalently, it is lensless if every geodesic is a boundary-to-boundary path and no two geodesics intersect more than once.

A CP graph with a lensless medial graph is called *critical* (in [1], criticality is a different condition, but it is equivalent to the medial graph being lensless). [1] proves that a circular planar graph is recoverable over positive linear conductances if and only if there is a compatible lensless medial graph, and in fact, if the medial graph has a lens, then the graph is Y-Δ equivalent to a graph with a parallel or series connection. [4] shows that critical circular planar (CCP) graphs are recoverable over bijective zero-preserving conductances. We will give an alternative proof of this last result by showing that CCP graphs are totally layerable.

We start with some definitions and lemmas: Suppose that C_{upper} and C_{lower} are two arcs which partition ∂D. Let B_{upper} and B_{lower} be the sets of vertices of G whose medial cells touch C_{upper} and C_{lower} respectively. Then B_{upper} and B_{lower} are called a *circular pair*. They intersect in at most two vertices. We will construct layerings between circular pairs.

Lemma 9.1. *Let G be CCP with medial graph $\mathcal{M}$. Let C_{upper}, C_{lower}, B_{upper}, and B_{lower} be as above. Suppose every geodesic has at least one endpoint on C_{upper}. Then there exists a layering from B_{upper} to B_{lower} with no lower-stub layers.*

Proof. We can assume that if a black medial cell $\mathcal{X}$ touches C_{upper}, then $\overline{\mathcal{X}} \cap C_{\text{upper}}$ is an arc of C_{upper}, and the same holds for the lower boundary. Indeed, if there is a black medial cell such that $\overline{\mathcal{X}} \cap C_{\text{upper}}$ consists of two or more arcs, then it is not hard to change the medial cell so that $\overline{\mathcal{X}} \cap C_{\text{upper}}$ has only one arc, and we can do this without affecting B_{upper} and B_{lower} or the hypotheses of the lemma, and the same holds for the lower boundary.

Our goal is to construct the layering inductively. First, we show that if there is at least one edge in G or one geodesic with both endpoints on the upper-boundary, then there exists an upper-boundary spike, upper-boundary edge, or upper-boundary stub. Observe the following:

- Suppose a geodesic consists of one medial edge and has both endpoints on the upper boundary. Then this geodesic together with an arc of C_{upper} bound a medial cell, and the corresponding vertex of G is an upper-boundary stub. We will call this cell a *upper-boundary stub-cell.*

- Suppose there is a triangular medial cell bounded by two medial edges and an arc of C_{upper}. Then two medial vertices of the cell are on C_{upper} and the other is interior. We call such a medial cell an *empty upper-boundary triangle.* Then the corresponding vertex of G is a valence 1 vertex in $B_{\text{upper}} \setminus B_{\text{lower}}$. Hence, the edge of G corresponding to the medial vertex of the triangle is an upper-boundary spike or upper-boundary edge.

- If there is a white empty upper-boundary triangle, and v is the interior medial vertex of the cell, then the edge of G corresponding to v is an upper-boundary edge.

Therefore, it suffices to show that there is an upper-boundary stub-cell or an empty upper-boundary triangle. If there is an upper-boundary stub-cell, we are done.

Suppose there is no upper-boundary stub-cell. Then there is at least one edge in G; then there is a medial vertex v at which two geodesics g_0 and g_0' intersect. Choose an orientation of g_0 with a starting point on C_{upper}, and let g_1 be the first geodesic which g_0 intersects. Since g_0 and g_1 have an endpoint on the upper boundary, there are geodesic arcs g_0' and g_1' contained in g_0 and g_1 such that g_0' and g_1' have one endpoint at v and one on C_{upper}. Since $\mathcal{M}$ is lensless, g_0' and g_1' only intersect at v. There there is an arc C_0 of the upper boundary such that C_0, g_0, and g_1 form a simple closed curve and by the Jordan curve theorem, they bound a triangular region T_0. Orient g_1' starting at C_{upper} and ending at v, and let g_2 be the first geodesic g_1' intersects. Then there is a triangular region $T_1 \subset T_2$ bounded by an arc of g_1 and an arc of g_2 and an arc of C_{upper}. Continuing inductively to define $g_3, g_4, \dots$ and $T_2, T_3, T_4, \dots$. Each T_n is a union of medial cells and edges, so there must be some n for which $T_n = T_{n+1}$. Let $R_0 = T_n$. Then the geodesic arcs which bound R_0 must consist of only one medial edge apiece.

So either R_0 is an empty upper-boundary triangle or it fully contains some geodesics. In the first case, we are done. In the second, R_0 must contain more than one geodesic; otherwise, it would contain a stub-cell, contrary to our assumption. Thus, we can we find an edge of G contained in R_0. Then we repeat the above argument to find a new triangle $R_1 \subset R_0$ bounded by two medial edges and an arc of C_{upper}. If this is not an empty upper-boundary triangle, then there is an $R_2 \subset R_1$, and so on. The process must terminate after finitely many steps because each R_n contains strictly fewer medial cells than R_{n-1}. Thus, some R_n is an empty boundary triangle. Therefore, G must have an upper-boundary spike, upper-boundary edge, or upper-boundary stub.

Given G, we construct the layering as follows: If every geodesic has one endpoint on each boundary and there are no interior vertices in the medial graph, then every vertex of G is on both boundaries, and G has no edges. So there exists a trivial layering of G. Otherwise, there is an upper-boundary spike, upper-boundary edge, or upper-boundary stub. Thus, we can write $G = G_1 \bowtie G'$ where G_1 is a vertical-edge, horizontal-edge, or upper-stub layer.

In fact, the subgraph partition can be constructed by dividing the disc into two pieces using a curve C_{upper}' with the same endpoints as C_{upper} and C_{lower}. In the case of an upper-boundary stub, we make C_{upper}' stay close to C_{upper} except near the upper-boundary stub-cell, so that the region bounded by C_{upper}' and C_{upper} contains the stub-cell, but all interior vertices of $\mathcal{M}$ are contained in the other region. Similarly, for an empty upper-boundary triangle, we make C_{upper}' stay close to C_{upper} except to "skirt" the triangle. Then the region D' bounded by C_{lower} and C_{upper}' is homeomorphic to D and has an embedded medial graph $\mathcal{M}'$ formed by intersecting the cells of $\mathcal{M}$ with D'. Then G' can be embedded in D' with medial graph $\mathcal{M}'$.

It is easy to verify that G' and $\mathcal{M}'$ satisfy the original hypotheses. Thus, we can continue inductively to divide G into into elementary layers until there are no more edges in G and no more geodesics with both endpoints on the upper boundary curve. □

Theorem 9.2. *Critical circular planar graphs are totally layerable.*

Proof. Let G be CCP with a lensless medial graph $\mathcal{M}$. Let e be any oriented edge of G, and let v be the corresponding medial vertex. Let g be one of the geodesics which intersects at v, and let r and s be its endpoints. Let r' be a point on the boundary circle on the counterclockwise side of r, so that r' is closer to r than any geodesic endpoint (other than r itself), and let s' similarly be on the counterclockwise side of s. Let C_{upper} be the counterclockwise arc from r' to s' and let C_{lower} be the counterclockwise arc from s' to r'.

Then g divides D into two regions; call the region bounded by g and C_{upper} "above g" and the other region "below g." Let C be a curve which starts at r' crosses g once and ends at s'; let it cross g on the medial edge incident to v which is closest to r'; we can arrange that C is so close to g that there are no medial vertices above g and below C, or below g and above C. Let C' start at r', cross g on the medial edge incident to v closest to s', and end at s'; we can arrange that C' is always above C and the only medial vertex in the region between them is v.

Then the subgraph of G corresponding to the region between C and C' is either a vertical-edge layer or horizontal-edge layer with C as the lower boundary curve and C' as the upper boundary curve. If it is a vertical-edge layer, then by putting r' and s' on the clockwise side of g instead and performing a similar contruction would produce a horiztonal-edge layer, and vice versa. Let G_0 be this elementary layer; let H be the subgraph in the region bounded by C and C_{lower} and let H' be the subgraph in the region bounded by C' and C_{upper}.

