Annular Plate Networks

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Abstract

I consider electrical networks formed by conducting plates in an annular region of the plane. These networks are mathematically similar to electrical networks on a graph with vertices and edges which is embedded in an annulus. I describe how to remove lenses from the medial graph by Y-Δ transformation. By partitioning the network into subnetworks, I prove analogues of the cut-point lemma with corresponding algebraic statements. I prove recoverability for certain classes of networks.

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1 Introduction

1.1 Annular Plate Networks

Let S_{inner} and S_{outer} be the regions of the plane contained in piecewise-smooth simple closed curves C_{inner} and C_{outer} respectively, with $S_{\text{inner}} \subset S_{\text{outer}}$, and let $S = S_{\text{outer}} \setminus \overline{S_{\text{inner}}}$. Let plates $P_1, \ldots, P_N$ be compact, simply connected subsets of $\overline{S}$ bounded by piecewise smooth curves such that

- The intersection of two plates is either empty or a single point,
- The point of intersection between two plates does not lie on a boundary curve,
- $\mathcal{P}_I \cap C_{\text{inner}}$ is either a empty or an interval of C_{inner} and $\mathcal{P}_I \cap C_{\text{outer}}$ is either empty or an interval of C_{outer}.

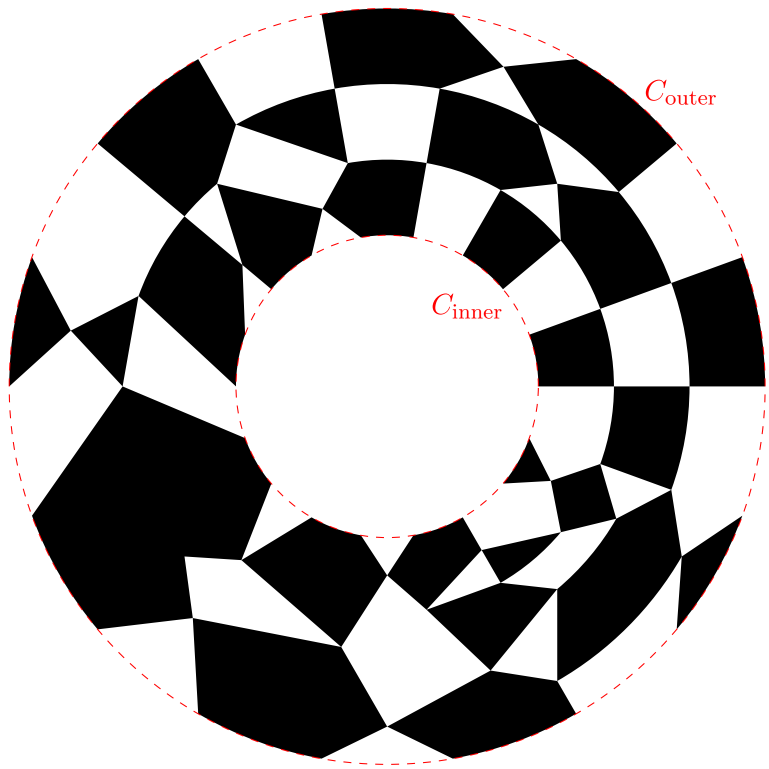
S is called the *region of embedding.* We will call it an annular region even though it is not technically an annulus, only homeomorphic to an annulus. C_{inner} and C_{outer} are called the *inner* and *outer boundary curves* respectively. S_{inner} is called the *hole.*

If two plates P and Q intersect at a point x, they are called adjacent, we write $P \sim Q$, and x is called a *juncture* between P and Q and is denoted PQ. A *directed juncture* $P \to Q$ is a juncture with specified order of the adjacent plates P and Q. The intersection of a plate and a boundary curve is called a *boundary interval.* If a plate intersects the boundary curve, we say it *touches* the boundary curve.

$\mathcal{P}$ is the collection of all plates, $\mathcal{J}$ is the collection of all directed junctures between adjacent plates, $\mathcal{I}$ is the collection of all boundary intervals. For each plate P, $\mathcal{I}(p)$ is the set of boundary intervals for P ($\mathcal{I}(P)$ contains either zero, one, or two elements). Let $\partial\mathcal{P}$ be the collection of *boundary plates* (plates touching any boundary curve), and let $\operatorname{int}\mathcal{P}$ be the collection of *interior plates* (plates not touching a boundary curve). Let $\mathcal{P}_{\text{inner}}$ and $\mathcal{P}_{\text{outer}}$ be the plates touching the inner and outer boundary respectively. A plate may be in both $\mathcal{P}_{\text{inner}}$ and $\mathcal{P}_{\text{outer}}$. Let $\mathcal{I}_{\text{inner}}$ and $\mathcal{I}_{\text{outer}}$ be the boundary intervals on the inner and outer boundaries respectively.

Together, S, $\mathcal{P}$, $\mathcal{J}$, and $\mathcal{I}$ form an *annular plate network* Γ. Of course, we can define plate networks on any planar region bounded by disjoint simple closed curves. In particular, we will sometimes need to consider networks in a simply connected region with one boundary curve. We will call them *circular planar* in accordance with standard terminology, even if the boundary curve is not strictly speaking a circle. For circular planar networks, it is reasonable

Figure 1: An annular plate network.



to assume each plate only intersects the boundary in *one* interval, although often we will allow two boundary intervals.

1.2 Conductivity Functions, Feasible Boundary Data, and Recovery

A *conductivity function* is a function γ which assigns a positive number, called the *conductance*, to each juncture, such that $\gamma(PQ) = \gamma(QP)$. We may think of γ as defined on undirected or directed junctures. If P and Q are not adjacent, we say $\gamma(PQ) = 0$.

A *voltage function* is a function $v : P \to \mathbb{R}$; the value assigned to each plate is a *voltage*. A *current function* is a function $c : \mathcal{J} \cup \mathcal{I} \to \mathbb{R}$. An *electrical function* $f = (v, c)$ consists of a voltage and a current function such that for every juncture $P \to Q$,

$$c(P \to Q) = \gamma(PQ)(v(P) - v(Q)).$$

An electrical function is called γ-*harmonic* if for every plate P,

$$\sum_{Q \sim P} c(Q \to P) + \sum_{I \in \mathcal{I}(P)} c(I) = 0.$$

For a boundary interval, positive current means current flowing into the plate P. If f_1 and f_2 are γ-harmonic, then so is $af_1 + bf_2$ for any $a, b \in \mathbb{R}$.

Suppose $\partial\mathcal{P} = \{P_1, \ldots, P_K\}$ and $\mathcal{I} = I_1, \ldots, I_L$. A vector $\mathbf{x} \in \mathbb{R}^{K+L}$ is called *feasible boundary data* if there exists a γ-harmonic function with $v(P_k) = x_k$ for each k and $c(I_\ell) = x_{K+\ell}$ for each ℓ. The space of all vectors which are feasible boundary data will be denoted F.

Suppose we are given a network Γ with a fixed conductivity function γ. We do not know γ, and we want to determine γ knowing only F. If γ can be determined from F, then γ is called *recoverable.* If *any* conductivity function γ can be recovered, we say the network Γ is recoverable. The problem of recovering the conductances of Γ is called the *inverse problem.*

1.3 Paths and Connections

A *path* is a sequence of plates $P_1, \ldots, P_N$ such that $P_n \sim P_{n+1}$; it may equivalently be viewed as a sequence of junctures. A 1-connection α between two boundary cells q_1 and q_2 is a path $P_1, \ldots, P_N$ with $P_1 = q_1$, $P_N = q_2$, and all other P_n interior cells, such that $P_i \neq P_j$ for all $i \neq j$. A path with one boundary cell q is considered a connection from q to q. A k-connection is a set of k disjoint 1-connections.

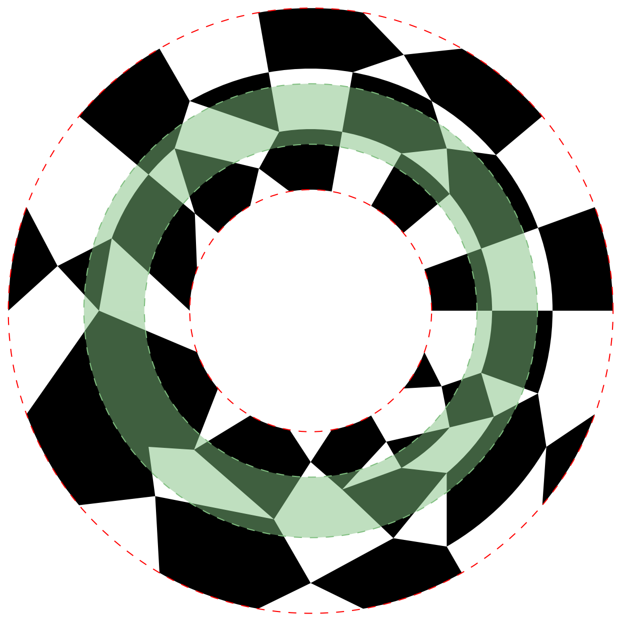
A network is connected if there exists a path between any two plates. Here we do *not* assume the network is connected. However, we assume that for every plate P, there exists a path from P to a boundary plate.

Suppose $\mathcal{U}$ and $\mathcal{V}$ are subsets of $\partial\mathcal{P}$. Let $M(\mathcal{U}, \mathcal{V})$ signify the largest k such that there is a k-connection between $\mathcal{U}$ and $\mathcal{V}$.

1.4 Subnetworks

Suppose U is an open proper subset of the region of embedding, such that ∂U consists of one or two nonintersecting, piecewise-smooth simple closed curves. Suppose no junctures of Γ lie on ∂U. Suppose that each plate P of Γ is either contained in U, it is contained in $\overline{S} \setminus \overline{U}$, or ∂U divides P into two

Figure 2: A layer of Γ.



or more smaller regions (called *subplates*). Suppose that for each subplate Q, $\partial U \cap Q$ is either empty or an interval of Q.

Then we can form a plate network Γ' with region of embedding U, whose plates are the plates and subplates of Γ contained in $\overline{U}$. Γ' is called a *subnetwork* of Γ. A *layer* of Γ is subnetwork Γ' such that

- the region of embedding U is bounded by two curves;
- $C_{\text{inner}}(\Gamma)$ lies inside $C_{\text{inner}}(\Gamma')$;
- either $C_{\text{inner}}(\Gamma')$ and $C_{\text{inner}}(\Gamma)$ are the same or they do not intersect;
- either $C_{\text{outer}}(\Gamma')$ and $C_{\text{outer}}(\Gamma)$ are the same or they do not intersect.

A *partition of* Γ *into subnetworks* is a collection of subnetworks $\Gamma_1, \ldots, \Gamma_K$ with regions of embedding $S_1, \ldots, S_K$, such that $S_i \cap S_j = \emptyset$ for $i \neq j$ and $\bigcup \overline{S_k} = \overline{S}$.

Theorem 1.1. *Let* $\Gamma_1, \ldots, \Gamma_K$ *be a parition of* Γ *with feasible data sets* $F_1, \ldots, F_K$. F *of* Γ *can be determined from* $F_1, \ldots, F_K$.

Proof. Any partition can be expressed in terms of partitions and subpartitions into two parts. Thus, by induction, it suffices to consider the case where $K = 2$.

Suppose Γ is partitioned into Γ_1 and Γ_2. Let C be the curve or union of curves $\partial S_1 \cap \partial S_2$. Let $P_1, \ldots, P_N$ be the plates of Γ_1 touching C and let $Q_1, \ldots, Q_N$ be the plates of Γ_2 touching C. Let $I_1, \ldots, I_M$ and $J_1, \ldots, J_M$ be the sets of boundary intervals of Γ_1 and Γ_2 which are subsets of C. (If P_n and Q_n form a boundary plate of Γ, then their boundary intervals in Γ_1 and Γ_2 are not included in $\{I_m\}$ or $\{J_m\}$.)

Let vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ represent boundary data on Γ_1 and Γ_2. Let $g_1(\mathbf{x}_1)$ be the restriction of $\mathbf{x}_1$ to $P_1, \ldots, P_N$ and $I_1, \ldots, I_M$ (We can think of restriction as a function). Define $g_2(\mathbf{x}_2)$ be function which restricts $\mathbf{x}_2$ to $Q_1, \ldots, Q_N$ and $J_1, \ldots, J_M$ and in addition changes the signs of the *current* entries. Define $h : F_1 \times F_2 \to \mathbb{R}^{N+M}$ by

$$h(\mathbf{x}_1, \mathbf{x}_2) = g_1(\mathbf{x}_1) - g_2(\mathbf{x}_2).$$

If $h(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{0}$, the voltage on P_n is equal to the voltage on Q_n. If P_n and Q_n form an interior plate of Γ, then the net current on the plate $P_n \cup Q_n$ is zero. We assumed that $\mathbf{x}_1$ and $\mathbf{x}_2$ were boundary data for some γ-harmonic functions f_1 and f_2 on Γ_1 and Γ_2. We can combine f_1 and f_2 into a γ-harmonic function f on Γ. Of course, if $h(\mathbf{x}_1, \mathbf{x}_2) \neq \mathbf{0}$, we cannot find such a γ-harmonic function on Γ.

We can compute the boundary data for f from $\mathbf{x}_1$ and $\mathbf{x}_2$. For plates other than P_n and Q_n, this is obvious. Suppose P_n and Q_n form a boundary plate of Γ with boundary interval I. It is obvious how to find the voltage of $P_n \cup Q_n$. The current on I can be found from the boundary currents on P_n and Q_n. Hence, we can define a function w "restricting" $\mathbf{x}_1$ and $\mathbf{x}_2$ to the boundary of Γ. Then $F = w(h^{-1}(\mathbf{0}))$. □

Theorem 1.2. *If a subnetwork of* Γ *is not recoverable, then* Γ *is not recoverable.*

Proof. Suppose Γ has a nonrecoverable subnetwork. Let $\Gamma_1, \ldots, \Gamma_K$ be a partition of Γ where Γ_1 is not recoverable. Let g be a function mapping the sets $F_1, \ldots, F_K$ to the set F; this function exists by the previous theorem. Let L be the function mapping γ to F. Γ is recoverable if and only if L is injective. Let γ_k be the restriction of γ to the junctures of Γ_k, and let $L_k(\gamma_k)$ be the function mapping γ_k to F_k. Then

$$L(\gamma) = g(L_1(\gamma_1), \ldots, L_K(\gamma_K)).$$

Since Γ_1 is not recoverable, we know L_1 is not injective, and so L is not injective either. $\square$

1.5 Electrical Similarity and Equivalence

Two networks Γ and Γ' with conductivity functions γ and γ' are *electrically similar* if there is a one-to-one correspondence M mapping each boundary plate or boundary interval of Γ to a boundary plate or boundary interval of Γ' such that

- P is an inner/outer boundary plate of Γ if and only if $M(P)$ is an inner/outer boundary plate of Γ';
- I is an inner/outer boundary interval of Γ if and only if $M(I)$ is an inner/outer boundary interval of Γ';
- I is a boundary interval of P if and only if $M(I)$ is a boundary interval of $M(P)$;
- M preserves the counterclockwise ordering of the inner/outer boundary plates;
- The set F is the same for Γ, γ and Γ', γ'.

They are called *electrically equivalent* if in addition

- Γ and Γ' have the same region of embedding.
- For each boundary interval I, $M(I)$ and I are the same curve.

The definitions of electrical similarity and equivalence for circular planar networks are similar except that there is only one boundary curve.

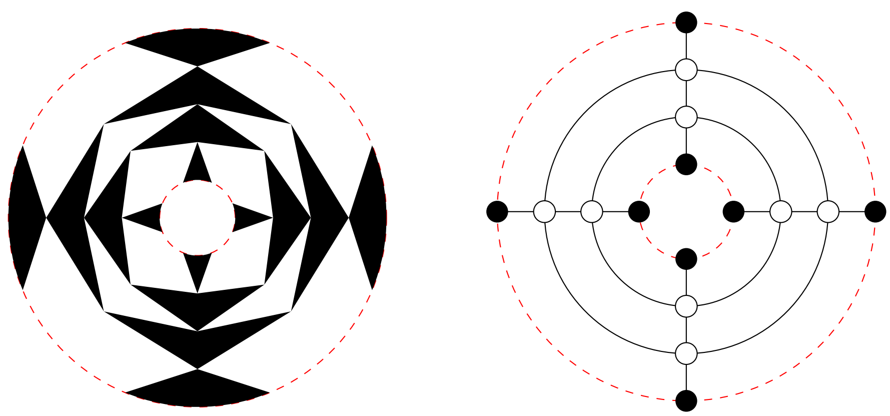
Electrical similarity and equivalence are transitive. As a consequence of Theorem 1.1,

Theorem 1.3. *Let $\Gamma_1, \ldots, \Gamma_K$ and $\Gamma'_1, \ldots, \Gamma'_K$ be a partitions of Γ and Γ' into subnetworks. If Γ_k and Γ'_k are electrically equivalent for each k, then Γ and Γ' are electrically equivalent.*

1.6 Comparison to Graph-Based Networks

The plate-based networks described here are similar to the graph-based electrical networks discussed by [1] and others. In a planar vertex-based network (and in particular, an annular planar network), we can construct the medial

Figure 3: A plate network and equivalent graph-based network.



graph and color the cells white and black, such that each black cell contains a vertex of the primal graph. The plates described here correspond to the black cells.