Since each other geodesic only intersects g once, we can arrange that no geodesic intersects C twice or C' twice. Then every geodesic in H' has one endpoint on C_{upper}. Hence, by the Lemma, there is a layering $H'_1, \ldots, H'_j$ of H' with C_{upper} as the upper boundary curve and C' as the lower boundary curve, and no lower-stub layers. Similarly, there is a layering $H_1, \ldots, H'_k$ of H with C as the upper boundary curve and C_{lower} as the lower boundary curve, and no upper-stub layers. Then $H'_1, \ldots, H'_j, G_0, H_1, \ldots, H_k$ is a layering of G from B_{upper} to B_{lower} which is an e-vertical or e-horizontal layering, depending on which type of layer G_0 is. Thus, there are e-horizontal and e-vertical layerings of G. □

Remark. The layerings $G_1, \ldots, G_K$ constructed in the above proofs have a corresponding division of D into regions $S_1, \ldots, S_K$ such that G_k is the subgraph of G corresponding to the region S_k. For each S_k, there are arcs $C_{\text{upper}}(S_k)$ and $C_{\text{lower}}(S_k)$ of the boundary curve which correspond to $B_{\text{upper}}(G_k)$ and $B_{\text{lower}}(G_k)$, and $C_{\text{lower}}(S_k) = C_{\text{upper}}(S_{k-1})$.

9.3 The Cut-Point Lemma

[1] shows the following: Suppose G is a CCP graph with positive linear conductances. If we know the response matrix Λ and the order of the boundary vertices and geodesic endpoints on the boundary circle, then we can determine the graph up to Y-Δ equivalence. The key observations, in my terminology, are:

1. Suppose P and Q are a circular pair corresponding to a partition C_{upper} and C_{lower} of the boundary circle. Then the maximum size connection $M(P,Q)$ can be determined from L.
2. The number of geodesics which have both endpoints on C_{upper} is uniquely determined by $M(P,Q)$.
3. If we know the number of geodesics with both endpoints on C_{upper} for all possible choices of C_{upper}, we can determine which points on the boundary circle are endpoints of the same geodesic.
4. This will determine G up to Y-Δ equivalence.

We will generalize (1) to nonlinear conductances and present another proof of the "cut-point lemma" used in (2). For (3) and (4), refer to [1].

Lemma 9.3. *Let G be CCP. If P and Q are a circular pair, then there is a layering from P to Q.*

Proof. If P and Q are a circular pair, then there are arcs C_{upper} and C_{lower} which partition the boundary circle. If a geodesic g has both endpoints on C_{upper}, then there is a region R_g bounded by g and some arc of C_{upper}; if h has both endpoints on C_{lower} it cannot intersect g; otherwise, it would have to both enter and exit R_g, and hence would intersect g in two places, creating a lens. Thus, the geodesics with both endpoints on the upper boundary and those with both endpoints on the lower boundary cannot intersect. As the reader may verify, it is possible to construct a simple curve C with endpoints r and s such that the geodesics with both endpoints on the upper boundary are contained in the region R_1 bounded by C and C_{upper}, the geodesics with both endpoints on the lower boundary are contained in the region R_2 bounded by C and C_{lower}, and no geodesic intersects C more than once. Then as in the previous section we construct a layering of the subgraph G_1 in R_1 with C_{upper} as the upper boundary curve and C as the lower boundary curve and no lower-stub layers, and a layering of the subgraph G_2 in the other region with no upper-stub layers. These join to form a layering of G from P to Q. □

Corollary 9.4. *Let G be CCP, and let Γ be a network on G with bijective continuous conductances. If P and Q are a circular pair, then $M(P,Q)$ is uniquely determined by L.*

Proof. This follows from Corollary 8.3. □

Corollary 9.5 (Cut-Point Lemma)**.** *Let C_{upper} and C_{lower} be a partition of the boundary circle into arcs. Let P and Q be the corresponding sets of boundary vertices. The number of geodesics with both endpoints on C_{upper} is $|P| - M(P,Q)$.*

Proof. Let $G_1, \ldots, G_K$ be a layering from P to Q constructed as above. As before, there are corresponding regions $S_1, \ldots, S_K$ in D and upper and lower boundary curves for each S_k. Let $G'_k = G_1 \bowtie \cdots \bowtie G_k$, and let S'_k be the corresponding region. For each k, let m_k be the maximum size connection in G_k from the upper boundary, let r_k be the number of geodesics with both endpoints on the upper boundary, and let $n_k = |B_{\text{upper}}(G_k)|$. Let j be the index where $C_{\text{upper}}(S_j) = C$. For $k \leq j$, we prove that $r_k = n_k - m_k$ by induction. Since $G_j \bowtie \cdots \bowtie G_K$ has no upper-stub layers, the maximum connection (that is, the width of the layering) is the number of upper-boundary vertices, so $m_j = n_j$; there are no geodesics with both endpoints on the upper boundary, so $r_j = 0 = n_j - m_j$, which completes the base case. If G_{k-1} is a vertical-edge or horizontal-edge layer, then $m_{k-1} = m_k$, $n_{k-1} = n_k$, and $r_{k-1} = r_k$. If G_{k-1} is an upper-stub layer, then G'_{k-1} has one more geodesic on the upper boundary than G'_k; it also has one more vertex on the upper boundary; but the maximum connection is unchanged. Hence, $r_{k-1} = r_k + 1 = n_k - m_k + 1 = n_{k-1} - m_{k-1}$. It follows that the number of geodesics with both endpoints on C_{upper}, which is r_1, equals $n_1 - m_1 = |P| - M(P,Q)$. □

The upshot is that we can determine the Y-Δ equivalence class of a CCP network from L and the arrangement of the boundary vertices for bijective continuous conductances. If they are bijective, continuous, and zero-preserving, we can in theory attempt the recovery process for each member of the Y-Δ equivalence class and discover by trial and error which ones could have been the original graph G. In the positive linear case, any member could have been the original graph, but this is not true in general. A question for further research is when exactly the original graph is uniquely determined by L.

10 Some Graph Contructions

10.1 Covers

Let G and H be graphs, with or without boundary. A *graph morphism* $f : H \to G$ consists of two functions $f_V : V(H) \to V(G)$ and $f_E : E(H) \to E(G)$ such that $\iota(f_E(e)) = f_V(\iota(e))$ and $\overline{f_E(e)} = f_E(\overline{e})$. A graph morphism is a *covering map* if for each vertex p, f_E restricted to $\{e : \iota(e) = p\}$ is bijective. For graphs with boundary, we require in addition that $f_V(p) \in B(G)$ if and only if $p \in B(H)$. If there is a covering map $f : H \to G$, then H is said to be a *cover* of G.

I will write f for both f_V and f_E since the meaning will be clear from the context. If $f : H \to G$ is a graph morphism and G' is a subgraph of G, then we define $f^{-1}(G')$ by

$$\begin{aligned} V(f^{-1}(G')) &= f^{-1}(V(G')), \\ E(f^{-1}(G')) &= f^{-1}(E(G')), \\ B(f^{-1}(G')) &= f^{-1}(B(G')). \end{aligned}$$

If G' is a two-boundary graph and is a subgraph of G as a graph with boundary, we define $f^{-1}(G')$ similarly, with

$$\begin{aligned} B_{\text{upper}}(f^{-1}(G')) &= f^{-1}(B_{\text{upper}}(G')), \\ B_{\text{upper}}(f^{-1}(G')) &= f^{-1}(B_{\text{upper}}(G')). \end{aligned}$$

Suppose $f : \tilde{G} \to G$ is a covering map. The following are easy to verify:

- If $G_1, \ldots, G_K$ are a subgraph partition of G, then $f^{-1}(G_1), \ldots, f^{-1}(G_K)$ are a subgraph partition of $\tilde{G}$.
- G is reducible to the empty graph if and only if $\tilde{G}$ is reducible to the empty graph.
- If $G = G_1 \bowtie G_2$, then $\tilde{G} = f^{-1}(G_1) \bowtie f^{-1}(G_2)$.
- The preimage of an elementary layer is an elementary layer of the same type.
- If $G_1, \ldots, G_K$ are a layering of G, then $f^{-1}(G_1), \ldots, f^{-1}(G_K)$ are a layering of $\tilde{G}$.
- If G is recoverably layerable, then so is $\tilde{G}$.
- If G is totally layerable, then so is $\tilde{G}$.

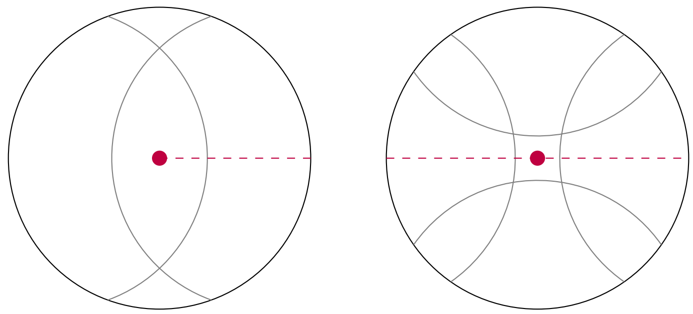
It is easy to construct covers of a given graph G. Choose an integer n, and define the sets vertices and edges for our cover by

$$\begin{aligned} V(\tilde{G}) &= V(G) \times \{1, \ldots, n\}, \\ E(\tilde{G}) &= E(G) \times \{1, \ldots, n\}, \\ B(\tilde{G}) &= B(G) \times \{1, \ldots, n\}. \end{aligned}$$

We still need to define ι and $^-$ for $\tilde{G}$. For $p \in V(G)$, denote the corresponding vertices of $\tilde{G}$ by $p_1, \ldots, p_n$, and for $e \in E(G)$, let $e_1, \ldots, e_n$ be the corresponding edges. Let $\iota(e_j) = (\iota(e))_j$. For each $e \in E(G)$, choose a permutation $\sigma_e \in S_n$, such that $\sigma_{\overline{e}} = \sigma_e^{-1}$, and let $\overline{e_j} = (\overline{e})_{\sigma_e(j)}$. Then setting $f(p_j) = p$ and $f(e_j) = e$ defines a covering map $\tilde{G} \to G$.

Actually, any (finite) cover of a connected graph G can be constructed this way. Suppose G is connected and $\iota(e) = p$ in G. For each $p' \in f^{-1}(p)$, there is exactly one $e' \in f^{-1}(e)$ with $\iota(e') = p'$. Thus, $f^{-1}(p)$ and $f^{-1}(e)$ have the same cardinality. Since G is connected, $f^{-1}(p)$ and $f^{-1}(e)$ have the same cardinality

Figure 4: Removing a lens using a covering map.



A lens in $\mathcal{M}$. Corresponding geodesics in $h^{-1}(\mathcal{M})$.

for all vertices and edges. Let n be the cardinality (called the *rank* or *number of sheets* of the cover). For each vertex p, let $p_1, \ldots, p_n$ be the elements of $f^{-1}(p)$ and for each edge e, let e_j be the element of $f^{-1}(e)$ with $\iota(e_j) = f^{-1}((\iota(e))_j)$. Note $^-$ is bijective and maps $f^{-1}(e)$ onto $f^{-1}(\overline{e})$; thus, there must be some permutation $\sigma_e \in S_n$ with $\overline{e_j} = (\overline{e})_{\sigma_e(j)}$, and of course, $\sigma_{\overline{e}} = \sigma_e^{-1}$.

This construction allows us to produce totally layerable graphs from other totally layerable graphs. For instance, we could start with a critical circular planar graph G, choose some large n, and choose appropriate permutations σ_e, so that the resulting cover $\tilde{G}$ is a large, non-circular-planar, but totally layerable graph.

Actually, a cover $\tilde{G}$ may be totally layerable even if G is not. Suppose for example that G is a connected circular planar graph, and it has a medial graph $\mathcal{M}$ with only one lens, which is formed by two geodesics as shown in Figure 10.1. Suppose G is embedded in the unit disc in $\mathbb{C}$ such that no vertex lies on the origin, and the lens contains the origin. Let $h : \mathbb{C} \to \mathbb{C}$ be given by $z \mapsto z^2$. Then $h^{-1}(G)$ is a two-fold cover of G embedded in the unit disc, and $h^{-1}(\mathcal{M})$ is a compatible lensless medial graph. This method will not work directly if there are multiple lenses which do not contain a common point, since removing one lens will create two copies of all the other lenses. However many times we repeat the process, there will still be more lenses.

In general, if S is a surface with boundary and $h : \tilde{S} \to S$ is a topological covering map, then for any graph G embedded on S, $\tilde{G} = h^{-1}(G)$ is a covering graph of G, and if $\mathcal{M}$ is a medial graph for G, then $h^{-1}(\mathcal{M})$ is a medial graph for $\tilde{G}$. For general surfaces with boundary, it is not known what conditions on the medial graph will guarantee total layerability; however, if such conditions are discovered, topological covering maps may be a useful tool for creating totally layerable covers of certain graphs. In particular, the map $h : z \mapsto z^n$ could be used for graphs embedded in the annulus $\{1 < |z| < 2\} \subset \mathbb{C}$. One could also consider constructing covering graphs in a purely graph-theoretical way to remove “obstacles” to total layerability.

10.2 Products

If G and H are graphs with boundary, let $G \times H$ be the graph with

- $V(G \times H) = V(G) \times V(H)$.
- $B(G \times H) = (B(G) \times V(H)) \cup (V(G) \times B(H))$.
- $E(G \times H) = (E(G) \times V(H)) \cup (V(G) \times E(H))$.
- If $e \in E(G)$ and $q \in V(H)$, then $\overline{(e,q)} = (\overline{e}, q)$ and $\iota((e,q)) = (\iota(e), q)$. Similarly, if $p \in V(G)$ and $e \in E(H)$, then $\overline{(p,e)} = (p, \overline{e})$ and $\iota((p,e)) = (p, \iota(e))$.

Layerings of G and H naturally produce layerings of $G \times H$. Suppose $G_1, \ldots, G_K$ is a layering of G. We construct a layering of $G \times H$ as follows: The first layer S_0 will be a horizontal edge layer with vertices $B_{\text{upper}}(G_1) \times V(H)$ and edges $B_{\text{upper}}(G_1) \times E(H)$. Then for each n,

- If G_k is an upper-stub, lower-stub, or horizontal-edge layer, add a layer S_k with
$$\begin{aligned} B_{\text{upper}}(S_k) &= (B_{\text{upper}}(G_k) \times V(H)) \cup (V(G) \times B(H)), \\ B_{\text{lower}}(S_k) &= (B_{\text{lower}}(G_k) \times V(H)) \cup (V(G) \times B(H)), \\ E(S_k) &= E(G_k) \times V(H). \end{aligned}$$
- If G_k is a vertical-edge layer, add a vertical-edge layer S_k with
$$\begin{aligned} B_{\text{upper}}(S_k) &= (B_{\text{upper}}(G_k) \times V(H)) \cup (V(G) \times B(H)), \\ B_{\text{lower}}(S_k) &= (B_{\text{lower}}(G_k) \times V(H)) \cup (V(G) \times B(H)), \\ E(S_k) &= E(G_k) \times I(H), \end{aligned}$$
then a horizontal-edge layer S'_k with
$$\begin{aligned} V(S'_k) &= (B_{\text{lower}}(S_k) \times V(H)) \cup (V(G) \times B(H)), \\ E(S'_k) &= (E(G_k) \times B(H)) \cup (B_{\text{lower}}(G_k) \times E(H)). \end{aligned}$$

We will call the layering given by the S_k's and S'_k's the *product layering induced by* $G_1, \ldots, G_n$. If $H_1, \ldots, H_N$ is a layering of H, then we can define a similar layering of $G \times H$, switching the roles of G and H. A consequence of these product layerings is that $G \times H$ is generally “at least as layerable as” G and H.

Proposition 10.1.

i. If either G or H is layerable, then so is $G \times H$.

ii. *If either G is recoverably layerable and H has no parallel edges, then $G \times H$ and $H \times G$ are recoverably layerable.*

iii. *If for every $e \in E(G)$ there is an e-vertical layering of G and for every $e \in E(H)$ there is an e-vertical layering of H, then $G \times H$ is totally layerable. In particular, this holds if G and H are totally layerable.*

Proof.

i. For any layering of G or H, there is an induced layering of $G \times H$.

ii. Suppose G is recoverably layerable; the other case is symmetrical. Let H' be the graph obtained from H by changing all the boundary vertices to interior. Then $G \times H$ is a subgraph of $G \times H'$, so it suffices to show $G \times H'$ is recoverably layerable.

Let $J_0, \ldots, J_N$ be a sequence of graphs with $J_0 = G$, and $J_N = \varnothing$, and each J_k obtained from J_{k-1} by a reduction operation, such that if J_k is obtained by contracting a spike e, then there is an e-horizontal layering of J_{n-1}, and if J_n is obtained by deleting a boundary edge e, then there is an e-vertical layering of J_{n-1}.

Suppose J_n is obtained from J_{n-1} by deleting a boundary edge e. Let $G_1, \ldots, G_K$ be an e-vertical layering of J_{n-1}. Then the induced layering of $J_{n-1} \times H'$ is an (e, q)-vertical layering for each $(e, q) \in \{e\} \times V(H')$. When we delete the boundary edges $\{e\} \times V(H')$ from $J_{n-1} \times H$, we obtain $J_n \times H'$.