There are important differences between the two constructions:

- A plate can touch both boundary curves, but a vertex must lie on one or the other.
- Unlike a vertex, a plate can have multiple boundary currents.
- We discuss trivial connections: A plate is considered to be connected to itself.
- We do not make the usual assumption that the network is connected, only that each plate has a path to the boundary.

With these changes, we will be able to consider subnetworks which would be too small to make sense in the graph-based system.

Graph-based and plate-based networks are algebraically equivalent when a plate does not touch both boundaries. Thus, many of the results about graph-based networks for [1] and others carry over to plate-based networks.

For instance, we know that the *Dirichlet problem* has a nearly unique solution. That is, for given voltages on $\partial\mathcal{P}$, the voltages on the network are uniquely determined. Boundary currents for each plate are uniquely determined except when the plate touches both boundary curves, in which case, the sum of its two boundary currents is uniquely determined. Similarly, the

Neumann problem has a unique solution for plate networks, that is, for every set of boundary currents which sum to zero on each connected component of the network, there is a γ-harmonic function with those boundary currents, which is unique up to an additive constant.

As with graph-based networks, we can discuss the *Kirchhoff matrix*: if each plate is assigned an index, then we define a $|\mathcal{P}| \times |\mathcal{P}|$ matrix K by

$$\kappa_{PQ} = \begin{cases} -\gamma(PQ), & P \neq Q \\ \sum_{R \sim P} \gamma(PR), & P = Q. \end{cases}$$

The *response matrix* Λ is the Dirichlet-to-Neumann map, that is, if ϕ is a vector representing boundary voltages, then $\Lambda\phi$ is a vector representing (sums of) boundary currents for the γ-harmonic function with boundary voltages ϕ. Λ is given by a Schur complement of K. The set F is equivalent information to Λ

Minors of Λ are related to connections in the graph by the *determinant-connection formula* (Lemma 3.12 of [1]). If $\mathcal{U}$ and $\mathcal{V}$ are disjoint sets of boundary plates $P_1, \ldots, P_k$ and $Q_1, \ldots, Q_k$, and α is a k-connection between $\mathcal{U}$ and $\mathcal{V}$, then τ_α is the permutation of the symmetric group S_k such that a 1-connection in α connects P_n and $Q_{\tau(n)}$ for each n. We define $\mathcal{W}_\alpha$ as the collection of plates which are not used in any 1-connection of α and let $D_\alpha = \det K(\mathcal{W}_\alpha; \mathcal{W}_\alpha)$. The determinant-connection formula says that

$$\det \Lambda(\mathcal{U}; \mathcal{V}) \cdot \det K(\operatorname{int} \mathcal{P}; \operatorname{int} \mathcal{P}) = (-1)^k \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \sum_{\substack{\alpha \\ \tau_\alpha = \tau}} \prod_{PQ \in \mathcal{J}_\alpha} \gamma(PQ) D_\alpha,$$

where the second sum is taken over k-connections which exist between $\mathcal{U}$ and $\mathcal{V}$.

2 Geodesics and Network Modifications

2.1 Geodesics and Lenses

Suppose ∂P is the boundary of a plate P and T is the union of the boundary intervals. The junctures of P divide $\partial P \setminus T$ into smaller curves, called *edges*. At a juncture y, four edges meet (two edges from each of two plates). If these edges are e_A, e_B, e_C, and e_D in counterclockwise order about y, then e_A is opposite e_C and e_B opposite e_D. Two edges are *adjacent* if they share a juncture.

Suppose that $e_1, e_2, \ldots, e_K$ is a sequence of edges such that $e_i \neq e_j$ for all $i \neq j$ and e_k is opposite e_{k+1} for all k. If either

Figure 4: Some geodesics.

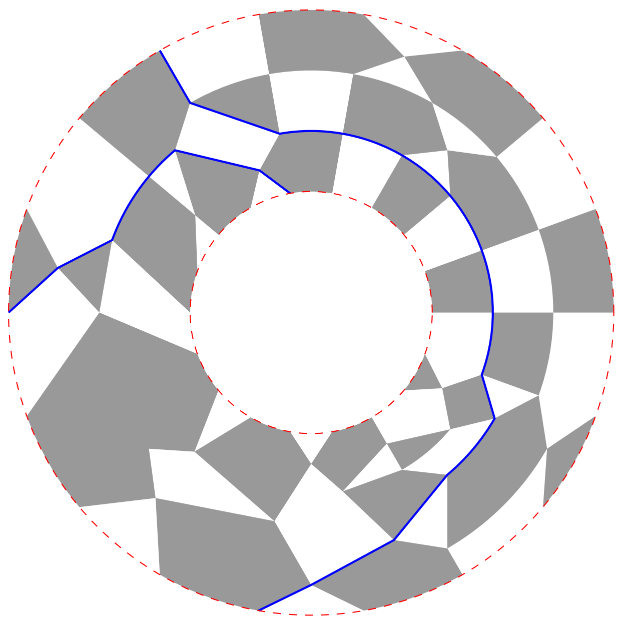
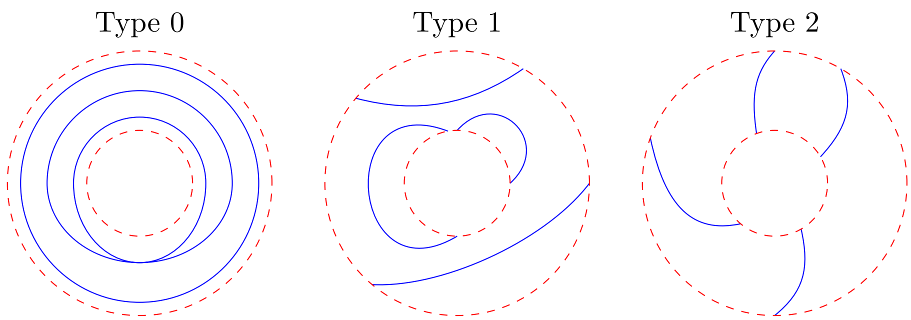


Figure 5: Types of geodesics.



- e_1 touches a boundary curve and e_K touches a boundary curve, or
- e_1 is opposite e_K,

then $e_1 \cup e_2 \cup \cdots \cup e_K$ is called a *geodesic*.

The intersection between two geodesics is also called a *crossing*. There are three types of geodesics:

- Type 0 geodesics are self-loops with no endpoints on a boundary curve.
- Type 1 geodesics have both endpoints on one boundary curve.
- Type 2 geodesics have one endpoint on each boundary curve.

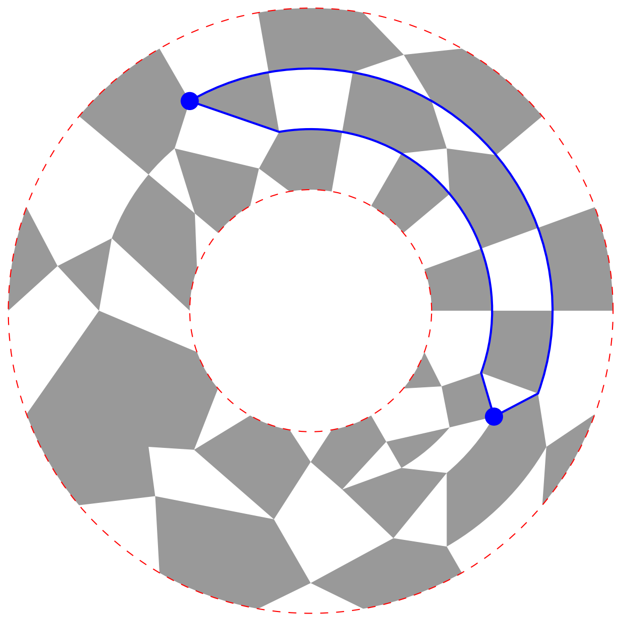
$\mathcal{G}_0$, $\mathcal{G}_1$, and $\mathcal{G}_2$ signify the collections of type 0, type 1, or type 2 geodesics. Type 1 geodesics are further divided into type inner ($\mathcal{G}_{\text{inner}}$) and type 1 outer ($\mathcal{G}_{\text{outer}}$) according to where their endpoints are.

When parametrizing a geodesic, we assume the following: A type 1 or type 2 geodesic g can be parametrized by a continuous function $\phi : [0,1] \to \mathbb{C}$. Assume $\phi(0)$ and $\phi(1)$ are the endpoints of g and ϕ is injective except at self-intersections of g. When parametrizing a type 0 geodesic, $\phi(0) = \phi(1)$ and ϕ is injective on $[0,1)$ except at self-intersections of g.

Suppose $e_1, \ldots, e_K$ is a sequence of edges with $e_i \neq e_j$ for each $i \neq j$, e_k and e_{k+1} are adjacent for each k, and e_1 is adjacent to e_K. The curve $e_1 \cup \cdots \cup e_K$ is called a *lens* if one of the following conditions is satisfied:

- In a *zero-pole lens*, each e_k is opposite e_{k+1} and e_1 is opposite e_k.
- In a *one-pole lens*, each e_k is opposite e_{k+1}, but e_1 and e_k are not opposite. The juncture between e_1 and e_K is the *pole* of the lens.

Figure 6: A two-pole lens.



- In a *two-pole lens*, each e_k is opposite e_{k+1} for $k \neq J$; e_1, e_K and e_J, e_{J+1} are not opposite. The junctures between e_1, e_K and e_J, e_{J+1} are the poles.

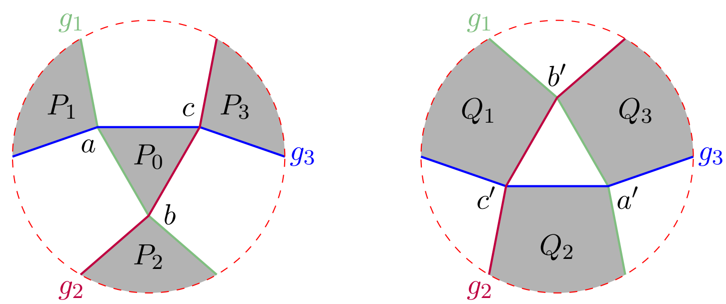
A lens is a closed curve formed by arcs of one or two geodesics.

2.2 Y-Δ Transformations

A *wye* (or Y) is a circular planar network with four plates, P_0, P_1, P_2, P_3 such that the boundary plates are P_1, P_2, and P_3, and the junctures are P_0P_1, P_0P_2, and P_0P_3. A *delta* (or Δ) is a circular planar network with three plates Q_1, Q_2, Q_3; all the plates are boundary plates, and all the plates are adjacent.

Given a wye, it is always possible to find an electrically equivalent delta and vice versa. Suppose that in the wye, $\gamma(P_0P_1) = a$, $\gamma(P_0P_2) = b$, and $\gamma(P_0P_3) = c$, and that in the delta, $\gamma(Q_2Q_3) = a'$, $\gamma(Q_1Q_3) = b'$, and $\gamma(Q_1Q_2) = c'$. Then the wye and the delta are electrically equivalent (with

Figure 7: A wye and a delta.



$M(P_j) = Q_j$ for $j = 1, 2, 3$) if and only if

$$a' = \frac{bc}{a+b+c}, \quad b' = \frac{ac}{a+b+c}, \quad c' = \frac{ab}{a+b+c}$$

if and only if

$$a = \frac{a'b' + b'c' + a'c'}{a'}, \quad b = \frac{a'b' + b'c' + a'c'}{b'}, \quad c = \frac{a'b' + b'c' + a'c'}{c'}.$$

Suppose Γ is a network with a subnetwork Σ which is a wye. Let Γ' be a network obtained from Γ by replacing Σ with a delta Σ'. The modifications changing Γ to Γ' and Γ' to Γ are called Y-Δ *transformations*.

A Y-Δ transformation may produce a network which does not fit our original definition of a plate network because multiple junctures join the same two plates or a plate has a self-juncture. For the purposes of this section, we extend our definition to allow such networks.

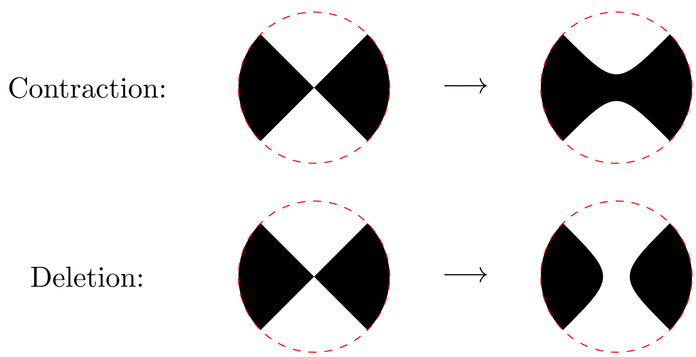
Two networks are Y-Δ-*equivalent* if one can be transformed into the other by Y-Δ transformations. If Γ and Γ' are Y-Δ equivalent and we are given the conductivity function γ, we can compute γ'. As a result, Γ is recoverable if and only if Γ' is recoverable.

A Y-Δ transformation alters the geodesics by changing the order in which they intersect one another. In a Y or Δ subnetwork, three geodesics meet; call them g_1, g_2, g_3. A Y-Δ transformation moves the crossing of g_1 and g_2 to the other side of g_3.

2.3 Juncture Removals and Trivial Modifications

A *single-juncture network* is a circular planar (sub)network with two plates, and one juncture, and one boundary interval on each juncture.

Figure 8: Juncture removals.



A *juncture deletion* removes a juncture from the network by replacing a single-juncture subnetwork with a network with two plates, one boundary interval on each plate, and *no juncture.* A *juncture contraction* replaces a single-juncture subnetwork with a network with one plate which has two boundary intervals and no junctures. Both these transformations are called *juncture removals.*

If two geodesics meet a juncture, then a juncture removal *uncrosses* them. If the geodesics g_1 and g_2 in the original network had endpoints x_1 and y_1, x_2 and y_2 respectively, then the geodesics in the modified network have endpoints x_1 and y_2, x_2 and y_1.

If a plate has a self-juncture, no current can ever flow across the juncture. Thus, changing the conductance of the juncture will not affect F, so the network is not recoverable. Deleting the juncture while keeping all other conductances the same will produce an electrically equivalent network.

An interior plate with only one juncture is called an *interior spike.* No current can flow across the juncture, so the network is not recoverable, and contracting this juncture (or *contracting the spike*) will produce an electrically equivalent network.

A *parallel network* is circular planar network in which there are two plates, both of which are boundary plates, and two junctures between the plates. A parallel network with conductances a and b is electrically equivalent to a network with only one juncture, with conductance $a + b$. A parallel network is not recoverable because any conductances a' and b' with $a' + b' = a + b$ will produce the same F.

A *series network* is a circular planar network with three plates, P_0, P_1, and P_2, where P_1 and P_2 are boundary plates, and there are two junctures P_0P_1 and P_0P_2. A series network with conductances a and b is electrically equivalent to a network with only two plates and one juncture, with conductance $ab/(a+b)$. A parallel network is not recoverable because any conductances a' and b' with $a'b'/(a'+b') = ab/(a+b)$ produce the same F.

Trivial modifications are the following network transformations:

- Deleting a self-juncture.
- Contracting an interior spike.
- Replacing a parallel with a single-juncture subnetwork.
- Replacing a series with a single-juncture subnetwork.

Self-junctures and interior spikes correspond to empty one-pole lenses. Parallel and series connections correspond to empty two-pole lenses.[1] A trivial modification removes the lens.

A lens is called *removable* if it can removed from the network by Y-Δ transformations and trivial modifications. Any network on which we can perform a trivial modification is unrecoverable (in fact, the inverse problem has infinitely many solutions). Since Y-Δ transformations preserve recoverability properties, we know that any network with removable lenses is unrecoverable.

2.4 Lens Removal I

Because removable lenses make a network unrecoverable, we want to determine what kinds of lenses are removable. We begin with the easiest case. A lens is called *simply connected* if it is contained within some simply connected subset of the region of embedding.