Suppose J_n is obtained from J_{n-1} by contracting a boundary spike e. To obtain $J_n \times H'$ from $J_{n-1} \times H'$, we must first delete the boundary edges $\{\iota(e)\} \times E(H')$, then contract the boundary spikes $\{e\} \times V(H')$. Let $G_1, \ldots, G_K$ be an e-horizontal layering of G with $e \in E(G_k)$. Choose $e' \in E(H')$. Note $\iota(e)$ is on both the upper and lower boundary of J_{n-1}, so $(\iota(e), \iota(e'))$ and $(\iota(e), \tau(e'))$ are on both boundaries of $J_{n-1} \times H'$ in the induced layering. Also, $(\iota(e), e')$ is in the initial layer S_0. We modify the layering as follows, relying on the fact that the only edges incident to $(\iota(e), \iota(e'))$ are $(e, \iota(e'))$ and edges in $\{\iota(e)\} \times E(H')$, and if $e'' \in E(H)$ with $\iota(e'') = \tau(e')$, then $(\iota(e), \iota(e''))$ is on both boundaries:

- Remove $(\iota(e), e')$ from $E(S_0)$, and for any $e'' \in E(H)$ incident to $\tau(e')$, remove $(\iota(e), e'')$.
- For $j = 0, \ldots, k-1$, remove $(\iota(e), \tau(e'))$ from each layer S_j and (where applicable) S'_j.
- Remove $(e, \tau(e'))$ from S_k.
- Insert a vertical-edge layer S^*_k with $(\iota(e), e')$ as a vertical edge.
- Insert a horizontal-edge layer with edges $(e, \tau(e'))$ and $(\iota(e), e'')$ for $e'' \in E(H)$ incident to $\tau(e')$.
- For $j = k+1, \ldots, k$, remove $(\iota(e), \iota(e'))$ from each layer S_j and (where applicable) S'_j.

This produces an $(\iota(e), e')$-vertical layering. When we delete the boundary edges $\{\iota(e)\} \times E(H')$ from $J_{n-1} \times H'$, the edges $\{e\} \times V(H')$ become boundary spikes, and the induced layering from $G_1, \ldots, G_K$ produces a (e, q)-horizontal layering for each $q \in V(H')$.

If J_n is obtained from J_{n-1} by deleting a disconnected boundary vertex, there is nothing to prove.

iii. Let G' and H' be the graphs obtained by changing the boundary vertices of G and H to interior. Choose $(e, q) \in E(G) \times V(H)$. If $G_1, \ldots, G_N$ is an e-vertical layering of G, then the induced product layering of $G \times H'$ is an (e, q)-vertical layering, and since $G \times H$ is a subgraph of $G \times H'$, there is an induced e-vertical layering of $G \times H$. To find an (e, q)-horizontal layering, choose $e' \in E(H)$ with $\tau(e') = q$. Let $H_1, \ldots, H_N$ be an e'-vertical layering of H with $e' \in E(H_k)$. In the induced layering of $G' \times H$, (e, q) occurs in the second product layer corresponding to H_k (called S'_k earlier). Thus, the induced layering of $G' \times H$ is an (e, q)-horizontal layering, and it induces an (e, q)-horizontal layering of $G \times H$. If $(p, e) \in V(G) \times E(H)$, then there are (p, e)-vertical and (p, e)-horizontal layerings of $G \times H$ by a symmetrical argument. □

Products thus provide another method of producing large and complicated totally layerable or recoverably layerable graphs from smaller ones. Less symmetrical graphs can be created by taking subgraphs of products.

11 More Signed Linear Conductances

11.1 The Electrical Linear Group EL_n

In [5], Lam and Pylyavskyy define an "electrical linear group" EL_{2n}, whose "positive part" acts on circular planar networks with $n+1$ boundary vertices with positive linear conductances. The group is isomorphic to the symplectic group. Here we define a slightly different electrical linear group. We also approach it differently since we have dealt with signed conductances on non-planar networks from the outset. We show its relationship to the symplectic group in a more explicit and elementary way using electrical networks rather than Lie theory.

Suppose Γ is a network on a vertical-edge layer G with a single edge, and the resistance of the edge is $\rho_e(t) = at$, where $a_e \neq 0$. Index the columns of the layer by $1, \ldots, n$, and let p_j and q_j be the upper and lower boundary vertices on the jth column. Let k be the index of the column with the edge, so that the edge connects p_k and q_k, and $p_j = q_j$ for $j \neq k$. Any $x \in \mathbb{R}^{2n}$ can represent upper boundary data on Γ; for $j = 1, \ldots, n$, we let x_j represent the potential on p_j and x_{n+j} the upper-boundary current on p_j (with the appropriate sign). Similarly, any element of $\mathbb{R}^{2n}$ can also represent data on the lower boundary. If x and y represent the upper- and lower- boundary data of a harmonic function on Γ, then

- For $j \neq k$, $x_j = y_j$ and $x_{n+j} = y_{n+j}$.
- $y_k = x_k + ax_{n+k}$ and $y_{n+k} = x_{n+k}$.

Hence,

$$y = \begin{pmatrix} I & aE_{k,k} \\ 0 & I \end{pmatrix} x,$$

where each block is $n \times n$ and $E_{k,k}$ is an $n \times n$ matrix with 1 on the k, k entry and zeroes elsewhere. We will call this $2n \times 2n$ matrix $V_k(a)$. Then $V_k(a + b) = V_k(a)V_k(b)$ and hence $V_k(a)^{-1} = V_k(-a)$.

Similarly, if Γ is a network on a horizontal-edge layer G with vertices $p_1, \dots, p_n$ and an edge between p_j and p_k with conductance $\gamma_e(t) = at$ for $a \neq 0$, and if x and y represent data on the upper and lower boundaries, then

$$y = \begin{pmatrix} I & 0 \\ a(E_{j,j} - E_{j,k} - E_{k,j} + E_{k,k}) & I \end{pmatrix} x.$$

We call this matrix $H_{j,k}(a)$. Then $H_{j,k}(a+b) = H_{j,k}(a)H_{j,k}(b)$ and $H_{j,k}(a)^{-1} = H_{j,k}(-a)$.

The *electrical linear group* EL_n is the group generated by the matrices $V_k(a)$ and $H_{j,k}(b)$ for real $a, b \neq 0$ and $j, k \in \{1, \dots, n\}$ with $j \neq k$. Suppose $\Xi \in EL_n$ is given by $\Xi = \Xi_M \dots \Xi_1$, where each Ξ_m is one of the generators $V_k(a)$ or $H_{j,k}(b)$. Then each Ξ_m corresponds to a vertical- or horizontal-edge layer G_m. By identifying $B_{\text{lower}}(G_m)$ and $B_{\text{upper}}(G_{m+1})$ according to the given indexing of the columns in each layer, we construct a network $G = G_1 \bowtie \cdots \bowtie G_M$. The matrix Ξ maps upper-boundary data on G to lower-boundary data on G.

Thus, each stubless-layerable network with n columns indexed by $1, \dots, n$ corresponds to a $\Xi \in EL_n$. However, multiple stubless-layerable graphs may have the same Ξ matrix. For example, we can join two vertical-edge layers together to produce $\Xi = V_k(a)V_k(b)$, but this is the same as $V_k(a + b)$, and hence could represent a network with a single vertical edge. Each Ξ matrix thus represents a large class of stubless-layerable networks. A stubless-layerable network with $B_{\text{upper}} = B_{\text{lower}}$ and no edges corresponds to the identity matrix.

There is another interpretation of EL_n in terms of reduction operations and their inverses. Suppose Γ is a network with n boundary vertices, indexed $1, \dots, n$ and Γ' is obtained by adjoining a spike e on vertex k with $\rho_e(t) = at$. Assume the new boundary vertex $\iota(e)$ inherits the index k from $\tau(e)$. Then if $x = (\phi, \psi) \in \mathbb{R}^{2n}$ represents the boundary data of a harmonic function on Γ, then the extension of (u, c) to Γ' has boundary data represented by $y = V_k(a)x$, and $L' = V_k(a)(L)$. $V_k(-a) = V_k(a)^{-1}$ means that in terms of L, adjoining a spike of resistance $-\rho_e$ is the inverse of adjoining a spike of resistance ρ_e, and it is the same as contracting a spike of resistance ρ_e. Similar statements hold for $H_{j,k}(a)$ and boundary edge additions, and adding a boundary edge of conductance $-\gamma_e$ is the inverse of adding a boundary edge with conductance γ_e.

The two interpretations of EL_n in terms of stubless-layerable networks and reduction operations are related. Suppose Γ is an electrical network with boundary vertices indexed $1, \dots, n$; consider Γ as a two-boundary network with $B_{\text{upper}} =$

$\varnothing$ and $B_{\text{lower}} = B$. If Γ' is obtained from Γ by adjoining a spike, then $\Gamma' = \Gamma \bowtie \Gamma^*$ where Γ^* is a vertical-edge layer. Similarly, adding a boundary edge corresponds to joining a horizontal-edge layer. Thus, a sequence of spike adjunctures and boundary-edge additions corresponds to a stubless-layerable network and vice versa.

If we let EG_n be the collection of sets $L \subset \mathbb{R}^{2n}$ representing boundary data of linear electrical networks, then EL_n acts on EG_n via $\Xi \cdot L = \Xi(L)$. The action corresponds to applying a sequence of inverse reduction operations (or joining elementary layers) to an electrical network with boundary data L.

An analogous group can be constructed with nonlinear electrical networks, where the potential-current relationship is given by a resistance function on the vertical edges and a conductance function on the horizontal edges. We leave the details to the reader.

11.2 Characterization of EG_n and EL_n

We say a $2n \times 2n$ matrix Ξ is *symplectic* if

$$\Xi^T \Omega \Xi = \Omega, \qquad \text{where } \Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The $2n \times 2n$ symplectic matrices form a group called $\mathrm{Sp}_{2n}(\mathbb{R})$.

Let $x_0 = (1, \ldots, 1, 0, \ldots, 0)^T$ represent the vector with the first n entries 1 and the last n entries 0. It is straightforward to show that any $\Xi \in EL_{2n}$ is symplectic and fixes x_0: This is true of $V_k(a)$ and $H_{j,k}(a)$ by direct computation, and hence true of any product of these matrices.

This makes sense given that $\Xi \in EL_n$ represents a sequence of inverse reduction operations. The *standard symplectic form* on $\mathbb{R}^{2n}$ is

$$\omega(x, y) = x^T \Omega y.$$

Ξ is symplectic if and only if $\omega(\Xi x, \Xi y) = \omega(x, y)$ for all $x, y \in \mathbb{R}^{2n}$. We saw earlier that L is the set of boundary data for an electrical network with signed linear conductances, then for $(\phi_1, \psi_1), (\phi_2, \psi_2) \in L$, we have $\phi_1 \cdot \psi_2 = \phi_2 \cdot \psi_1$. This says exactly that if $x, y \in L$, then $\omega(x, y) = 0$. The fact that Ξ is symplectic guarantees that this property is preserved under a sequence of inverse reduction operations. And Ξ must fix x_0 because a harmonic function with constant potential 1 on an electrical network extends to a harmonic function with constant potential 1 when we add boundary spikes or boundary edges to the network. Ξ must also map a vector $x \in \mathbb{R}^{2n}$ whose "current" entries sum to zero to another vector whose "current" entries sum to zero; indeed, the sum of the "current" entries is $\omega(x_0, x)$, and $\omega(x_0, \Xi x) = \omega(\Xi x_0, \Xi x) = \omega(x_0, x)$.

These properties also make sense if we view Ξ as two-boundary map for a stubless-layerable network. Actually, if Γ is any two-boundary network with linear conductances with a well-defined bijective map Ξ from upper-boundary data to lower-boundary data, then Ξ must be symplectic; this follows from rewriting $\phi_1 \cdot \psi_2 = \phi_2 \cdot \psi_1$ in terms of the upper and lower boundaries. Ξ

must also fix x_0 because the function with constant potential 1 is harmonic. And the sum of the upper-boundary currents must equal to the sum of the lower-boundary currents.

With these facts in hand, we are ready to characterize both EG_n and EL_n.

Theorem 11.1. *$L \subset \mathbb{R}^{2n}$ is an element of EG_n if and only if*

i. L is an n-dimensional linear subspace;

ii. For $(\phi, \psi) \in L$, the entries of ψ sum to zero;

iii. For (ϕ_1, ψ_1) and $(\phi_2, \psi_2) \in L$, we have $\phi_1 \cdot \psi_2 = \phi_2 \cdot \psi_1$.

Proof. We have already proved that any $L \in EG_n$ must satisfy these properties. Suppose L satisfies (i), (ii), and (iii). Let $(\phi_1, \psi_1), \ldots, (\phi_n, \psi_n)$ be a basis for L. Let $\pi_1 : \mathbb{R}^{2n} \to \mathbb{R}^n$ be the projection onto the first n entries. Let $\ell = \dim \pi_1(L)$ and $m = n - \ell$. Let M be the matrix

$$M = \begin{pmatrix} \phi_1 & \phi_2 & \ldots & \phi_n \\ \psi_1 & \psi_2 & \ldots & \psi_n \end{pmatrix},$$

so that L is the column space of M. Observe that $\ell \geq 1$; if ℓ were zero, then the top half of the matrix would be 0; but the bottom half has column sums zero, and hence is not invertible; so $\ell = 0$ implies the (ϕ_j, ψ_j)'s are not linearly independent.

By applying Gaussian elimination to M using column operations, and then reindexing B if necessary, we can assume M has the form

$$\begin{pmatrix} I & 0 \\ * & 0 \\ * & * \\ * & ** \end{pmatrix},$$

where the dimensions of the blocks are

$$\begin{pmatrix} \ell \times \ell & \ell \times m \\ m \times \ell & m \times m \\ \ell \times \ell & \ell \times m \\ m \times \ell & m \times m \end{pmatrix}.$$

Note: Reindexing B means reindexing the rows in the upper half and reindexing the rows in the lower half in the same way. This does not affect the hypotheses of the theorem. Since we are only concerned with the column space of M, we are free to perform any invertible column operations, which correspond to changing our basis for L.

I claim that if $\ell < n$ the $**$ block of M is invertible. Suppose not. Then the columns of $**$ are linearly dependent, so there is a nontrivial linear combination $\psi^* = \sum_{j=\ell+1}^{n} \psi_j$ such that the entry $(\psi^*)_j = 0$ for $j = \ell + 1, \ldots, n$. Now $(0, \psi^*) \in L$. For $k = 1, \ldots, \ell$, observe by (iii) that

$$0 = 0 \cdot \psi_k = \phi_k \cdot \psi^* = \sum_{j=1}^{\ell} (\phi_k)_j \cdot (\psi^*)_j + \sum_{j=\ell+1}^{n} (\phi_k)_j \cdot (\psi^*)_j = (\psi^*)_k + 0.$$

Hence, $(\psi^*)_k = 0$. This is true for all k, so $\psi^* = 0$. But this implies that $\psi_{\ell+1}, \ldots, \psi_n$ are linearly dependent, contradicting our choice of $(\phi_1, \psi_1), \ldots, (\phi_n, \psi_n)$. Hence, $**$ must be invertible. Thus, by performing column operations on the columns $\ell + 1, \ldots, n$, we can put M in the form

$$\begin{pmatrix} I & 0 \\ * & 0 \\ * & * \\ * & I \end{pmatrix}.$$

Let $M' = V_{\ell+1}(1)V_{\ell+2}(1)\ldots V_n(1)M$, and let $L' = V_{\ell+1}(1)V_{\ell+2}(1)\ldots V_n(1)(L)$ be its column space. Because each $V_j(1)$ is symplectic and fixes x_0, we know that L' satisfies (i), (ii), and (iii). M' has the form

$$\begin{pmatrix} I & 0 \\ * & I \\ * & * \\ * & I \end{pmatrix},$$

and further column operations will reduce M' to a matrix

$$M'' = \begin{pmatrix} I & 0 \\ 0 & I \\ * & * \\ * & I \end{pmatrix}.$$

Let A be the lower half of M''. Property (iii) implies A is symmetric, and (ii) implies it has column sums zero. Thus, we can write A in the form

$$A = \sum_{i=1}^{\ell} \sum_{j=i+1}^{n} a_{i,j}(E_{i,i} - E_{i,j} - E_{j,i} + E_{j,j}).$$

Then by direct computation,

$$M'' = \begin{pmatrix} I \\ K \end{pmatrix} = \left(\prod_{i=1}^{\ell} \prod_{j=i+1}^{k} H_{i,j}(a_{i,j}) \right) \begin{pmatrix} I \\ 0 \end{pmatrix}$$