Theorem 2.1. *Every simply connected lens is removable.*

Proof. A simply connected lens is contained within some subnetwork in a simply connected region. Theorem 8.3 of [1] shows that all lenses can be removed from a circular planar network. The proof is similar but simpler than the later lens removal arguments of this paper. □

[1] An empty zero-pole lens would either be an interior plate with no junctures or a junctureless hole in a plate. We assume that these configurations do not exist in the original network, and we know they cannot be produced by Y-Δ transformations.

Figure 9: Trivial modifications.

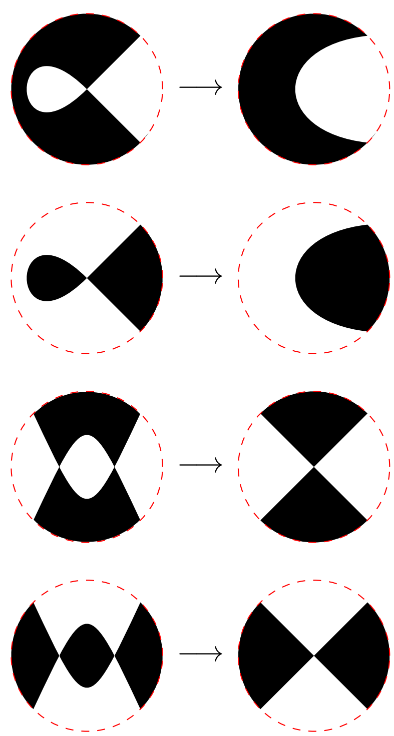
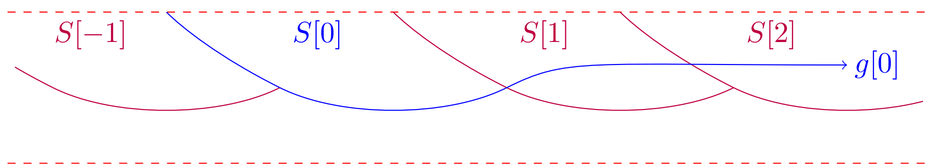


Figure 10: Proof of Lemma 2.2.



Lemma 2.2. *A self-intersecting type 2 geodesic forms a simply connected lens.*

Proof. We will use the universal cover of the annulus, which is a strip extending infinitely to the left (clockwise) and right (counterclockwise). For a geodesic g or region R in the annulus, we use choose one "copy" in the universal cover to be $g[0]$ or $R[0]$, and we index the other "copies" from left to right by the integers as $g[n]$ or $R[n]$.

Suppose g is a self-intersecting type 2 geodesic with parametrization ϕ such that $\phi(0)$ is on the inner boundary and $\phi(1)$ is on the outer boundary. Let ϕ_0 be the corresponding parametrization of $g[0]$ in the universal cover. Choose the smallest t_1 such that $\phi([0, t_1])$ intersects itself and the $t_0 \in (0, t_1)$ such that $\phi(t_0) = \phi(t_1)$. Let $A = \phi([0, t_0])$ and $B = \phi([t_0, t_1])$. Let $A[0]$ and $B[0]$ be the corresponding arcs of $g[0]$ in the universal cover.

If the self-loop $\phi([t_0, t_1])$ does not encircle the hole, it is a simply connected lens. Suppose $\phi([t_0, t_1])$ encircles the hole, and assume without loss of generality that it is counterclockwise. In the universal cover, $A[0]$, $B[0]$, and $A[1]$ together with the inner boundary enclose a region $S[0]$. For small positive ϵ, $\phi_0(t_1 + \epsilon) \in S[1]$. To reach the outer boundary, ϕ_0 must exit $S[1]$, but it will form a simply connected lens unless it exits along $A[2]$, entering $S[2]$. Continuing inductively, we see ϕ_0 must enter $S[n]$ for all positive n, which is a contradiction. □

Lemma 2.3. *If a type 1 inner and a type 1 outer geodesic intersect, they form a simply connected lens.*

Proof. Suppose $g_1 \in \mathcal{G}_{\text{inner}}$ with parametrization ϕ and $g_2 \in \mathcal{G}_{\text{outer}}$ with parametrization ψ intersect. Let t^* be the first time $\psi(t)$ intersects g_1 and let u^* be the value such that $\phi(u^*) = \psi(t^*)$.

Since $\psi([0, t^*))$ does not intersect g_2, we can assume without affecting the lenses formed by g_1 and g_2 that $\psi([0, t^*])$ does not intersect itself. If

$\phi([0, u^*])$ intersects itself, we can join $\phi([0, u^*])$ and $\phi([0, t^*])$ into a self-intersecting curve from the inner boundary to the outer boundary, which must have a simply connected lens by the previous lemma. No pole of this lens can lie on $\psi([0, t_0])$ by assumption, so g_1 forms a simply connected lens. Therefore, suppose $\phi([0, u^*])$ does not intersect itself.

If g_1 is not self-intersecting, the proof is easy, so assume g_1 is self-intersecting. Choose the smallest u_1 such that $\phi([0, u_1])$ is self-intersecting. Assume the loop winds counterclockwise; the other case is similar. Let u_0 be the number in $[0, u_0)$ with $\phi(u_0) = \phi(u_1)$. Let $A = \phi_1([0, u_0])$ and $B = \phi_2([u_0, u_1])$. In the universal cover, $A[0]$, $B[0]$, and $A[1]$ enclose a region $S[0]$. For small positive ϵ, $\psi_0(t^* + \epsilon)$. As in the previous lemma, ψ_0 must eventually hit the outer boundary, but cannot exit $S[n]$ without forming a simply connected lens except by passing across $A[n+1]$ into $S[n+1]$. □

Lemma 2.4. *If two type 2 geodesics g and h intersect without forming a simply connected lens, then g always crosses h in the same direction (always counterclockwise or always clockwise).*

Proof. Suppose $g[0]$ crosses $h[0]$ counterclockwise; the other case is similar. Let $S[n]$ be the fundamental domain between $h[n]$ and $h[n+1]$. After entering $S[0]$, $g[0]$ cannot exit along $h[0]$ without forming a simply connected lens. Thus, it must either reach the outer boundary from $S[0]$ or enter $S[1]$. By the same argument, if $g[0]$ enters $S[n]$, it cannot exit clockwise across $h[n]$. Eventually, $g[0]$ reaches the outer boundary, and it has never crossed any $h[n]$ clockwise. □

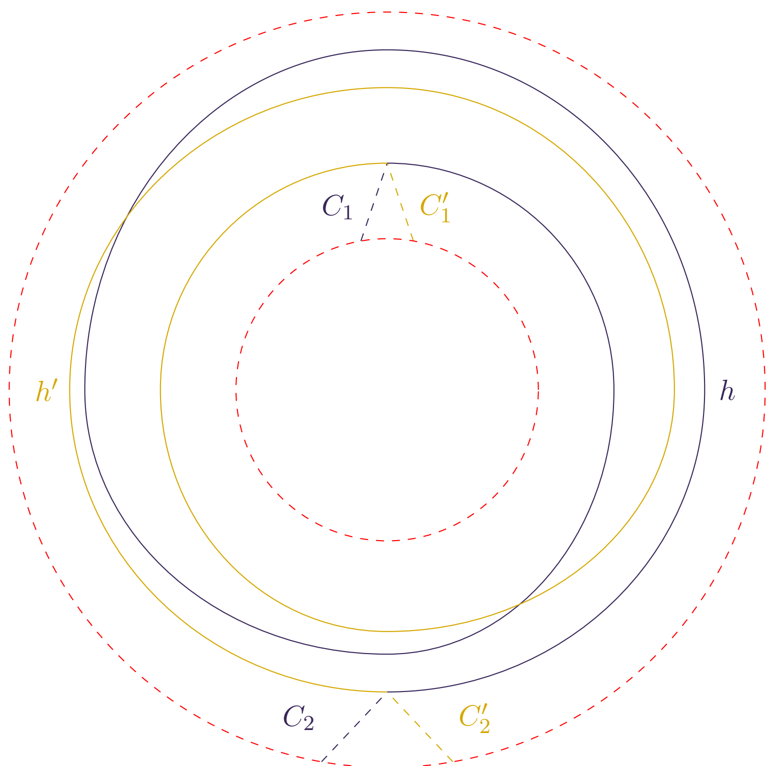
Definition 2.5. Let $i(g)$ be the number of self-intersections of a geodesic or curve g. Let $i(g_1, g_2)$ for $g_1 \neq g_2$ be the number of intersections between g_1 and g_2.

Definition 2.6. A type 0 geodesic can be parametrized in two directions and has a well-defined winding number around the hole for each one. Let $w(g)$ be the nonnegative winding number.

Lemma 2.7. *If a type 0 geodesic g does not form a simply connected lens, then $w(g) = i(g) + 1$.*

Proof. Let x be a point on g such that two curves C_1 and C_1' connect the inner boundary to x without intersecting g or each other except at x. Let $y \neq x$ be a point on g such that two curves C_2 and C_2' connect the outer boundary to x without intersecting g or each other except at y. Parametrize g by a function ϕ with $\phi(0) = \phi(1) = x$. Let $t_0 = \phi^{-1}(y)$ and let $h =$

Figure 11: Proof of Lemma 2.7.



$\phi([0, t_0])$ and $h' = \phi([t_0, 1])$. C_1, C'_1, h, and h' meet at x. Assume without loss of generality that C_1 is opposite h and C'_1 is opposite h' at x. Assume also that at y, C_2 is opposite h and C'_2 is opposite h'.

Let $C = C_1 \cup h \cup C_2$ and $C' = C'_1 \cup h' \cup C'_2$. If C and C' form a simply connected lens, then so does g. This is obvious if the poles of the lens are not x or y. If x or y is the pole of a lens of C and C' and the other pole is not y or x, then there is a simply connected one-pole lens in g. If x and y are the poles of a simply connected lens, then g forms a simply connected zero-pole lens.

Suppose g does not form a simply connected lens; then neither do C and C'. By Lemma 2.2, C and C' do not have self-intersections, so $i(g) = i(C, C') - 2$. By Lemma 2.4, one of C and C' always crosses the other counterclockwise, so $w(g) = i(C, C') - 1$. □

Before removing other types of lenses, we must discuss empty boundary triangles and stubs.

2.5 Empty Boundary Triangles and Stubs

An *empty boundary triangle* is a triangle formed by two edges of plates and an interval of a boundary curve. For an empty boundary triangle, there are two possibilities:

1. The triangle is a plate. In this case, the plate is called a *boundary spike*.
2. The triangle is adjacent to two plates. Then, the juncture at the vertex of the triangle is called a *boundary juncture*.

A type 1 geodesic is called *empty* if it does not intersect any other geodesics. A *stub* is a boundary plate which touches only one boudary curve and is not adjacent to any other plates; its edge is an empty geodesic. Since we assumed that a plate intersects the each boundary curve in at most one interval, there cannot be an empty geodesic which is not the edge of a stub.

Here we prove the existence of an empty boundary triangles or stubs in certain families of geodesics, which is an essential step for the rest of this paper's arguments.

Lemma 2.8. *Suppose g be a type 1 inner geodesic which does not form a simply connected lens. Suppose that at a juncture point y, two opposite edges e_1 and e_2 are in g and another edge e_3 touches the inner boundary. Then e_3 forms a triangle with some arc of g and some arc of the inner boundary.*

Proof. The juncture point y splits g into two segments (possibly intersecting). Choose a curve C which begins at the outer boundary and ends at a non-juncture point z on g such that C only intersects g once. Parametrize g by ϕ such that $\phi(t_0) = y$ and $\phi(t_1) = z$ with $t_0 < t_1$. Then $C \cup \phi([0, t_1])$ forms a curve from the inner boundary to the outer boundary. By Lemma 2.2, the curve cannot intersect itself without forming a simply connected lens, but we know no pole of such a lens can lie on C. Hence, $\phi([0, t_1])$ does not intersect itself, so $\phi([0, t_0])$ and e_3 form a triangle with some arc of the boundary. □

Definition 2.9. A family of geodesics $\mathcal{F}$ is connected if

- For any two points x on $g \in \mathcal{F}$ and y on $h \in \mathcal{F}$, there is a path from x to y along arcs of geodesics in $\mathcal{F}$.

- If g and h intersect and $g \in \mathcal{F}$, then $h \in \mathcal{F}$.

Theorem 2.10. *Suppose $\mathcal{F}$ is a family of type 1 inner and type 2 geodesics with no simply connected lenses and at least one intersection or at least one type 1 inner geodesic. There is an empty boundary triangle or empty geodesic on the inner boundary.*

Proof. First, consider the case where the family of geodesics is connected.

Let g_0 be a geodesic which intersects some other geodesic. By hypothesis, g_0 has one endpoint x_0 on the inner boundary. Orient g_0 so that the positive direction moves from x to g_0's other endpoint. Let y_0 be the first juncture along g_0, let g_1 be the other geodesic at y_0, and let $\widehat{x_0 y_0}$ be the open arc of g_0 from x_0 to y_0. If g_1 is type 2, it cannot intersect itself, and so it must form a triangle on the inner boundary with g_0. If g_1 is type 1, then by the previous lemma, it forms a triangle with g_0.

In either case, let T_0 be the triangle, let x_1 be an endpoint of g_1 on the inner boundary which is the vertex of the triangle, and let $x_1 y_0$ be the open arc of g_1 from x_1 to y_0.

If T_0 is not empty, let y_1 be the first intersection point along g_1, and let g_2 be the other geodesic intersecting g_1. There is an arc s_2 of g_2 which lies inside T_0 and has both endpoints on the boundary of T_1. Since s_2 cannot intersect $x_0 y_0$ and it cannot intersect $x_1 y_0$ more than once without forming a simply connected lens, s_2 must have its other endpoint on R_0. Letting $x_2 y_1 = s_2$, we have a triangle T_1 formed by $x_2 y_1$, $x_1 y_1$, and an arc of the boundary curve R_1, and $T_1 \subsetneq T_0$.

If T_1 is not an empty boundary triangle, repeat the above construction to find T_2, T_3, ... There are only finitely many junctures, so eventually T_n will be an empty boundary triangle.

Now consider the case where there are multiple connected families of geodesics. Since one them must have an intersection or type 1 inner geodesic, one of them has a boundary triangle which is empty with respect to other geodesics in that family. If the triangle is not completely empy, then it contains some other connected family of geodesics (which must all be type 1). In that case, we can repeat the argument. We will eventually reach an empty boundary triangle or empty geodesic. □

Corollary 2.11. *Suppose $\mathcal{F}$ is a lensless family of geodesics in a simply connected region with at least one crossing. Suppose R is an arc of the boundary curve such that every geodesic has an endpoint on R. Then there is an empty boundary triangle or empty geodesic on R.*

Proof. Suppose g and h intersect. Then an arc of g, an arc of h, and an arc of R form a triangle. By the previous argument, this triangle must contain an empty boundary triangle or empty geodesic. □

Corollary 2.12. *Suppose $\mathcal{F}$ is a family of type 1 and type 2 geodesics with no simply connected lenses at least one type 1 inner geodesic. There is an empty boundary triangle or empty geodesic on the inner boundary.*

Proof. By Lemma 2.3, we know the type 1 inner and type 1 outer geodesics do not intersect. We can construct a curve which partitions the network into two layers, one of which contains all the type 1 inner geodesics. Then apply the lemma to the subnetwork. □

2.6 Lens Removal II

Definition 2.13. For a point x, let $w(x)$ be the sum of the winding numbers about x of all type 0 geodesics, parametrized counterclockwise.

Lemma 2.14. *Suppose a type 2 geodesic g is parametrized by ϕ with $\phi(0)$ on the inner boundary and $\phi(1)$ on the outer boundary. If there are no simply connected lenses, then $w(\phi(t))$ is weakly decreasing.*

Proof. Consider the case with only one type 0 geodesic h. Let t_0 and t_1 be the first and last times ϕ crosses h; let $x = \phi(t_0)$ and $y = \phi(t_1)$. Then h can be broken into two curves A and B which each begin at x and end at y, parametrized by ψ_A and ψ_B. By the argument for Lemma 2.7, we know A and B are not self-intersecting and by the argument of Lemma 2.4, we know that one of them, say ψ_A, always crosses g counterclockwise and the other, ψ_B, always crosses g clockwise. Orienting h counterclockwise gives the same orientation as ψ_A and the opposite orientation from ψ_B, which implies h always crosses g counterclockwise. Hence, h always crosses g counterclockwise, which implies $w(\phi(t))$ is weakly decreasing.

If there are several type 0 geodesics $h_1, \ldots, h_N$, then $w(\phi(t))$ is the sum of weakly decreasing functions $w_n(\phi(t))$, where $w_n(x)$ is the winding number of h_n about x. □

Definition 2.15. Let A be a continuous oriented curve consisting of oriented arcs $A_1, \ldots, A_N$ of type 0 geodesics $h_1, \ldots, h_N$. We say A is a counterclockwise (respectively clockwise) curve if the orientation of each A_n matches the counterclockwise (respectively clockwise) orientation of h_n.

Lemma 2.16. *If a type 1 and type 0 geodesic intersect, there is a removable lens.*

Proof. Assume there are no simply connected lenses. Suppose a type 1 inner geodesic intersects a type 0 (the other case is similar). The type 0 geodesics $h_0, \ldots, h_N$ divide the region of embedding into simply connected or annular subregions $S_1, \ldots, S_K$. Order the subregions by weakly decreasing $w(S_k)$ ("innermost to outermost"). S_0 is the region touching the inner boundary, and S_K is the region touching the outer boundary.

For $0 < k < K$, I claim that ∂S_k oriented positively can be partitioned into a counterclockwise curve and a clockwise curve. Consider uncrossing all the type 0 geodesics without changing the orientation of any arc. At each intersection x, four edges e_1, e_2, e_3, and e_4 meet. If e_1 comes before e_2 in counterclockwise order along one geodesic, and e_3 comes before e_4 on the other, then uncross to join e_1 with e_4 and e_2 with e_3. When all crossings are removed, the type 0 geodesics are nonintersecting simple closed curves which each wind around the hole once. In the modified network, the claim is clearly true, (although the subregions have changed), and the claim remains true when we reverse each uncrossing.

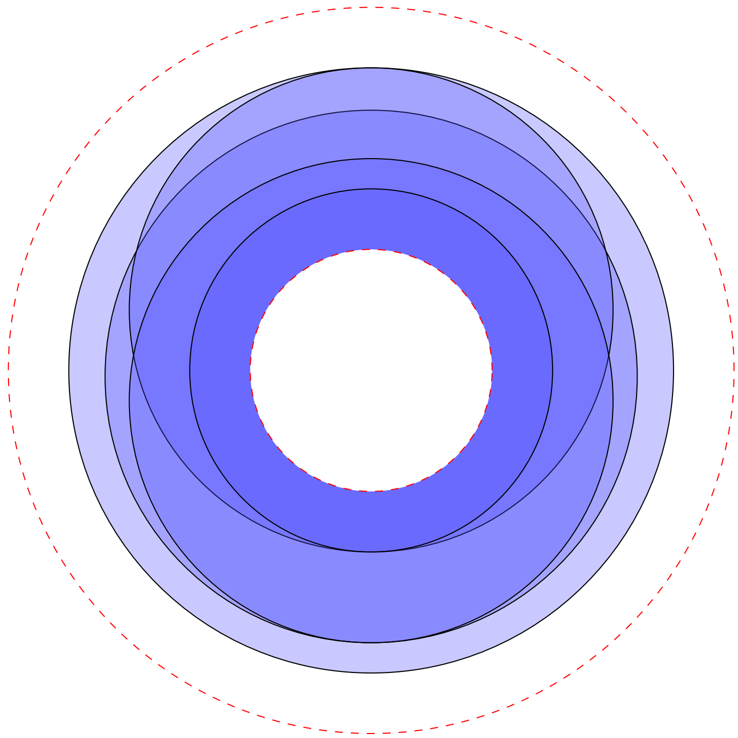
Let $g_1, \ldots, g_M$ be the type 1 inner geodesics which intersect type 0 geodesics. Let J be the first *positive* number such that $\bigcup_{k=0}^{J} \overline{S_k}$ fully contains some g_m. We will show that all crossings of type 1 and type 2 geodesics can be removed from $S_1, \ldots, S_{J-1}$.

If $1 < J$, divide ∂S_1 into a clockwise curve A and a counterclockwise curve B. All geodesics which enter S_1 across A must exit across B. Consider a subnetwork Σ_1 whose region of embedding lies inside S_1 and which contains all the junctures inside S_1. The boundary of Σ_1 consists of two curves A_1' and B_1' which cross the same geodesics as A_1 and B_1 respectively. By either Lemma 2.10 or Corollary 2.11, Σ_1 has an empty boundary triangle along A_1' (it cannot have a stub because there are no "type 1" geodesics in Σ_1). This implies that two geodesics in Γ form an empty triangle with A_1. A_1 must consist of a single geodesic arc by construction of S_1. Thus, the crossing at the vertex of the triangle can be moved out of S_1 by a Y-Δ transformation.

Continue the process until all crossings are removed from S_1. Then consider S_2, A_2, and B_2. By the same argument, two geodesics g_1 and g_2 which cross within S_2 form an empty triangle with A_2. There may be a crossing of some h_{n_1} and h_{n_2} on the side of the triangle along A_2. In that case, because there are no crossings within S_1, the crossing of h_{n_1} and h_{n_2} can be freely moved across g_1 or g_2, so that the A_2-side of the triangle formed by g_1, g_2, and A_2 contains no crossings. Move the crossing of g_1 and g_2 out of S_2 and into either S_0 or S_1. If the crossing is in S_1, move it into S_0 by the procedure of the previous paragraph.

Continue inductively until $S_1, \ldots, S_{J-1}$ contain no crossings. Let Σ_J be

Figure 12: The regions $S_1, \dots, S_K$ in Lemma 2.16. Darker color indicates higher $w(S_k)$.



the subnetwork inside S_J. By Lemma 2.12, Σ_J will always have an empty boundary triangle or empty geodesic on A'_J. Move crossings out of Σ_J as in the previous paragraph until there is an empty geodesic g_0 of Σ_J with endpoints on A'_J, and the A_J-side of the "biangle" formed by g_0 and A_J contains no crossings. The lens can be removed by a trivial modification. □

Lemma 2.17. *If two type 0 geodesics intersect, there is a removable lens.*

Proof. Let $\mathcal{F}_1, \dots, \mathcal{F}_K$ be the connected families of type 0 geodesics (where "connected families" is defined by considering only type 0 geodesics, not type 1 or 2). Let $\mathcal{F}_J$ be the innermost family with more than one geodesic. Consider a layer Σ which contains $\mathcal{F}_J$ and no other $\mathcal{F}_k$.

Let h be a geodesic of $\mathcal{F}_J$ such that there exists a curve C from some point of h to the inner boundary of Σ which does not cross any other type 0 geodesic. Construct a curve D which begins at the inner boundary, crosses h once, crosses back across h, then returns to the inner boundary, such that D does not intersect a type 0 geodesic anywhere else and D partitions Σ into two subnetworks. Then one of the subnetworks Σ' is annular and has a type 1 geodesic h' which is an arc of h. In Σ', h' is a type 1 geodesic which intersects a type 0 geodesic, so Σ' has a removable lens. □

Theorem 2.18. *A network with no removable lenses can be partitioned into three layers such that*

- *the first layer contains all type 1 inner geodesics;*
- *the second layer contains all type 0 geodesics;*
- *the third layer contains all type 1 outer geodesics.*

Proof. It follows from the previous lemmas. □

2.7 Layered Form

Definition 2.19. A network is in layered form if it can be partitioned into four layers such that

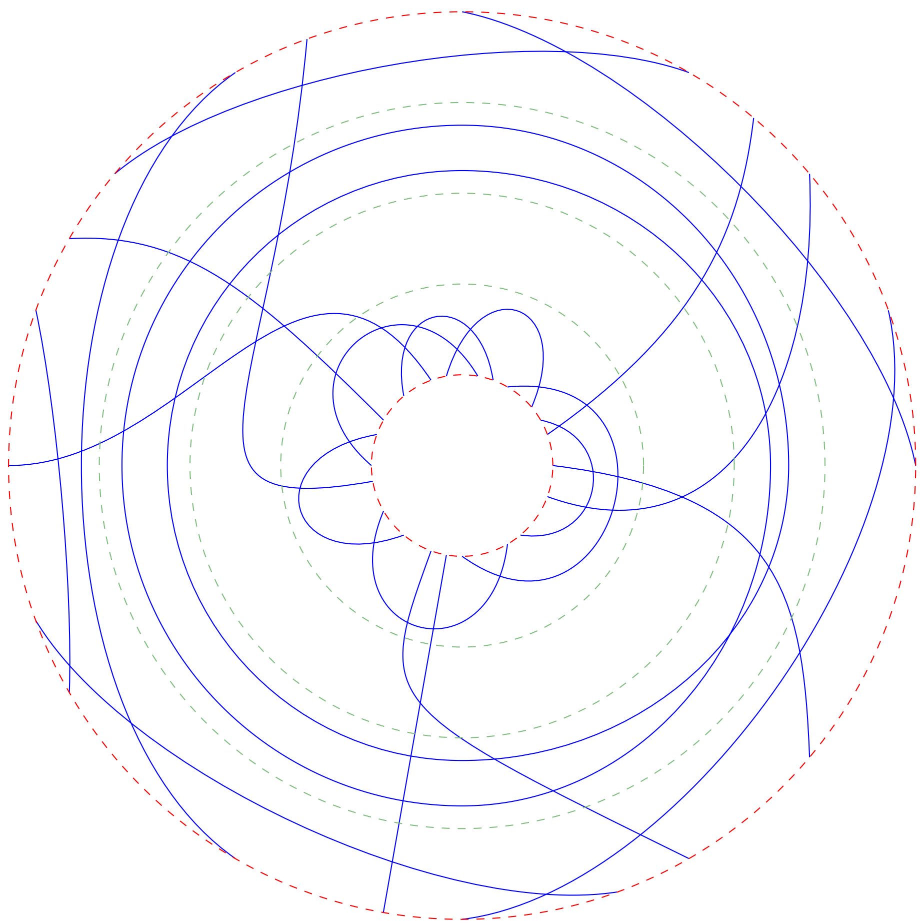
- the first layer contains all type 1 inner geodesics;
- the second layer contains all crossings between type 2 geodesics;
- the third layer contains all type 0 geodesics;
- the fourth layer contains all type 1 outer geodesics.

Theorem 2.20. *A network with no removable lenses can be put into layered form by Y-Δ transformations.*

Proof. By the previous theorem, we only have to show that all crossings of type 2 geodesics can be moved into one layer. Assume there are at least two type 2 geodesics (there must be an even number).

Consider moving them out of the layer with type 1 inner geodesics. For each type 1 geodesic in the universal cover, let $R(g[n])$ be the bounded region enclosed by $g[n]$ and one of the boundary curves. Choose a g such that no geodesic is fully contained in $R(g[0])$. This is possible because otherwise we could construct an infinite sequence of geodesics g_n with $R(g_n[0]) \subsetneq R(g_{n-1}[0])$. If $R(g[0])$ contains no crossings of type 2 geodesics, then $g[0]$ is

Figure 13: Geodesics of a network in layered form.



irrelevant for the proof, and we can work on a subnetwork which is outside of $R(g[n])$ for all n.

So suppose $R(g[0])$ contains some crossings. Consider a subnetwork Σ inside $R(g[0])$ which contains all the junctures in the region. In Σ, all geodesics have one endpoint on the upper boundary and one on C. By Corollary 2.11, there is an empty boundary triangle of Σ along C, which means that in Γ some geodesics form a triangle with g. By a Y-Δ, we can move the crossing out of $R(g[0])$. The crossing will not enter $R(g[n])$ for any integer n and type 1 geodesic g. The Y-Δ transformation in the universal cover corresponds to one in the annulus, and so we can apply the same transformation at each period in the universal cover. By repeating this argument, we can move all crossings out of $R(g[0])$, which includes crossings of type 2 geodesics.

Repeat the last two paragraphs until crossings of type 2 geodesics are removed from $R(g[0])$ for all $g \in \mathcal{G}_{\text{inner}}$. Then apply the same argument to the outer boundary. To move the crossings of type 2 geodesics inside the layer of type 0 geodesics, use the same argument as in Lemma 2.16. $\square$

3 The Relationship between the Two Boundaries

The cut-point lemma of [1] is a *geometric* statement relating connections and geodesics, but it has clear *algebraic* implications by way of the determinant-connection formula. Ian Zemke in fact uses linear algebra to prove a cut-point lemma for infinite graphs [5]. Here I develop an analogue of the cut-point lemma in which the partition of the boundary separates the inner and outer boundaries. Both geometric and algebraic statements are proved using partitions into elementary networks.

3.1 Solution Spaces

Let $P_1, \ldots, P_N$ be the plates of $\mathcal{P}_{\text{inner}}$ in counterclockwise order and let $I_1, \ldots, I_N$ be the corresponding inner boundary intervals. For each γ-harmonic function f, let f_{inner} be the restriction of f to $\mathcal{P}_{\text{inner}}$ and $\mathcal{I}_{\text{inner}}$ (f_{inner} assigns a voltage to each P_n and a current to each I_n). Let f_{inner} be written as a vector in $\mathbf{x} \in \mathbb{R}^{2|\mathcal{P}_I|}$ where $x_1, x_2, \ldots, x_N$ represent the voltages on $P_1, \ldots, P_N$, and $x_{N+1}, x_{N+2}, \ldots, x_{2N}$ represent the currents on $i_1, \ldots, i_N$.

A vector $\mathbf{x} \in \mathbb{R}^{2N}$ is called *feasible inner boundary data* if there exists a γ-harmonic function f with $f_{\text{inner}} = \mathbf{x}$. Let F_{inner} be the set of vectors which are feasible inner boundary data, and let F_{outer}, the set of feasible outer boundary data, be defined similarly. A vector $\mathbf{x} \in F_{\text{inner}}$ and $\mathbf{y} \in F_{\text{outer}}$ are compatible if there exists a γ-harmonic function with $f_{\text{inner}} = \mathbf{x}$ and

$f_{\text{outer}} = \mathbf{y}$. An $\mathbf{x} \in F_{\text{inner}}$ is called *zero-compatible* if it is compatible with $\mathbf{0} \in F_{\text{outer}}$, and a similar definition holds for $\mathbf{y} \in F_{\text{outer}}$. Let Z_{inner} be the set of zero-compatible vectors in F_{inner} and let Z_{outer} be the set of zero-compatible vectors in F_{outer}.

Obviously, F_{inner}, F_{outer}, Z_{inner}, and Z_{outer} can be determined from F. All these sets are examples of what I will call *solution spaces* for the network, and I will discuss other solution spaces later. An immediate, purely algebraic fact about these solution spaces is

Theorem 3.1. $\dim F_{\text{inner}} - \dim Z_{\text{inner}} = \dim F_{\text{outer}} - \dim Z_{\text{outer}}$.

Proof. Let $N = |\mathcal{P}_{\text{inner}}|$ and $M = |\mathcal{P}_{\text{outer}}|$. F_{inner} and Z_{inner} are linear subspaces of $\mathbb{R}^{2N}$ and F_{outer} and Z_{outer} are linear subspaces of $\mathbb{R}^{2M}$. Let $Z_{\text{inner}}^{\perp}$ be the orthogonal complement of Z_{inner} in $\mathbb{R}^{2N}$ with respect to the standard basis and inner product and let $Z_{\text{outer}}^{\perp}$ be the orthogonal complement of Z_{outer} in $\mathbb{R}^{2M}$. Let $U = Z_{\text{inner}}^{\perp} \cap F_{\text{inner}}$ and $V = Z_{\text{outer}}^{\perp} \cap F_{\text{outer}}$.

There is a one-to-one correspondence between vectors in U and vectors in V. For suppose $\mathbf{x} \in U$. Then there is a γ-harmonic function f such that $f_{\text{inner}} = \mathbf{x}$. Then $\mathbf{u} = f_{\text{outer}}$ is in F_{outer} and can be written uniquely as the sum of some $\mathbf{v} \in Z_{\text{outer}}$ and some $\mathbf{y} \in Z_{\text{outer}}^{\perp}$. Since $\mathbf{u}$ and $\mathbf{v}$ are in F_{outer}, so is $\mathbf{y}$, and so $\mathbf{y} \in V$. To show $\mathbf{y}$ is unique, suppose $\mathbf{y}_1$ and $\mathbf{y}_2$ are both in V and compatible with $\mathbf{x}$. Then $\mathbf{y}_1 - \mathbf{y}_2$ is compatible with zero, and it is in V because V is a linear subspace of $\mathbb{R}^{2N}$. Thus, $\mathbf{y}_1 - \mathbf{y}_2$ is in both Z_{outer} and $Z_{\text{outer}}^{\perp}$. Hence, $\mathbf{y}_1 = \mathbf{y}_2$.