Thus,

$$L' = \left(\prod_{i=1}^{\ell} \prod_{j=i+1}^{n} H_{i,j}(a_{i,j}) \right) (\mathbb{R}^n \times \{0\}^n),$$

and

$$L = \left(\prod_{k=\ell+1}^{n} V_k(-1) \right) \left(\prod_{i=1}^{\ell} \prod_{j=i+1}^{n} H_{i,j}(a_{i,j}) \right) (\mathbb{R}^n \times \{0\}^n).$$

But $\mathbb{R}^n \times \{0\}^n$ is the set of boundary data for an electrical network with n boundary vertices and no interior vertices, and each transformation $H_{i,j}(a_{i,j})$ or

$V_k(-1)$ represents adding a boundary edge or boundary spike (except if $a_{i,j} = 0$, then $H_{i,j}(a_{i,j}) = I$ and we do not add any edge to the network). Equivalently, we can think of A as the Kirchhoff matrix of a network with no interior vertices; this network has boundary data L'; then adding spikes to vertices $\ell + 1, \ldots, k$ produces a network with boundary data L. □

Several other results fall out of the proof:

Corollary 11.2. *Let Γ be a signed linear electrical network. There exists $P \subset B$ such that potentials on P and net currents on $B \setminus P$ uniquely determine the other boundary data.*

Proof. Index the vertices $1, \ldots, n$, and let M and ℓ and m be as above. By further column operations, we can put M in the form

$$\begin{pmatrix} I & 0 \\ * & 0 \\ * & * \\ 0 & I \end{pmatrix}.$$

Let $P = \{1, \ldots, \ell\}$. The columns (ϕ_j, ψ_j) are a basis for L, and if $(\phi, \psi) = \sum_{j=1}^{n} \alpha_j(\phi_j, \psi_j)$, then $\alpha_1, \ldots, \alpha_\ell$ represent potentials on P and $\alpha_{\ell+1}, \ldots, \alpha_n$ represent net currents on $B \setminus P$. □

Corollary 11.3. *Over the signed linear conductances, every network is electrically equivalent to a layerable network with $\leq \frac{1}{2}n(n-1) + 1$ edges, where $n = |B|$.*

Proof. Let Γ be any network, and index the boundary vertices by $1, \ldots, n$, then $L \in EG_n$. Let Γ' be the network with the same L constructed in the proof of the theorem. The number of vertical edges added is $m = n - \ell$ and the number of horizontal edges is the number of nonzero entries of A above the diagonal, which is at most $\frac{1}{2}\ell(\ell - 1) + \ell m$. Thus, the total number of edges is at most

$$m + \frac{1}{2}(n-m)(n-m-1) + (n-m)m = \frac{1}{2}n(n-1) - \frac{1}{2}m(m-3) \leq \frac{1}{2}n(n-1) + 1.$$

□

Corollary 11.4. *EG_n is a smooth manifold of dimension $\frac{1}{2}n(n-1)$.*

Proof. The networks constructed in the theorem in fact give us parametrizations of the EG_n. Let $Y = \{(i,j) : 1 \leq i < j \leq n\}$, which has $\frac{1}{2}n(n-1)$ elements. For $Z \subset \{1, \ldots, n\}$, let $F_Z : \mathbb{R}^Y \to EG_n$ be given by

$$\{a_{i,j}\}_{(i,j) \in Y} \longmapsto \left(\prod_{k \in Z} V_k(-1) \right) \left(\prod_{(i,j) \in Y} H_{i,j}(a_{i,j}) \right) (\mathbb{R}^n \times \{0\}^n).$$

It follows from the proof of the theorem that the images of the F_Z's cover EG_n.

To complete the proof, it suffices to show that $F_Z^{-1} \circ F_{Z'}$ is well-defined and C^∞ on $F_{Z'}^{-1} \circ F_Z(\mathbb{R}^Y)$. Suppose that $a \in \mathbb{R}^Y$, and let $\Gamma_1(a)$ be the network with no interior vertices and Kirchhoff matrix $K_1(a) = \sum_{(i,j) \in Y} a_{i,j}(E_{i,i} - E_{i,j} - E_{j,i} + E_{j,j})$; let $L_1(a)$ be its set of boundary data. Let

$$L_2(a) = \left(\prod_{k \in Z'} V_k(1) \right) (F_Z(a)) = \left(\prod_{k \in Z} V_k(1) \right) \left(\prod_{k \in Z} V_k(-1) \right) (L_1(a)).$$

If $a \in F_{Z'}^{-1} \circ F_Z(\mathbb{R}^Y)$ and $F_{Z'}(b) = F_Z(a)$, then $L_2(a)$ will be the set of boundary data of the network with no interior vertices and Kirchhoff matrix $K_1(b)$. Hence, $a \in F_{Z'}^{-1} \circ F_Z(\mathbb{R}^Y)$ if and only if the relationship $L_2(a)$ can be described by a Dirichlet-to-Neumann map, which will be the Kirchhoff matrix of a network $\Gamma_1(b)$, and the entries of the Kirchhoff matrix will be the entries of $F_{Z'}^{-1} \circ F_Z(\mathbb{R}^Y) \in \mathbb{R}^V$. Note $L_2(a)$ is the relationship for the network $\Gamma_2(a)$ formed by taking $\Gamma_1(a)$ and adjoining spikes of conductance -1 to vertices in Z, then spikes of conductance 1 to vertices in Z'. Let $K_2(a)$ be the Kirchhoff matrix of $\Gamma_2(a)$. If $L_2(a)$ is given by a Dirichlet-to-Neumann map $\Lambda_2(a)$, then the boundary potentials uniquely determine the boundary net currents, and since $\Gamma_2(a)$ is layerable, these uniquely determine the potentials and currents on the whole network; thus, the Dirichlet problem has a unique solution. Hence, $K_2(a)_{I,I}$ is invertible, and $\Lambda_2(a) = K_2(a)/K_2(a)_{I,I}$ depends smoothly on a. But the entries of any b with $F_{Z'}(b) = F_Z(a)$ must be the above-diagonal entries of $\Lambda_2(a)$; so there is a unique $b \in F_{Z'}^{-1} \circ F_Z(a)$, and it depends smoothly on a. □

Thus, EG_n is a submanifold of the Grassmann manifold $G_{n,2n}$ which is the set of n-dimensional subspaces of $\mathbb{R}^{2n}$. We suggest calling EG_n the *electrical Grassmann manifold*. For $\Xi \in EL_n$, the mapping $L \mapsto \Xi(L)$ is a diffeomorphism $EG_n \to EG_n$, so EL_n is a diffeomorphism group acting on EG_n.

Next, we characterize EL_n:

Theorem 11.5. *EL_n is the group of symplectic matrices which fix x_0.*

Proof. We have already shown that any $\Xi \in EL_n$ is symplectic and fixes x_0. We only have to show that any matrix with those properties is in EL_n. The proof goes by induction on n. For $n = 1$, it is easy because any symplectic matrix that fixes $(1, 0)^T$ must be of the form

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = V_1(a).$$

For the induction step, suppose $n \geq 2$ and Ξ is a symplectic $2n \times 2n$ matrix fixing $x_0 \in \mathbb{R}^2 n$. Our goal is to find $\Xi_1, \ldots, \Xi_K \in EL_n$ such that $\Xi' = \Xi_K \ldots \Xi_1 \Xi$ is of the form:

$$\Xi' = \begin{pmatrix} * & 0 & * & 0 \\ 0 & 1 & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where each "$*$" is $(n-1)\times(n-1)$. In other words, if χ_j is the standard basis vector in $\mathbb{R}^{2n}$ with a 1 on the jth entry, we want the nth and $2n$th rows to be χ_n and χ_{2n} and the same for the columns. This is the behavior we would expect from a stubless-layerable network where the nth column had a single disconnected boundary vertex which was on both the upper and lower boundaries. Assuming we can obtain such a Ξ', we let Ξ'' be the matrix obtained by deleting the nth and $2n$th rows and columns. Then Ξ'' is a $2(n-1)\times 2(n-1)$ symplectic matrix that fixes $x_0 \in \mathbb{R}^{2n-2}$, so by the induction hypothesis, Ξ'' is the two-boundary map of a stubless-layerable network with $n-1$ columns. By adding another column with a single vertex, we obtain Ξ' as the two-boundary map of a stubless-layerable network. Hence, $\Xi' \in EL_n$ and $\Xi = \Xi_1^{-1}\Xi_2^{-1}\dots\Xi_K^{-1}\Xi'$ is in EL_n.