A similar argument shows that for each $\mathbf{y} \in V$, there is a unique compatible $\mathbf{x} \in U$. Thus, there is a bijection between U and V. This bijection is obviously linear because if $\mathbf{x}_1, \mathbf{x}_2 \in U$ have compatible vectors $\mathbf{y}_1, \mathbf{y}_2 \in B$, then $a\mathbf{x}_1 + b\mathbf{x}_2 \in U$ is compatible with $a\mathbf{y}_1 + b\mathbf{y}_2 \in V$ because the sum of two γ-harmonic functions is γ-harmonic. Therefore, $\dim U = \dim V$. But $\dim U = \dim F_{\text{inner}} - \dim Z_{\text{inner}}$ and $\dim V = \dim F_{\text{outer}} - \dim Z_{\text{outer}}$. $\square$

Statements about solution spaces like F_{inner}, Z_{inner}, F_{outer}, and Z_{outer} have interpretations in terms of the response matrix. For example, if no plate touches both boundaries, then $\mathcal{P}_{\text{inner}}$ and $\mathcal{P}_{\text{outer}}$ are a partition of $\partial\mathcal{P}$ and we can write Λ in block form as

$$\Lambda = \begin{pmatrix} \Lambda_{II} & \Lambda_{IO} \\ \Lambda_{OI} & \Lambda_{OO} \end{pmatrix},$$

where the first row/column deals with the inner boundary and the second row/column deals with the outer boundary. Then

Proposition 3.2.

$$\dim Z_{\text{inner}} = \dim\ker \Lambda_{OI}, \quad \dim F_{\text{inner}} = |\mathcal{P}_{\text{inner}}| + \operatorname{rank} \Lambda_{OI}.$$

Proof. A vector $\mathbf{x} = (\mathbf{x}_v, \mathbf{x}_c)$ is in Z_{inner} if and only if

$$\begin{pmatrix} \Lambda_{II} & \Lambda_{IO} \\ \Lambda_{OI} & \Lambda_{OO} \end{pmatrix} \begin{pmatrix} \mathbf{x}_v \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{x}_c \\ \mathbf{0} \end{pmatrix},$$

which is true if and only if $\mathbf{x}_v \in \ker \Lambda_{OI}$ and $\mathbf{x}_c = \Lambda_{II}\mathbf{x}_v$. Hence, there is a bijective linear transformation from $\mathbf{x} \in Z_{\text{inner}}$ and $\mathbf{x}_v \in \ker \Lambda_{OI}$, so that $\dim Z_{\text{inner}} = \dim\ker \Lambda_{OI}$.

Let U be the set of all $\mathbf{y}$ such that

$$\begin{pmatrix} \Lambda_{II} & \Lambda_{IO} \\ \Lambda_{OI} & \Lambda_{OO} \end{pmatrix} \begin{pmatrix} \mathbf{y}_v \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{y}_c \\ * \end{pmatrix}.$$

Then U is a linear subspace of $\mathbb{R}^{2N}$, where $N = |\mathcal{P}_{\text{inner}}|$. For any $\mathbf{y}_v$, there is a unique $\mathbf{y}_c$ which satisfies the above equation, namely $\mathbf{y}_c = \Lambda_{ii}\mathbf{y}_v$. Hence, $\dim U = N$. Let V be the set of all $\mathbf{z} = (\mathbf{0}, \mathbf{z}_c)$ such that

$$\begin{pmatrix} \Lambda_{II} & \Lambda_{IO} \\ \Lambda_{OI} & \Lambda_{OO} \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{z}_c \\ * \end{pmatrix}$$

for some $\mathbf{w}$. V is isomorphic to the image space of Λ_{OI}, so $\dim V = \operatorname{rank} \Lambda_{OI}$. Every vector in F_{inner} can be uniquely written as $\mathbf{y} + \mathbf{z}$ where $\mathbf{y} \in U$ and $\mathbf{z} \in V$. Hence, $\dim F_{\text{inner}} = |\mathcal{P}_I| + \operatorname{rank} \Lambda_{OI}$. □

3.2 Partition into Elementary Networks

To get deeper results about solution spaces and connections, we can partition the network into specific types of subnetworks. An *elementary network* is any of the following four types:

1. An *elementary boundary-juncture network* is a network in which
   - every plate touches both boundaries,
   - there is exactly one juncture.
2. An *elementary spike* network is a network in which
   - every plate touches the inner boundary except one plate P
   - every plate touches the outer boundary except one plate Q

- there is a juncture point between P and Q
- there are no other junctures.

3. An *elementary inner-stub network* is a network in which

   - every plate touches the inner boundary
   - every plate except one (a stub) touches the outer boundary
   - there are no junctures.

4. An *elementary outer-stub network* is a network in which

   - every plate touches the outer boundary
   - every plate except one (a stub) touches the inner boundary
   - there are no junctures.

5. A *trivial network* is a network in which

   - Every plate touches both boundaries.
   - There are no junctures.

6. A *zigzag network* is a network in which

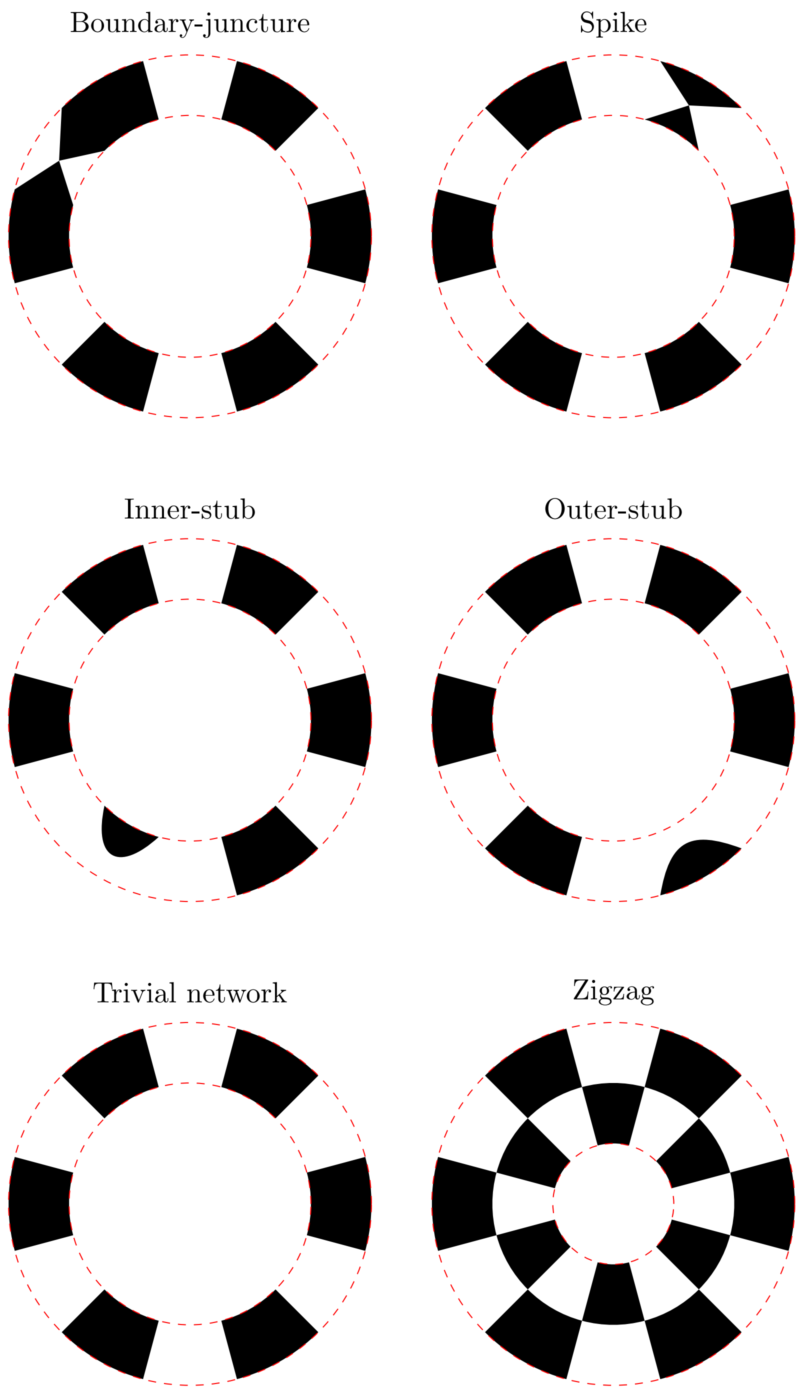
   - There are $2N$ plates, each of which touches exactly one boundary curve.
   - The inner boundary plates are $P_1, \ldots, P_N$ in counterclockwise order, the outer boundary plates are $Q_1, \ldots, Q_N$.
   - The junctures in the network are P_nQ_n and P_nQ_{n+1} for $n = 1, \ldots, N$ with indices reduced modulo N.

Any network can be partitioned into elementary networks of the first four types, but for our purposes, the most important fact is

Lemma 3.3. *Suppose* Γ *has no removable lenses and all geodesics are type 1 inner or type 2. Then* Γ *can be partitioned into elementary boundary-juncture, spike, and inner-stub networks. The number of inner-stub networks is equal to the number of type 1 geodesics.*

Proof. Let $\Sigma_0 = \Gamma$. We will define subnetworks $\Sigma_0, \Sigma_1, \ldots$ inductively. Each Σ_m will be partitioned into an elementary network Γ_{m+1} and a layer Σ_{m+1}, where Γ_{m+1} is next to the inner boundary of Σ_m and Σ_{n+1} is next to the outer boundary.

Figure 14: Elementary networks.



If Σ_m is not the trivial network, there are three cases. Let S_m be the region of embedding for Σ_m. The cases are not mutually exclusive, but they should be considered in the order given here (if more than one case is satisfied, follow the directions on the case listed first):

1. There is a stub on the inner boundary. Let P be the stub. Let C be the inner boundary of $S_m \setminus P$. Choose C' on the outside of C and so close to C that no junctures lie between C and C' and C' does not intersect any edges not intersected by C. Then C' partitions Σ_m into Γ_{m+1} and Σ_{m+1}, where Γ_{m+1} is an inner-stub network.

2. There is a boundary-juncture on the inner boundary. Let T be the inner boundary triangle at the juncture, and let C be the inner boundary of $S_m \setminus T$. Construct C', Γ_{m+1}, and Σ_{m+1} from C as in the previous case. Γ_{m+1} is an elementary boundary-juncture network.

3. There is a spike on the inner boundary. Let P be the spike and Q be the adjacent plate. We assume since P is a spike that it does not touch the outer boundary. Since we assumed there was no inner-boundary juncture, we know Q does not touch the inner boundary. Let C be the inner boundary of $S_m \setminus P$ and construct Γ_{m+1} and Σ_{m+1}. Γ_{m+1} is a spike network.

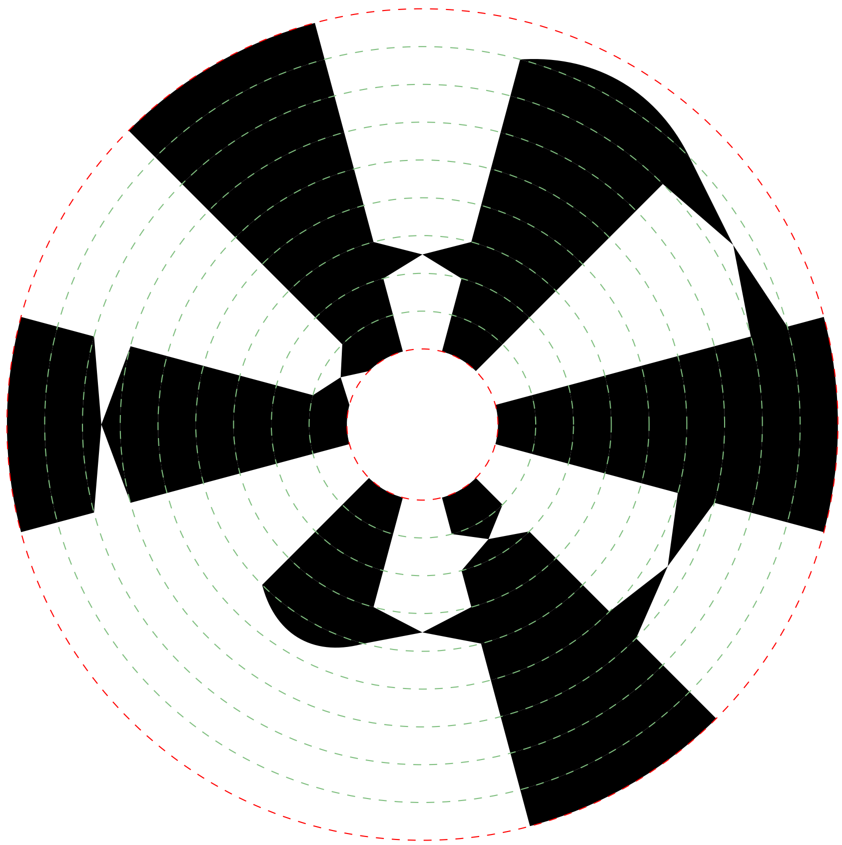
These are the only three cases because by Theorem 2.10, a nontrivial network with only type 1 inner and type 2 geodesics has an empty boundary triangle or stub and because if Σ_m has only type 1 inner and type 2 geodesics, then the same is true of Σ_{m+1}.The construction will continue until Σ_M is the trivial network. Then $\Gamma_1, \ldots, \Gamma_{M-1}, \Sigma_{M-1}$ are the desired partition.

The number of geodesic endpoints on each boundary is twice the number of plates. If K is the number of inner-stub networks, then there are K more plates on the inner boundary than the outer boundary. Since all the geodesics are type 1 inner or type 2, the number of type 1 inner geodesics is K, which is the same as the number of inner-stub networks. □

3.3 Geodesics and Connections

Lemma 3.4. *Suppose Γ has no removable lenses, and all geodesics are type 1 inner or type 2. Then $2 \cdot M(\mathcal{P}_{\text{inner}}, \mathcal{P}_{\text{outer}}) = |\mathcal{G}_2|$. The same is true if all geodesics are type 1 outer and type 2.*

Figure 15: A network partitioned into elementary boundary-juncture, spike, and inner-stub networks.



Proof. Consider the case where all geodesics are type 1 inner or type 2. The other case is symmetrical.

Γ can be partitioned into elementary boundary-juncture, spike, and inner-stub networks $\Gamma_1, \ldots, \Gamma_M$. Let $P_{1,1}, P_{2,1} \ldots, P_{N,1}$ be the outer boundary plates. Define sequences $\{P_{1,k}\}, \ldots, \{P_{N,k}\}$ inductively as follows:

For each n, k, let $\Gamma_{n,k}$ be the innermost subnetwork of the partition which includes a subplate of $P_{n,k}$. Then $\Gamma_{n,k}$ must be a spike, boundary-juncture, or inner stub network. If $P_{n,k}$ touches the inner boundary of $\Gamma_{n,k}$, then $P_{n,k}$ touches the inner boundary, and we let $P_{n,k}$ be the last plate of its sequence. Otherwise $\Gamma_{n,k}$ is a spike network and $P_{n,k}$ has a juncture in $\Gamma_{n,k}$. We let $P_{n,k+1}$ be the plate which meets $P_{n,k}$ at this juncture.

The sequences $\{P_{1,k}\}, \ldots, \{P_{N,k}\}$ define N disjoint paths from the outer to the inner boundary. They form an N-connection using all the plates of the outer boundary, so there cannot be any larger k-connection. N is half the number of type 2 geodesics. □

Lemma 3.5. *Suppose Γ has no removable lenses, and all geodesics are type 2 or type 0. Then $2 \cdot M(\mathcal{P}_{\text{inner}}, \mathcal{P}_{\text{outer}}) = |\mathcal{G}_2|$.*

Proof. As in the proof of Lemma 2.16, we can uncross the type 0 geodesics, so as to preserve the orientation of each arc of a type 0 geodesic. We are left with a network Γ' in which all the type 0 geodesics are simple closed curves winding once about the hole. Removing junctures cannot create any connections, only break them. Hence, if we can show that Γ' has a k-connection between all vertices on the inner boundary and all vertices on the outer boundary, the proof will be complete.

For each type 0 geodesic g, draw a closed curve along the inside of g so close to g that there are no junctures between g and the curve. Draw another closed curve along the outside of g. We can create such curves next to every type 0 geodesic in such a way that the curves do not intersect. The curves partition Γ' into $\Gamma_1, \ldots, \Gamma_M$, where each Γ_m is either a zigzag or has only type 2 geodesics.

It is obvious that a zigzag has a k-connection between all inner and all outer boundary plates. If Γ_m is not a zigzag, then the previous lemma shows that it has such a k-connection. By similar reasoning as in the previous lemma, we join the k-connections of the subnetworks into a k-connection of Γ' with the desired size. □

Theorem 3.6. *If Γ has no removable lenses, $2 \cdot M(\mathcal{P}_{\text{inner}}, \mathcal{P}_{\text{outer}}) = |\mathcal{G}_2|$.*

Proof. By Theorem 2.18, Γ can be partitioned into three layers Γ_1, Γ_2, and Γ_3, where Γ_1 contains the type 1 inner geodesics, Γ_2 contains the type 0, and Γ_3 contains the type 1 outer. By the preceding lemmas, Γ_1 has a k-connection from some plates of the inner boundary to all plates of the outer boundary, Γ_2 has a k-connection from the whole inner boundary to the whole outer boundary, and Γ_3 has a k-connection from the whole inner boundary to some subset of the outer boundary. The theorem follows. □

3.4 Geodesics and Solution Spaces

Lemma 3.7. *If Γ is an elementary boundary-juncture or spike network with N boundary intervals on each boundary, then* $\dim F_{\text{inner}} = \dim F_{\text{outer}} = 2N$ *and* $\dim Z_{\text{inner}} = \dim Z_{\text{outer}} = 0$*.*

Proof. Consider a boundary-juncture network with plates $P_1, \ldots, P_N$ and juncture between P_1 and P_2. To show any data is feasible on the inner boundary, suppose we are given inner boundary voltages and currents, and we will find a γ-harmonic function with that boundary data. The voltages are all determined. If I_n and J_n are the inner and outer boundary intervals for P_n, then we let $c(J_n) = -c(I_n)$ for $n \neq 1, 2$. We let $c(P_1 \to P_2) = \gamma(P_1P_2)(v(P_1) - v(P_2))$, and $c(J_1) = -c(I_1) + c(P_1 \to P_2)$ and $c(J_2) = -c(I_2) + c(P_2 \to P_1)$.

On the other hand, if all the outer boundary data is zero, then no current can flow across the juncture, and all the boundary currents are zero. Thus, the only zero-compatible inner boundary data is $\mathbf{0}$. The arguments for F_{outer} and Z_{outer} are symmetrical.

The argument for a spike network is similar and is left to the reader. □

Lemma 3.8. *For an elementary inner-stub network with $N+1$ plates,* $\dim F_{\text{outer}} = 2N$*,* $\dim Z_{\text{outer}} = 0$*,* $\dim F_{\text{inner}} = 2N+1$*, and* $\dim Z_{\text{inner}} = 1$*.*

Proof. Obviously, any data on the outer boundary is feasible and if voltages and currents are zero on the inner boundary, they must be zero on the outer boundary. Any data on the inner boundary is feasible so long as the current on the stub is zero. If voltages and currents on the outer boundary are zero, then all voltages and currents on the inner boundary must be zero except the voltage on the stub. □

Lemma 3.9. *Suppose Γ has no removable lenses, and all geodesics are type 1 inner or type 2. Then*

$$\begin{aligned} \dim F_{\text{outer}} &= |\mathcal{G}_2|, & \dim Z_{\text{outer}} &= 0, \\ \dim F_{\text{inner}} &= |\mathcal{G}_2| + |\mathcal{G}_{\text{inner}}|, & \dim Z_{\text{inner}} &= |\mathcal{G}_{\text{inner}}|. \end{aligned}$$

Proof. By Lemma 3.3, Γ can be partitioned into elementary boundary juncture, boundary spike, and inner stub networks $\Gamma_1, \ldots, \Gamma_M$, ordered from *outermost* to *innermost*. Let Σ_m be the subnetwork consisting of $\Gamma_1, \ldots, \Gamma_m$. We show that the theorem is true for each Σ_m by induction. The base case follows from the previous lemmas. The induction step is broken into four claims:

1. $\dim F_{\text{outer}} = |\mathcal{G}_2|$. Any outer boundary data which was feasible for Σ_m is still feasible for Σ_{m+1}; it does not matter what the inner boundary data of Σ_m is because any outer boundary data is feasible for Γ_{m+1}.

2. $\dim Z_{\text{outer}} = 0$. Suppose that a γ-harmonic function on Σ_{m+1} has inner boundary data zero. Then the outer boundary data on Γ_{m+1} must be zero. This implies that the inner boundary data on Σ_m is zero, and so by inductive hypothesis, the inner boundary data on Σ_m must be zero.

3. $\dim F_{\text{inner}} = |\mathcal{G}_2| + |\mathcal{G}_{\text{inner}}|$. If Γ_{m+1} is a boundary-juncture or spike network, then there is a unique compatible vector of outer boundary data for each vector of inner boundary data, so there is a linear isomorphism between F_{inner} and F_{outer} of Γ_{m+1}. This implies that there is a linear isomorphism between F_{inner} of Σ_{m+1} and F_{inner} of Σ_m, so $\dim F_{\text{inner}}$ is the same for both networks.

   If Γ_{m+1} is an inner stub network, then the set of feasible voltages and currents on the inner boundary plates and intervals of Σ_{m+1} *other than the stub* is exactly the same as F_{inner} of Σ_m. Any voltage is feasible on the stub, and it will not affect the rest of the network; however, the current on the stub must be zero. Thus, $\dim F_{\text{inner}}$ of Σ_{m+1} is one more than $\dim F_{\text{inner}}$ of Σ_m. Σ_{m+1} also has one more type 1 inner geodesic than Σ_m.

4. $\dim Z_{\text{inner}} = |\mathcal{G}_{\text{inner}}|$. The argument is the same as the previous claim.

□

Theorem 3.10. *If Γ has no removable lenses or type 0 geodesics, then*

$$\dim F_{\text{inner}} = |\mathcal{G}_2| + |\mathcal{G}_{\text{inner}}|, \quad \dim Z_{\text{inner}} = |\mathcal{G}_{\text{inner}}|,$$
$$\dim F_{\text{outer}} = |\mathcal{G}_2| + |\mathcal{G}_{\text{outer}}|, \quad \dim Z_{\text{outer}} = |\mathcal{G}_{\text{outer}}|.$$

Proof. Γ can be partitioned into a network Γ_1 with only type 1 inner and type 2 geodesics and Γ_2 with only type 1 outer and type 2. We apply the previous lemma to these networks (we can switch roles of the inner and outer boundaries and the lemma is still true). Since any data is feasible on the inner boundary of Γ_2, F_{inner} is the same for Γ as for Γ_1. Since the only zero-compatible data on the inner boundary of Γ_2 is $\mathbf{0}$, Z_{inner} is the same for Γ as for Γ_1. The argument for the other claims is symmetrical. $\square$

Theorem 3.11. *Suppose Γ has no removable lenses, but has type 0 geodesics. If $\frac{1}{2}|\mathcal{G}_2|$ is odd, Theorem 3.10 holds.*

Proof. We will use the determinant-connection formula. We know that no plate touches both boundaries because there is a type 0 geodesic. By Theorem 3.6, the maximum size k-connection between the inner and outer boundaries is $K = \frac{1}{2}|\mathcal{G}_2|$. Let $\mathcal{U}$ and $\mathcal{V}$ be sets of plates on the inner and outer boundaries respectively such that there is a K-connection between $\mathcal{U}$ and $\mathcal{V}$. If α is any K-connection between $\mathcal{U}$ and $\mathcal{V}$, then τ_α must be a cyclic permutation of the form $\tau_\alpha(n) \equiv n + J \mod K$ for some integer J. Otherwise, the paths in α would intersect as a consequence of the Jordan curve theorem. Since K is odd, all such permutations are even.

Thus, by the determinant-connection formula, $\det \Lambda(\mathcal{U}; \mathcal{V})$ is strictly negative. This implies $\operatorname{rank} \Lambda_{OI} = K$ (the rank cannot be any larger because no larger k-connections exist). By applying Proposition 3.2 and by counting geodesics and boundary plates, we see

$$\begin{aligned} \dim F_{\text{inner}} &= |\mathcal{P}_{\text{inner}}| + \operatorname{rank} \Lambda_{OI} = \tfrac{1}{2}|\mathcal{G}_2| + |\mathcal{G}_{\text{inner}}| + \tfrac{1}{2}|\mathcal{G}_2| \\ \dim Z_{\text{inner}} &= |\mathcal{P}_{\text{inner}}| - \operatorname{rank} \Lambda_{OI} = \tfrac{1}{2}|\mathcal{G}_2| + |\mathcal{G}_{\text{inner}}| - \tfrac{1}{2}|\mathcal{G}_2|, \end{aligned}$$

and the corresponding statements for Λ_{IO} and the outer boundary. $\square$

Theorem 3.12. *Suppose $\frac{1}{2}|\mathcal{G}_2|$ is even. If Theorem 3.10 does not hold, then*

$$\begin{aligned} \dim F_{\text{inner}} &= |\mathcal{G}_2| + |\mathcal{G}_{\text{inner}}| - 1, \quad \dim Z_{\text{inner}} = |\mathcal{G}_{\text{inner}}| + 1, \\ \dim F_{\text{outer}} &= |\mathcal{G}_2| + |\mathcal{G}_{\text{outer}}| - 1, \quad \dim Z_{\text{outer}} = |\mathcal{G}_{\text{outer}}| + 1. \end{aligned}$$

Proof. There exists a $K-1$-connection from the inner to the outer boundary, and $K - 1$ is odd, so by the argument in the previous theorem, $\operatorname{rank} \Lambda_{OI} = K-1$, and the statements about dimensions follow from Proposition 3.2. $\square$

The preceding theorem shows that for annular networks the algebraic versions of the cut-point lemma sometimes require stronger hypotheses than the corresponding geometric statements. Dimensions of solution spaces are not always what we would expect based on the connection properties. However, in this case, the matrix is "almost" invertible:

Proposition 3.13. *Suppose the network has no removable lenses. Suppose $K = \frac{1}{2}|\mathcal{G}_2|$ is even and there is at least one type 0 geodesic. Suppose there exists a K-connection between $\mathcal{U} \subset \mathcal{P}_{\text{inner}}$ and $\mathcal{V} \subset \mathcal{P}_{\text{outer}}$. Then every $K-1 \times K-1$ minor of $\Lambda(\mathcal{U};\mathcal{V})$ is strictly negative.*

Proof. Since $K-1$ is odd, it suffices to show that every $K-1$ connection from a subset of $\mathcal{U}$ to a subset of $\mathcal{V}$ exists. Since Y-Δ transformations do not affect connections, assume the network is in layered form. As in Lemma 3.5, uncross the type 0 geodesics until they have no self-intersections; this cannot create any new connections. It now suffices to show that all $K-1$-connections exist in a single zigzag.

Suppose the plates on the inner boundary are $P_1, \ldots, P_K$ and on the outer boundary $Q_1, \ldots, Q_K$ such that there are junctures between P_n and Q_n and P_n and Q_{n+1} with indices reduced modulo K. Assume without loss of generality that the inner-boundary plates in the desired connection are $P_1, \ldots, P_{K-1}$. Suppose we want to connect them with Q_n for $n \neq J$. For $n < J$, connect P_n and Q_n. For $n \geq J$, connect P_n and Q_{n+1}. □

A similar proof will show that, in general, if the sum of the winding numbers of the type 0 geodesics is N, and if m is an odd integer with $K - N \leq m \leq K$, then every $m \times m$ minor of $\Lambda(\mathcal{U};\mathcal{V})$ is strictly negative. This is true whether K is odd or even.

4 Cuts of One Boundary

4.1 Definitions

Let $I_1, \ldots, I_N$ be the boundary intervals of a network. A *cut R of the inner boundary* is an arc of C_{inner} whose endpoints are not the endpoints of any I_n. We will denote by R^C the union of C_{outer} and the arc in C_{inner} which is complementary to R in C_{inner}. (For circular planar networks, R^C is simply the complementary arc of the boundary curve.) We assume that R contains at least one endpoint of a geodesic, and does *not* contain *all* geodesic endpoints on the inner boundary.

The *boundary intervals of R* (denoted $\mathcal{I}_R$) include any I_n which is a subset of R, and the *plates of R* (denoted $\mathcal{P}_R$) include all plates with these boundary intervals. The endpoint of R may fall within some boundary interval J corresponding to a plate P. The endpoint divides J into two intervals

J_A and J_B with $J_A \subset R$ and $J_B \not\subset R$. In that case, J_A is considered a boundary interval of R and J_B is a boundary interval of R^C. P is considered a plate both of R and of R^C.

The *solution spaces of* R are as follows. Let $P_1, \ldots, P_K$ be the plates of a cut R. Let $J_1, \ldots, J_K$ be the corresponding boundary intervals of the cut. Let $\mathbf{x}$ be a vector in $\mathbb{R}^{2K}$. We say $\mathbf{x} \in F_R$ if there exists a γ-harmonic function with voltages $x_1, \ldots, x_K$ on $P_1, \ldots, P_K$ and currents $x_{K+1}, \ldots, x_{2K}$ on $J_1, \ldots, J_K$. Suppose J_k is at one of the endpoints of R and that some boundary interval I of Γ was split into J_k and $J^* \subset R^C$ by an endpoint of R. Then we consider $c(I) = c(J_k) + c(J^*)$. An $\mathbf{x} \in F_R$ may have any current on J_k because we can always choose $c(J^*)$ to make $c(I)$ correct for a γ-harmonic function on the whole network.

Define the maximum connection $M(R)$ as $M(\mathcal{P}_R, \mathcal{P}_{R^C})$. Define the family of *reentrant* geodesics $\mathcal{R}_R$ as the collection of all geodesics with both endpoints in R. A reentrant geodesic g forms a closed curve C with some interval of R. C can be oriented and its winding number about the hole can be computed. If the winding number is nonzero, g is called *reentrant around the hole*.

4.2 Circular Planar Case

We can prove the cut-point lemma for circular planar graphs by "changing the region of embedding into an annulus" and using the results of the previous section. Here we assume that each boundary plate of the given circular planar graph has only *one* boundary interval.

Consider a cut R of the boundary of a circular planar network in a region S. Let C_1 be an arc of R which is slightly shorter at the endpoints, but contains all the same geodesic endpoints. Let x_1 and y_1 be the endpoints of C_1. Let C_2 be a similar arc of R^C with endpoints x_2 and y_2. Construct a curve C_1' which connects x_1 and y_1 and remains outside $\overline{S}$ and a curve C_2' which connects x_2 and y_2 which connects x_2 and y_2, such that $C_1 \cup C_1'$ and $C_2 \cup C_2'$ are nonintersecting simple closed curves which form the boundary of an annular region S', and such that $C_1 \cup C_1'$ is inside $C_2 \cup C_2'$.

Let Γ' be the plate network in S'. Do not change the shape of any plates, even if a plate contains an endpoint of R. For each plate of Γ which contains an endpoint of R, Γ' will have an extra "type 2" geodesic. Otherwise, the geodesics will not change. F_R of Γ is exactly F_{inner} of Γ', Z_R is Z_{inner}, $M(R)$ is $M(\mathcal{P}_{\text{inner}}, \mathcal{P}_{\text{outer}})$, and $\mathcal{R}_R$ is $\mathcal{G}_{\text{inner}}$.

By applying the results of the previous section to Γ', we have

$$M(\mathcal{P}_{\text{inner}}, \mathcal{P}_{\text{outer}}) = |\mathcal{G}_2|, \quad \dim F_{\text{inner}} = |\mathcal{G}_2| + |\mathcal{G}_{\text{inner}}|, \quad \dim Z_{\text{inner}} = |\mathcal{G}_{\text{inner}}|.$$

The first two are not convenient formulae because we modified the number of "type 2" geodesics. But we can express $|\mathcal{G}_2|$ as

$$|\mathcal{G}_2| = 2|\mathcal{P}_{\text{inner}}| - 2|\mathcal{G}_{\text{inner}}| = 2|\mathcal{P}_R| - 2|\mathcal{R}_R|,$$

which yields

Theorem 4.1 (Cut-Point Lemma)**.** *Let Γ be a circular planar network and let R be a cut of the boundary curve. Then*

$$\begin{aligned} M(R) &= |\mathcal{P}_R| - |\mathcal{R}_R|, \\ \dim F_R &= 2|\mathcal{P}_R| - |\mathcal{R}_R|, \\ \dim Z_R &= |\mathcal{R}_R|. \end{aligned}$$

We could have proved this directly by partitioning the simply connected region into "elementary layers;" the proof given in [1] by uncrossing empty boundary triangles can be interpreted as constructing such a partition.