Thus, it suffices to show that for $n \geq 2$, we can obtain Ξ' from Ξ by multiplying by elements of EL_n. Our first goal is to find $\Xi_1,\dots,\Xi_m$ such that $\Xi_m\dots\Xi_1\Xi$ fixes χ_{2n}. This is the behavior we would expect from a stubless-layerable network where the nth column had only one vertex on both boundaries, but the vertex was not necessarily disconnected from the other columns. Let $x = \Xi\chi_{2n}$; it suffices to show that by multiplying by elements of EL_n we can transform x into χ_{2n}. There are several cases:

1. Suppose that $x_n \neq 0$ and that $x_{n+1},\dots,x_{2n-1} \neq 0$. Let
$$y = \left(\prod_{k=1}^{n-1} V_k(x_k/x_{n+k})\right) x.$$
Then $y_1,\dots,y_{n-1} = 0$, $y_n = x_n \neq 0$. Next, let
$$z = \left(\prod_{k=1}^{n-1} H_{k,n}(y_{n+k}/y_n)\right) y.$$
Then $z_1,\dots,z_{n-1} = 0$ and $z_{n+1},\dots,z_{2n-1} = 0$. But $\omega(x_0,z) = \omega(x_0,x) = 1$, so $z_{2n} = 1$. Thus, multiplying by $V_n(-z_n)$ will make the nth entry zero, yielding χ_{2n}.

2. If $x_n = 0$ but $x_{2n} \neq 0$ and $x_{n+1},\dots,x_{2n-1} \neq 0$, then we can multiply by $V_n(1)$ to make $x_n \neq 0$, then proceed to Case 1.

3. Suppose that some of "currents" $x_{n+1},\dots,x_{n+k}$ are zero, but the "potentials" $x_1,\dots,x_n$ are not all equal. If $x_{n+j} = 0$, we can find a k with $x_j \neq x_k$. Then multiply by $H_{j,k}(c)$ to make it nonzero; if $x_{n_k} \neq 0$, then we can choose c so that it will still be nonzero. Once we have done this for every j, proceed to Case 2.

4. Suppose that $x_1,\dots,x_n$ are all equal to c. Since $x_0 = (c,\dots,c,0,\dots,0)^T$ is fixed by Ξ and all matrices in EL_n, it is not possible that $x_{n+1},\dots,x_{2n}$ are all zero. Hence, we can multiply by some $V_k(1)$ to make the new $x_k \neq c$. Then proceed to Case 3.

Our next task is find $\Xi_{m+1} \dots \Xi_\ell$ such that $\Xi' = \Xi_\ell \dots \Xi_{m+1}\Xi^*$ fixes both χ_n and χ_{2n}. Let $x = \Xi\chi_n$, and consider the following cases:

1. Suppose that the "currents" $x_{n+1}, \dots, x_{2n}$ are all nonzero. Observe

$$x_n = \omega(x, \chi_{2n}) = \omega(\Xi^*\chi_n, \Xi^*\chi_{2n}) = \omega(\chi_n, \chi_{2n}) = 1.$$

   Let
$$y = \prod_{k=1}^{n-1} V_k(-x_k/x_{n+k}),$$
   so that $y_1, \dots, y_{n-1} = 0$ and $y_n = 1$. Then let
$$z = \prod_{k=1}^{n-1} H_{k,n}(y_{n+k}).$$
   Then $z_1 = y_1, \dots, z_n = y_n$, and $z_{n+1}, \dots, z_{2n-1} = 0$. But $\omega(x_0, z) = \omega(x_0, \chi_n) = 1$, so $z_{2n} = 0$. Hence,
$$z = \prod_{k=1}^{n-1} H_{k,n}(-y_{n+k}) \prod_{k=1}^{n-1} V_k(-x_k/x_{n+k})x = \chi_n.$$

2. If some of "currents" $x_{n+1}, \dots, x_{n+k}$ are zero, but the "potentials" $x_1, \dots, x_n$ are not all equal, we can multiply by $H_{j,k}$'s to make all the "currents" nonzero. Then proceed to Case 1.

3. Suppose that $x_1, \dots, x_n$ are all equal to 1. One of the "currents" must be nonzero; in fact, at least two of them are nonzero. Hence, we can multiply by $V_k(1)$ for some $k \neq n$ to make the new $x_k \neq 1$. Then proceed to Case 2.

In all these cases, we never multiplied by a $V_n(a)$ matrix. Thus, each $\Xi_{m+1}, \dots, \Xi_\ell$ fixes χ_{2n} and Ξ' fixes χ_{2n} as well as χ_n.

Because $(\Xi')^T\Omega\Xi' = \Omega$, we know $(\Xi')^T = \Omega^{-1}(\Xi')^{-1}\Omega$. Since Ξ' fixes χ_n and χ_{2n}, we know $(\Xi')^T$ fixes $\Omega^{-1}\chi_n = -\chi_{2n}$ and $\Omega^{-1}\chi_{2n} = \chi_n$. Thus, the nth and $2n$th columns of Ξ' are χ_n and χ_{2n}, and the n and $2n$th columns of $(\Xi')^T$ are χ_n and χ_{2n}. So Ξ' has the desired form. □

We can view the process in the above proof as a fancy form of row reduction using the symplectic matrices $H_{j,k}(a)$ and $V_k(a)$ instead of elementary matrices. We showed that any symplectic matrix fixing x_0 could be reduced to the identity multiplying by these "electrical elementary matrices." This provides a non-standard proof that the determinant of a symplectic matrix is 1; since each of the electrical elementary matrices has determinant 1, it is true for $\Xi \in EL_n$. But it is not hard to show that Ω and EL_n generate Sp_{2n}, and $\det \Omega = 1$.

Another corollary is that EL_n is a smooth manifold of dimension $n(2n-1)$. This can be proved using Lie theory, but we can also find explicit parametrizations in the same vein as Corollary 11.4. We sketch the process and leave the details to the reader. For each Ξ_0, construct a factorization as in the theorem. We parametrize a neighborhood of Ξ_0 in EL_n, taking as our parameters the conductance/resistance coefficients from Case 1 of each step. These are uniquely determined and depend smoothly on the entries of Ξ in a neighborhood of Ξ_0.

At the nth step of the induction, there were $n-1+n-1+1 = 2n-1$ edges in the first part (finding $\Xi_1, \ldots, \Xi_m$ from Ξ), and $n-1+n-1 = 2n-2$ edges in the second part (finding $\Xi_{m+1}, \ldots, \Xi_\ell$). That makes for $4n-3$ parameters in the nth induction step. And in the base case $n = 1$, there was $1 = 4-3$ edges. Summing over the induction steps gives the total number of parameters:

$$\sum_{j=1}^{n}(4j-3) = 4 \cdot \frac{1}{2}n(n+1) - 3n = 2n^2 + 2n - 3n = n(2n-1).$$

This is the same as $\dim EG_{2n}$, the number of parameters we would expect for a network with $2n$ boundary vertices.

The action of EL_n on EG_n is transitive, that is, for every $L_1, L_2 \in EG_n$, there is a $\Xi \in EL_n$ with $\Xi(L_1) = L_2$. Indeed, we saw in Theorem 11.1 that are $\Xi_1, \Xi_2 \in EL_n$ with $L_1 = \Xi_1(\mathbb{R}^n \times \{0\})$ and $L_2 = \Xi_2(\mathbb{R}^n \times \{0\})$. Hence, $L_2 = \Xi_2^{-1}\Xi_1(L_1)$. However, the action is not faithful: There exist nontrivial elements of EL_n which fix every element of EG_n. These elements are the kernel of the homomorphism Υ from EL_n to the group of bijections $EG_n \to EG_n$ given by $\Xi \mapsto F_\Xi$, where $F_\Xi : EG_n \to EG_n : L \mapsto \Xi(L)$. The reader can verify that the kernel consists of matrices of the form

$$\begin{pmatrix} I + \mathbf{1}\alpha^T & \mathbf{1}\beta^T + \beta\mathbf{1}^T \\ 0 & I - \alpha\mathbf{1}^T \end{pmatrix},$$

where $\mathbf{1}$ is the vector with every entry 1 and $\alpha, \beta \in \mathbb{R}^n$ with $\sum_{k=1}^n \alpha_k = 0$.