4.3 Annular Case

In the annular case, the connections and solution spaces of a cut are not as easy to describe. However, there are certain cases where the same formulas hold:

Lemma 4.2. *Suppose Γ has only type 2 and type 1 inner geodesics, no removable lenses, and no self-intersecting geodesics, and that the type 2 geodesics do not intersect each other. Let R be a cut of the inner boundary with no reentrant geodesics such that at least one type 2 geodesic does not have an endpoint in R. Then $M(R) = |\mathcal{P}_R|$, $\dim F_R = 2|\mathcal{P}_R|$, and $\dim Z_R = 0$.*

Proof. Let A and B be curves which begin at the clockwise and counterclockwise endpoints of R and ends at some points on the outer boundary, such that

- A and B do not intersect themselves.
- A and B do not intersect each other.
- Neither one contains a juncture.
- Neither one intersects any type 2 geodesic. This is possible because the type 2 geodesics do not intersect each other.

- Neither one intersects the same geodesic twice. This is possible because of our assumptions about lenses.

(See Figure 16.)

Let U be the region of the annulus which is counterclockwise of A and clockwise of B, and let V be the region which is counterclockwise of B and clockwise of A. Let Γ_U and Γ_V be the subnetworks in these regions. Since we assumed there was a type 2 geodesic with no endpoint on R, we know that no plate of Γ_V touches both A and B.

No geodesic of Γ_U has both endpoints on R because no geodesic of Γ does. On the other hand, all geodesics of Γ_V have an endpoint on R' by the following argument: No geodesic of Γ_V has both endpoints on the outer boundary, on A, or on B by construction. A geodesic with an endpoint on the outer boundary must have been a type 2 geodesic of Γ, so it must not intersect A or B. If a geodesic had one endpoint on A and one on B, it would been a type 1 geodesic of Γ, and would have both endpoints on R, which is impossible.

Let C and C' be the arcs of the outer boundary along Γ_U and Γ_V respectively. By Theorem 4.1, there is a k-connection α of Γ_U from R to $C \cup A \cup B$ which uses all boundary plates of R. Similarly, there is a k-connection α' of Γ_V from R' to $C' \cup A \cup B$ which uses all boundary plates of $C' \cup A \cup B$.

We construct a connection β in Γ from R to R^C which uses all the plates of R in the following way. Let $P_1, \ldots, P_N$ be the plates along R in Γ_U. For each P_n, let α_n be the path in α connecting P_n to some plate Q_n of Γ_V. We know Q_n touches $C \cup A \cup B$. There are two cases:

- If Q_n touches C, let $\beta_n = \alpha_n$ (except that if Q_n is a *subplate* in Γ, we replace it with the *whole plate*).

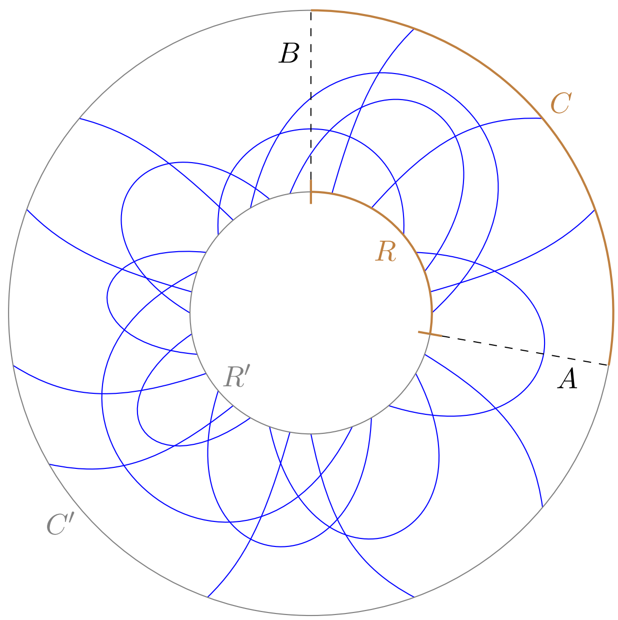
- If Q_n touches A or B, but not C, there is a plate Q'_n of Γ_V such that $Q_n \cup Q'_n$ forms one plate of Γ. Since Q_n does not touch C, neither does Q'_n. Let α'_n is the path of α' connecting Q'_n to some plate touching R', and join α_n and α'_n to form β_n.

The paths $\beta_1, \ldots, \beta_N$ are distinct and form an N-connection from R to R^C. This proves the first assertion in the theorem.

For the other two claims, repeat the above argument using the theorems about solution spaces instead of the theorems about connections. □

Theorem 4.3. *Suppose* Γ *has no removable lenses and no self-intersecting type 1 inner geodesics. Let* R *be a cut of the inner boundary with no geodesics*

Figure 16: Curves in the proof of Lemma 4.2.



reentrant around the hole, such that at least one type 2 geodesic does not have an endpoint on R. Then $M(R) = |\mathcal{P}_R| - |\mathcal{R}_R|$.

Proof. We can assume that Γ is in layered form without changing the connections or solution spaces.

Let C be a curve which begins at the clockwise endpoint of R and proceeds to the counterclockwise endpoint without intersecting itself, intersecting the same geodesic twice, or including any junctures, such that the region T bounded by $R \cup C$ is a simply connected subset of the annulus. Let Γ_T be the subnetwork in this region. Notice T contains all the reentrant geodesics of R, and no geodesic of Γ_T is reentrant for C.

Divide the rest of Γ into two layers Γ_U and Γ_V (in regions U and V) such that Γ_U contains all type 1 inner geodesics and Γ_V contains all type 0 and type 1 outer geodesics and crossings between type 2 geodesics.

By Theorem 3.6, there is a k-connection between the plates touching the inner boundary of Γ_V and some subset of the plates on the outer boundary. By the previous lemma, there is a k-connection between all the plates of Γ_U along C and some of the plates of Γ_U in C^C. Thus, we have a connection in $\Gamma_{U\cup V}$ from the plates touching C to the some of the plates in C^C.

By Theorem 4.1, the maximum k-connection in Γ_T from the plates along R to the plates touching C is $2|\mathcal{P}_R| - |\mathcal{R}_R|$. We can join the paths in a maximal k-connections with the paths of the connections in the previous paragraph. Thus, the maximum k-connection in Γ from the plates along R to the plates along R^C is the same size. □

Theorem 4.4. *Let* Γ *and R be in the previous theorem. Suppose that Theorem 3.10 holds. Then*

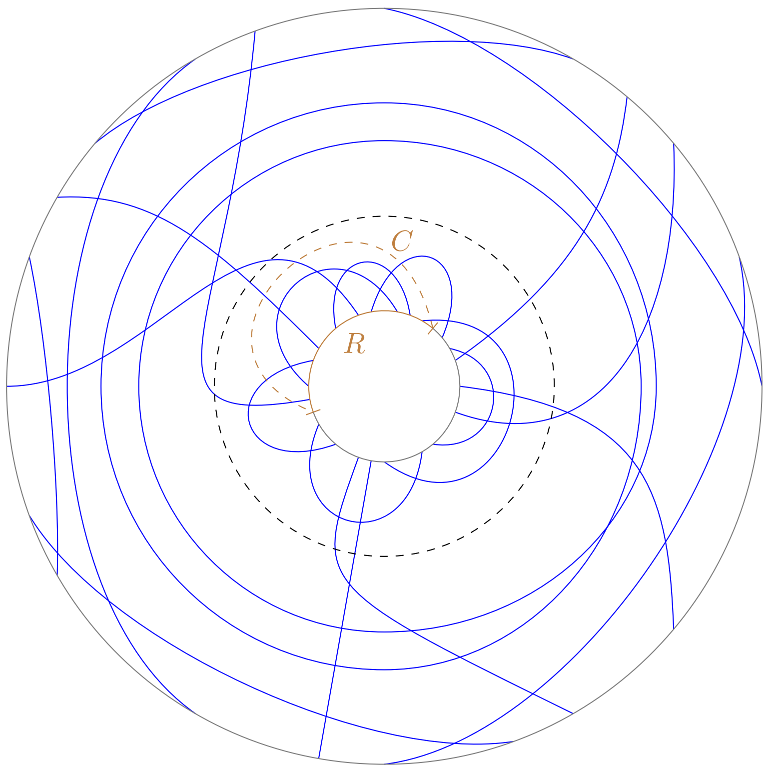
$$\dim F_R = 2|\mathcal{P}_R| - |\mathcal{R}_R|,$$
$$\dim Z_R = |\mathcal{R}_R|.$$

Proof. Let Γ_T, Γ_U, and Γ_V be as in the previous proof. For Γ_V, any data is feasible for the plates on the inner boundary because we assumed Theorem 3.10 holds. For Γ_U, any data is feasible for the plates touching C. Hence, for $\Gamma_{U\cup V}$, any data is feasible for the plates touching C. For Γ_T, $\dim F_R = 2|\mathcal{P}_R| - |\mathcal{R}_R|$ by Theorem 4.1. Hence, the same is true for Γ. □

4.4 Partial Recovery by Removal of Type 1 Geodesics

We can recover boundary junctures and spikes in networks with type 1 geodesics using the algebraic "cut-point lemma."

Figure 17: Proof of Theorem 4.3. The black dashed curve divides U and V.



Lemma 4.5. *Suppose* Γ *has no removable lenses or self-intersecting type 1 inner geodesics and suppose Theorem 3.10 holds. Let* P *be a spike on the inner boundary with juncture* PQ *connecting it to another plate* Q*. Suppose that* g*, one of the geodesics touching* P*, is type 1 and is not part of a two-pole lens. Suppose there is a type 2 geodesic which does not intersect* g*. Then the conductance of* PQ *is recoverable.*

Proof. The geodesics g splits the region of embedding into two subregions; let T be the simply connected subregion. Let A_T be the arc of the inner boundary curve which is part of ∂T. Let $R \supset A_T$ be an arc which contains the same geodesic endpoints as A_T such that the endpoints of R are not endpoints of a geodesic. Let C be a curve with endpoints at the endpoints of R which remains close enough to g that no junctures lie between C and g. Let C' be a curve with endpoints at the endpoints of R which remains between g and C except that C' crosses g twice along the boundary of P. C' enters P and then immediately exits P.

Let U be the region bounded by R and C', let V be the region bounded by C and C', and let W be the annular region bounded by R^C and C. Let Γ_U, Γ_V, and Γ_W be the subnetworks in these regions.

Consider the boundary value problem for Γ where all voltages and currents on R^C are 0 and the voltage of P is 1. I claim this problem has a solution and that the voltage of Q is 0.

For Γ_W, C is a cut of the inner boundary with no reentrant geodesics. Hence, by Theorem 4.4 the only zero-compatible data on C is zero. In particular, the boundary data for our problem force the voltage of Q to be zero. $\Gamma_{V\cup W}$ is the same as Γ_W except with a stub P' on the inner boundary. Thus, to have zero-compatible data on C', we need all voltages and currents zero except that P' can have whatever voltage we want. Thus, we can set the voltage of P' to 1.

Finally, notice that for Γ_U, C' is a cut of the boundary with no reentrant geodesics, so any boundary data is feasible, and in particular, we can have zero voltage and current everywhere except voltage 1 on P'. Thus, for Γ, there is zero-compatible data on R with voltage 1 on P', which is the same thing as voltage 1 on P.

Therefore, there is a boundary value problem which forces the voltage of Q to be zero and the voltage of P to be 1. Knowing F, we can compute the current on the boundary interval of P, and it is equal to the conductance of PQ. □

Lemma 4.6. *Suppose* Γ *has no removable lenses or self-intersecting type 1 inner geodesics and suppose Theorem 3.10 holds.* PQ *be a juncture between* *inner boundary plates P and Q. Suppose that g, one of the geodesics at PQ, is type 1 and is not part of a two-pole lens. Suppose there is type 2 geodesic which does not intersect g. Then the conductance of PQ is recoverable.*

Proof. Assume without loss of generality that P is inside g and Q is outside g. Construct R, C, C', U, V, and W as in the previous proof. Consider the boundary value problem with all voltages and currents zero on R^C and voltage 1 on P. By the same argument, this problem has a solution, and all voltages and currents on Γ_U are zero. In particular, the current is zero on all junctures of Q other than PQ. Knowing F, we can find the boundary current on Q, and it is negative the conductance of PQ. □

Theorem 4.7. *Let Γ be a network in layered form with no removable lenses and no lenses involving type 1 geodesics. Suppose Theorem 3.10 holds. Suppose that for every type 1 geodesic there is type 2 geodesic which does not intersect it. Then all conductances in the layers with type 1 geodesics can be recovered.*

Proof. Consider the type 1 geodesics on the inner boundary first. Let Γ_{inner} be the layer which contains all the type 1 inner geodesics. By Theorem 2.10, Γ_{inner} has an empty boundary triangle or stub. Remove all stubs from the network. Then there is an empty boundary triangle, and one of the geodesics must be type 1. By the previous two lemmas, the conductance of the juncture at this triangle is recoverable.

Uncross the triangle and update the solution spaces (or response matrix) to match the modified network. The modified network will still satisfy the hypotheses of this theorem. Repeat the above argument and keep removing stubs and uncrossing empty boundary triangles until all conductances in the layer are recovered. The conductances in the layer with type 1 outer geodesics can be recovered in the same way. □

This theorem has implications for networks which are not in layered form because any network with no removable lenses can be put into layered form by Y-Δ transformations. Knowing the conductances of a Y-Δ-equivalent network is (theoretically) just as good as knowing the conductances of the original network. However, at this point we have only *partially* recovered the network. After removing the type 1 geodesics, we still have to recover the conductances of the remaining network, and that is the focus of the next section.

5 Radial Networks

A *radial network* is an annular network with only type 2 geodesics. Radial networks with no removable lenses have special properties, as we would expect from the theorems of §3. They can be partitioned into elementary boundary-juncture and spike networks (Lemma 3.3). There is a k-connection from all the plates on the inner boundary to all plates on the outer boundary (Theorem 3.6). Any data is feasible on the inner boundary or the outer boundary, and the only zero-compatible data is zero (Theorem 3.10). Thus, complete voltage and current data on one boundary curve determines a unique γ-harmonic function on the network.

In this section, I explore the geometric and electrical properties of radial networks in greater depth and prove recoverability for a certain class of networks.

5.1 Structure

In a radial network, there is a canonical way to classify the junctures. Orient all the geodesics so the positive direction moves from the inner to the outer boundary. At each juncture point y, designate the four edges as

- a counterclockwise inner edge e_1,
- a counterclockwise outer edge e_2,
- a clockwise inner edge e_3,
- a clockwise outer edge e_4,

where e_1 and e_4 appear in that order on one of the geodesics, e_2 and e_3 appear in that order on the other geodesic, and the counterclockwise ordering of the four edges about the point y is e_1, e_2, e_4, e_3.

There are two possibilities:

1. e_1 and e_2 are edges of the same plate and e_3 and e_4 are edges of the same plate. In this case, y is called a *counterclockwise-clockwise* juncture.

2. e_1 and e_3 are edges of the same plate and so are e_2 and e_4. Then y is called an *inward-outward* juncture.

By Lemma 3.3, a radial network with no removable lenses can be partitioned into boundary-juncture and boundary-spike networks. In any partition, the counterclockwise-clockwise junctures are part of an elementary boundary-juncture network, and the inward-outward edges are part of an elementary spike network. As a result,

Theorem 5.1. *In a radial network with no removable lenses, there is exactly one k-connection between $\mathcal{P}_{\text{inner}}$ and $\mathcal{P}_{\text{outer}}$, and the junctures in the connection are exactly the inward-outward junctures.*

Proof. Let $\Gamma_0, \Gamma_1, \dots, \Gamma_M$ be a partition of Γ where Γ_0 is a trivial network and $\Gamma_1, \dots, \Gamma_M$ are elementary boundary-juncture and spike networks listed from innermost to outermost. Let Σ_m be the subnetwork consisting of $\Gamma_0, \dots, \Gamma_m$. We prove the theorem by induction for each Σ_m. The base case is trivial.