11.3 Generators of EL_n and Circular Planarity

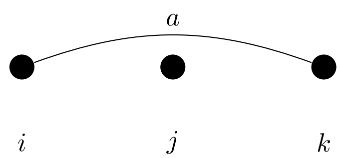
We defined EL_n with the generators $V_k(a)$ and $H_{j,k}(a)$ for $j \neq k$ and $a \in \mathbb{R}\backslash\{0\}$. However, it would have been sufficient to include only the $H_{j,k}(a)$'s with $k = j+1$ (which is in fact what Lam and Pylyavsky did):

Proposition 11.6. *EL_n is generated by elements of the form $V_k(a)$ and $H_{k,k+1}(a)$.*

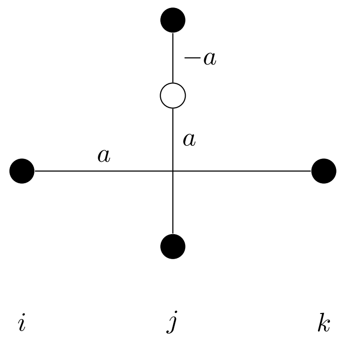
Proof. It suffices to show that $H_{j,k}(a)$ can be written as a product of elements of the form $V_m(a)$ and $H_{m,m+1}(a)$. To do this, we use the following identity:

$$H_{i,k}(a) = V_j(-1/a)H_{i,j}(-a)V_j(1/2a)H_{j,k}(2a)V_j(-1/4a)H_{i,j}(2a)V_j(1/2a)H_{j,k}(-a).$$

We begin with an elementary layer representing $H_{i,k}(a)$; for simplicity, I will show only the columns i, j, k; the conductance coefficient is printed next to the edge:

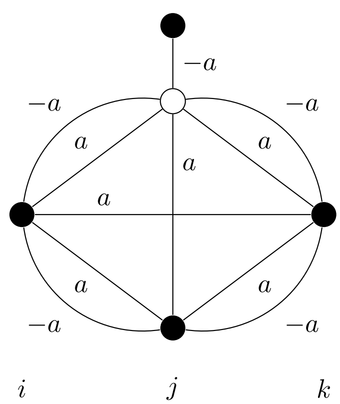


This is equivalent to $V_j(-\frac{1}{a})\,V_j(\frac{1}{a})\,H_{i,k}(a)$:



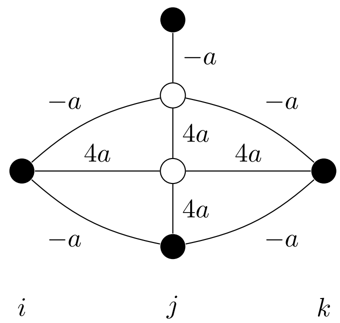
We insert cancelling horizontal edges to obtain

$$V_j(-\tfrac{1}{a})\,H_{i,j}(-a)\,H_{j,k}(-a)\,H_{i,j}(a)\,H_{j,k}(a)\,V_j(\tfrac{1}{a})\\ H_{i,k}(a)\,H_{i,j}(-a)\,H_{j,k}(-a)\,H_{i,j}(a)\,H_{j,k}(a).$$



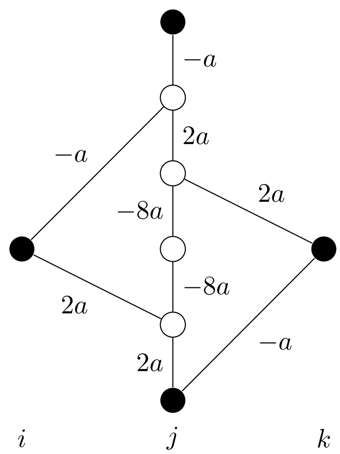
By a ★-$\mathcal{K}$ transformation, this is equivalent to

$$V_j(-\tfrac{1}{a})\,H_{i,j}(-a)\,H_{j,k}(-a)\,V_j(\tfrac{1}{4a})\,H_{i,j}(4a)\,H_{j,k}(4a)\,V_j(\tfrac{1}{4a})\,H_{i,j}(-a)\,H_{j,k}(-a).$$



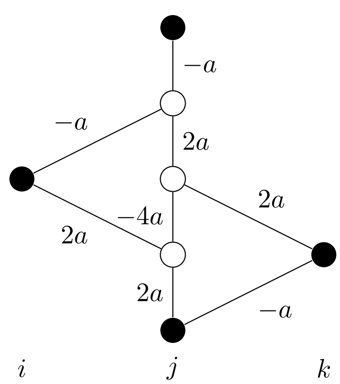
Although not strictly necessary, we simplify with two Y-Δ moves:

$$V_j(-\tfrac{1}{a})\,H_{i,j}(-a)\,V_j(\tfrac{1}{2a})\,H_{j,k}(2a)\,V_j(-\tfrac{1}{8a})\,V_j(-\tfrac{1}{8a})\,H_{i,j}(2a)\,V_j(\tfrac{1}{2a})\,H_{j,k}(-a)$$



and a series reduction to

$$V_j(-\tfrac{1}{a})\,H_{i,j}(-a)\,V_j(\tfrac{1}{2a})\,H_{j,k}(2a)\,V_j(-\tfrac{1}{4a})\,H_{i,j}(2a)\,V_j(\tfrac{1}{2a})\,H_{j,k}(-a)$$



Thus, for any j, k with $j < k-1$, we can write $H_{j,k}$ in terms of $H_{j,k-1}$'s, $H_{k-1,k}$'s, and V_{k-1}'s. Then proceeding inductively, we can write $H_{j,k-1}$ in terms of $H_{j,k-2}$'s, $H_{k-2,k-1}$'s, and V_{k-2}'s, and so on. Any $H_{j,k}$ can be expressed in terms elements of the form $H_{m,m+1}(a)$ and $V_m(a)$. □

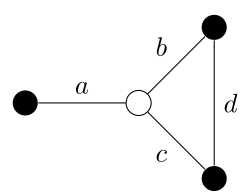
The significance is that if G is circular planar with the boundary vertices embedded in counterclockwise order, then adjoining a boundary edge between k and $k+1$ will preserve circular planarity. Thus, we have the following result in the spirit of [7]:

Corollary 11.7. *Over the signed linear conductances, every network is electrically equivalent to a circular planar network. Every $\Xi \in EL_n$ can be represented by circular planar stubless-layerable network.*

Proof. We already showed that any $L \in EG_n$ could be represented by a layerable network. The layerable network can be obtained from a network with n disconnected boundary vertices by adding boundary spikes and boundary edges. By the Proposition, we can find an equivalent sequence of boundary spike and boundary edge additions such that boundary edges are only added between adjacent columns. Since a network with n disconnected boundary vertices is circular planar, so is the network obtained by applying these operations.

Similarly, every stubless-layerable network is equivalent to a stubless-layerable network where horizontal edges only occur between adjacent columns, which is circular planar. □

However, not every network is equivalent to *critical* circular planar network. Consider the following network:



Suppose that $a + b + c = 0$ and $1/b + 1/c + 1/d = 0$. Then the network is both Dirichlet-singular and Neumann-singular. However, there does not exist a critical circular planar network, or indeed any network recoverable over positive linear conductances, which has three boundary vertices and is both Dirichlet- and Neumann-singular. To be Dirichlet-singular, it must have an interior vertex, and the interior vertex must have degree ≥ 3. Since any such network cannot have more than 3 edges, the only possibility is a Y. However, a Y cannot be Neumann-singular. This example also shows that not every network is equivalent to a network with $\leq \frac{1}{2}n(n-1)$ edges, as we might hope.

Admittedly, the construction in the above proposition is rather inefficient for finding a circular planar network equivalent to a given network, in the sense that it produces many extra edges, and these edges are difficult to remove by Y-Δ transformations. The final network also has no relationship to the original network since we discarded it and started instead with the representative of its boundary data L from Theorem 11.1. Thus, one goal for future research might be to find efficent ways of transforming a signed linear network into a circular planar network using local electrical equivalences.

References

[1] Edward B. Curtis and James A. Morrow. *Inverse Problems for Electrical Networks.* World Scientific. 2000.

[2] Robin Forman. "Determinants of Laplacians on Graphs." *Topology* Vol. 32, No. 1, pp. 35-46, 1993.

[3] David Ingerman. "Theory of Equivalent Networks and Some of its Applications."
`http://www.math.washington.edu/~reu/papers/1992/ingerman/ingerman.pdf`

[4] Will Johnson. "Circular Planar Resistor Networks with Nonlinear and Signed Conductors."
`http://arxiv.org/abs/1203.4045`

[5] Thomas Lam and Pavlo Pylyavkyy. "Electrical Networks and Lie Theory."
`http://arxiv.org/pdf/1103.3475v2.pdf`

[6] Jeffrey Russell. "★ and $\mathcal{K}$ Solve the Inverse Problem."
`http://www.math.washington.edu/~reu/papers/2003/russell/recovery.pdf`

[7] Konrad Schroder. "Mixed-Sign Conductor Networks."
`http://www.math.washington.edu/~reu/papers/1993/schroder/sign.pdf`