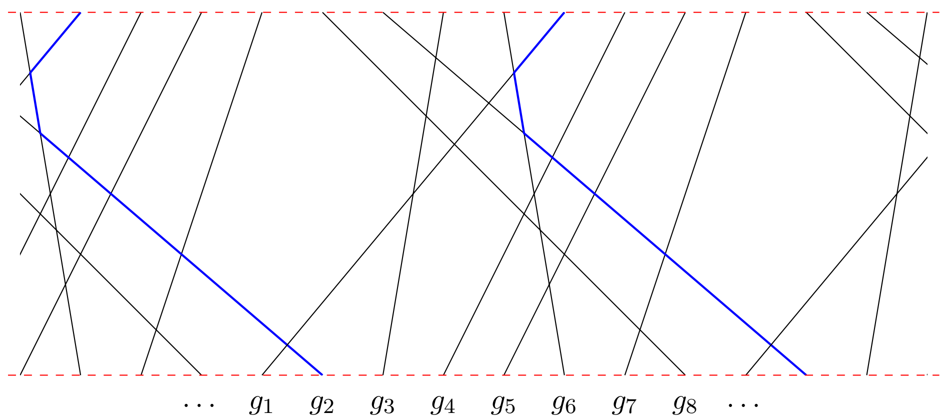
Suppose the theorem is true of Σ_m. If Γ_{m+1} is a spike network, then its juncture is inward-outward. There is exactly one k-connection from the inner to the outer boundary of Σ_m. To find the k-connection for Σ_{m+1}, simply add the spike of Γ_{m+1} to the appropriate path. If Γ_{m+1} is a boundary-juncture network, then its juncture is counterclockwise-clockwise. The juncture cannot be used in an interboundary k-connection in Σ_{m+1} because it is a boundary juncture. Thus, the set of interboundary k-connections for Σ_{m+1} is the same as that of Σ_m. □

Each path in this single connection will be called a *ray* (by analogy with the "circles and rays" networks of [2]). We will index the rays $1, \dots, N$ in some counterclockwise order. The collection of plates along the nth ray will be called $\mathcal{P}_n$, and the plates of $\mathcal{P}_n$ will be $P_{n,1}, P_{n,2}, \dots P_{n,K_n}$, ordered from innermost to outermost. In the following sections, we assume that the indexing of the inner and outer boundary plates is consistent with the indexing of the rays. We label the inner boundary intervals $I_1, \dots, I_N$, and the outer boundary intervals $J_1, \dots, J_N$.

5.2 Pseudo-Geodesics and Dominant Geodesics

Let y_0 be a geodesic endpoint on the inner boundary. Let e_1 be a plate edge with endpoint y_0. Let y_1 the other endpoint of e_1. For $k = 1, 2, \dots$, let e_{k+1} be the counterclockwise outer edge at y_k, and let y_{k+1} be the other endpoint of e_{k+1}. Continue inductively until y_K is on the outer boundary. The curve formed by $e_1, \dots, e_K$ is a *counterclockwise outward* pseudo-geodesic. A *clockwise outward pseudo-geodesic* is defined in a symmetrical way. For *counterclockwise* and *clockwise inward* pseudo-geodesics, we perform the same process but begin at the *outer* boundary and choose counterclockwise or clockwise *inner* edges.

Figure 18: Geodesics in the universal cover. A counterclockwise outward pseudo-geodesic is shown in blue. g_2 and g_8 are counterlclockwise-dominant; g_1, g_4, g_5, and g_6 are clockwise-dominant.



A geodesic is called *counterclockwise-dominant* if it is identical to a counterclockwise outward pseudo-geodesic. Equivalently, g is counterclockwise-dominant if for every geodesic h which crosses g, g crosses h counterclockwise and h crosses g clockwise. Similarly, a geodesic is *clockwise-dominant* if it is identical to a clockwise outward pseudo-geodesic.

The *slant* of a geodesic or pseudo-geodesic is defined as follows. Let the geodesic endpoints on the upper and lower boundaries of the universal cover be y_j and z_j for all integers j. Index them from left to right, and such that y_j and z_j are on the same ray in the universal cover and the same side of the ray. Let g be a (pseudo-)geodesic with endpoints y_i and z_j. The slant of g is $j - i$.

The outward counterclockwise pseudo-geodesic beginning at y is the path of maximal slant out of all paths consisting of positively oriented edges which begin at y and end at the outer boundary.

As we will see in the next section, pseudo-geodesics and dominant geodesics are important for analyzing information propagation and recoverability. To do so, we need the following results:

Lemma 5.2. *Let h be a counterclockwise outward pseudo-geodesic. Let $g_1, \ldots, g_K$ be the geodesics which share edges with h, listed in outward order along h. Then g_K is counterclockwise-dominant; g_K is either the same as g_1 or the first counterclockwise-dominant geodesic intersected by g_1.*

Proof. Consider the geodesics in the universal cover. The lower endpoint of h must lie to the left of the lower endpoint of each g_k. If h crosses any geodesic g, then h must cross g counterclockwise or g must intersect h in an edge. Suppose that some geodesic g^* intersects g_K clockwise. Then the endpoint of g^* is to the left of the endpoint of g_K, so g^* must not intersect h in an edge. Thus, g^* must cross h counterclockwise, which is impossible. Therefore, g_K is counterclockwise-dominant.

Suppose g^* is a counterclockwise-dominant geodesic such that g_1 intersects g^* before g_K. Then g^* crosses h counterclockwise. The argument of the previous paragraph shows this is impossible. □

Corollary 5.3. *The endpoints and slants of pseudo-geodesics are unaffected by Y-Δ transformations.*

Proof. The endpoint of h is determined by g_K, the first counterclockwise-dominant geodesic which is intersected by g_1. No two counterclockwise-dominant geodesics intersect because if they did, then one would cross the other counterclockwise. Also, g_1 must cross any counterclockwise-dominant geodesic in the clockwise direction. Thus, Y-Δ transformations cannot change the order in which g_1 intersects these counterclockwise-dominant geodesics. Therefore, they do not affect which geodesic is g_K.

A symmetrical argument holds for the other types of pseudo-geodesics. □

Lemma 5.4. *Let $g_1, \ldots, g_{2N}$ be the geodesics of Γ, ordered counterclockwise by their endpoints on the outer boundary. If Γ is not the trivial network, there exists a j such that g_j is counterclockwise-dominant and intersects g_{j-1} (indices reduced modulo $2N$).*

Proof. Suppose that no such geodesic exists. Let g_J be a geodesic of maximal slant. If a geodesic h intersected g_J counterclockwise, then h would have greater slant. Thus, g_J is counterclockwise-dominant. Since g_{J-1} does not intersect g_J, the slant of g_{J-1} is greater than or equal to the slant of g_J, so g_{J-1} must also have maximal slant. Proceeding by induction, we see that all the geodesics have maximal slant and are counterclockwise-dominant, which is impossible unless Γ is the trivial network. □

Lemma 5.5. *A nontrivial radial network with no removable lenses is Y-Δ equivalent to one which has an empty boundary triangle on the outer boundary, such that one of the geodesics forming the triangle is counterclockwise-dominant.*

Proof. Let g_j be a counterclockwise-dominant geodesic which intersects g_{j-1}. Let y be their point of intersection closest to the outer boundary. Then there is a triangle T with a vertex at y formed be g_j, g_{j-1}, and an arc of the outer boundary, such that T lies within a simply connected region of the network. Any geodesic which intersects T must enter T along g_{j-1} and exit along g_j. By Y-Δ transformations, we can remove all crossings out of T. After that, we can empty T of all geodesics. Then T is an empty boundary triangle. $\square$

5.3 Principal Electrical Functions

For any inner boundary plate $P_{j,1}$, there is a unique γ-harmonic function f with $v(P_{j,1}) = 1$ and all other voltages and all currents on the inner boundary equal to zero. We will call f the *principal electrical function for* $P_{j,1}$.

Let y_A and y_B the the clockwise and counterclockwise endpoints of I_j. Let h_A be the outward clockwise pseudo-geodesic with endpoint y_A and let h_B be the outward counterclockwise pseudo-geodesic with endpoint y_B. Let R be the arc of the inner boundary complementary to I_j. Let S be the region of embedding.

For $P_{j,1}$, the *zone of no propagation* U is the component of $S \setminus (h_A \cup h_B)$ which touches the inner boundary and does not intersect $P_{j,1}$. The *zone of propagation* V is $S \setminus U$. Let Q_A and Q_B be the outer boundary plates along h_A and h_B. See Figure 19 (W is explained later).

The zone of propagation is called *simple* if it is simply connected and $Q_A \neq Q_B$. This means that If either Q_A or Q_B is in $\overline{U}$, then there must be at least one geodesic endpoint outside of V.

Theorem 5.6. *Let f be the principal electrical function for $P_{j,1}$. Suppose the zone of propagation is simple and that the rays are indexed with $Q_B = P_{1,K_1}$. If $P_{j,1}$ has at least two junctures, then*

- *If $P_{n,k}$ intersects U, then $v(P_{n,k}) = 0$.*
- *If $P_{n,k} \subset \overline{V}$, then $(-1)^{n+j} v(P_{n,k}) > 0$.*
- *If the juncture $P_{n,k}P_{n,k+1} \in U$, then $c(P_{n,k} \to P_{n,k+1}) = 0$.*
- *If $P_{n,k}P_{n,k+1} \in V$, then $(-1)^{n+j} c(P_{n,k} \to P_{n,k+1}) < 0$.*
- *If $J_n \subset \overline{U}$, then $c(J_n) = 0$.*
- *If J_n intersects $\overline{V}$, then $(-1)^{n+j+1} c(J_n) < 0$.*

Figure 19: Zones of propagation for an inner boundary plate.

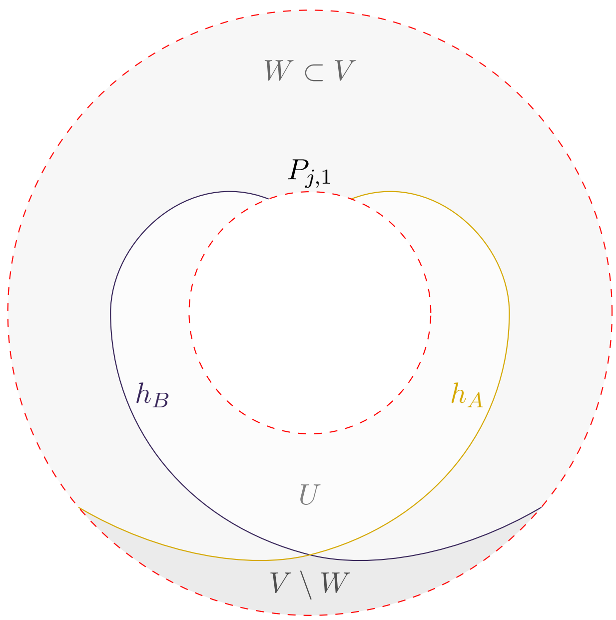
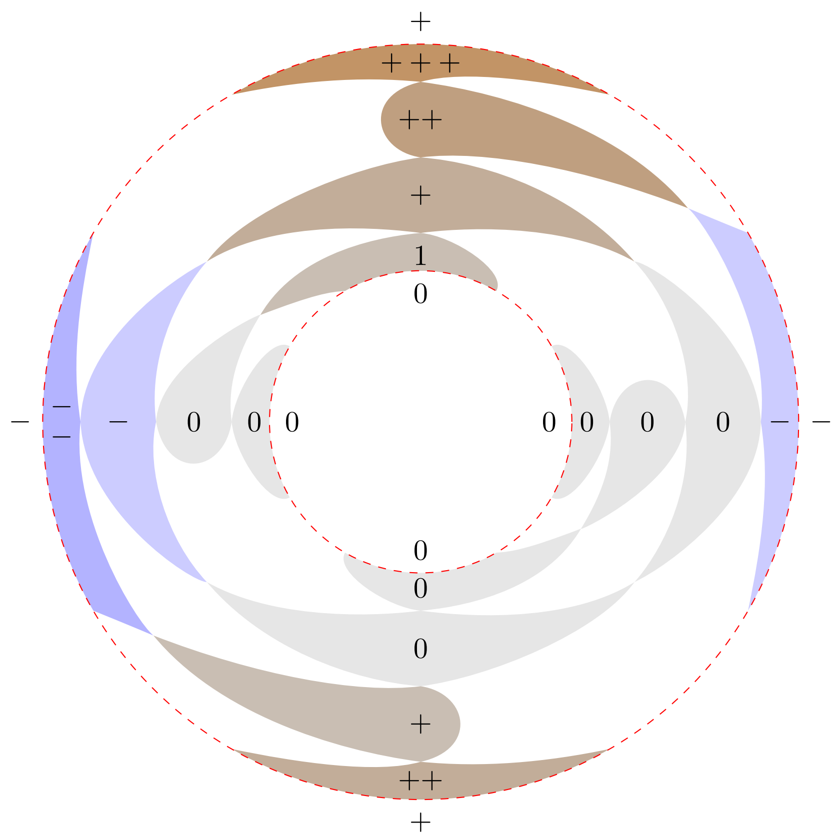


Figure 20: The principal electrical function of a boundary plate.



Otherwise, we make the following exceptions:

- *If* $P_{j,1}$ *touches both boundaries and has no junctures, then* $c(J_1) = 0$.
- *If* $P_{j,1}$ *is a boundary spike, then* $c(P_{j,1} \to P_{j,2}) = 0$.
- *If* $P_{j,1}$ *is a boundary spike, and* $P_{j,2}$ *is on the outer boundary and has no junctures besides* $P_{j,1}P_{j,2}$, *then* $c(J_1) = 0$.

Roughly speaking, this means that f is zero in the zone of no propagation, and in the zone of propagation, the voltages are positive and increasing or negative and decreasing on each ray, in an alternating pattern.

Proof. Let $\Gamma_0, \ldots, \Gamma_M$ be a partition of Γ where Γ_0 is a trivial network and $\Gamma_1, \ldots, \Gamma_M$ are elementary boundary-juncture and spike networks. Let Σ_m consist of $\Gamma_0, \ldots, \Gamma_m$. Let $J_{n,m}$ be the outer boundary interval for Σ_m on the nth ray. Notice that the zone of no propagation for Σ_m is exactly the part of U which intersects Σ_m's region of embedding, and the same is true for the zone of propagation.

We show that the theorem is true for Σ_m by induction. The base case is trivial. Next, we show that if the theorem holds for Σ_m, it holds for Σ_{m+1}.

Suppose Γ_{m+1} is a spike network with juncture $P_{n,k}P_{n,k+1}$. There are several cases:

1. $P_{n,k} = P_{j,1}$ and $P_{j,1}$ is a spike. In this case, $c(J_{n,m}) = 0$, so $c(P_{n,k} \to P_{n,k+1}) = 0$ and $v(P_{n,k}) = 1 > 0$.

2. $P_{n,k} = P_{j,2}$ and $P_{j,1}$ is a spike. We know that $P_{j,2}$ has junctures with other plates besides $P_{j,1}$ and $P_{j,3}$ because otherwise it would form a series connection, which corresponds to a removable lens. Thus, by inductive hypothesis, $c(J_{n,m}) < 0$. For the rest of the argument, see the next case.

3. $P_{n,k}P_{n,k+1} \in V$. Then $(-1)^{n+j}c(P_{n,k} \to P_{n,k+1}) = (-1)^{n+j+1}c(J_{n,m})$, which is negative by hypothesis because $J_{n,m}$ intersects $\overline{V}$. Also by hypothesis, $(-1)^{n+j}v(P_{n,k}) \geq 0$, and $(-1)^{n+j}v(P_{n,k+1}) > (-1)^{n+j}v(P_{n,k})$ because of the sign of $c(P_{n,k} \to P_{n,k+1})$.

4. $P_{n,k}P_{n,k+1} \in U$. In this case, $J_{n,m} \subset \overline{U}$, so its current is 0 by inductive hypothesis. This implies $c(P_{n,k} \to P_{n,k+1}) = 0$. Since $P_{n,k}$ intersects U, its voltage is zero, so $v(P_{n,k+1}) = 0$.

Suppose Γ_{m+1} is a boundary juncture network with juncture $P_{n,k}P_{n+1,k'}$ with $1 \leq n \leq N-1$. There are two cases:

1. In the case where $P_{n,k}P_{n+1,k'} \in U$, both plates intersect U, so their voltages are zero. Thus, the outer boundary data of Σ_{m+1} is the same as that of Σ_m.

2. Suppose $P_{n,k}P_{n+1,k'} \in V$. The outer boundary voltages for Σ_{m+1} are the same as for Σ_m. By inductive hypothesis, $(-1)^{n+j}v(P_{n,k}) \geq 0$ and $(-1)^{n+j+1}v(P_{n+1,k'}) \geq 0$. At least one of the plates is in the zone of propagation, so at least one of the inequalities is strict. This implies $(-1)^{n+j}c(P_{n,k} \to P_{n+1,k'}) > 0$. Both boundary intervals $J_{n,m+1}$ and $J_{n+1,m+1}$ intersect $\overline{V}$. Because of the sign of $c(P_{n,k} \to P_{n+1,k'})$,

$$(-1)^{j+n+1}c(J_{n,m+1}) > (-1)^{j+n+1}c(J_{n,m}),$$

which is nonnegative by hypothesis. Similarly,

$$(-1)^{j+n}c(J_{n+1,m+1}) > (-1)^{j+n}c(J_{n+1,m}) \geq 0.$$

Finally, if Γ_{m+1} is a boundary-juncture network with juncture $P_{N,k}P_{1,k'}$, then the juncture is in U because of the zone of propagation is simple and because of how we indexed the rays. This completes the induction argument and hence the proof. □

The *principal electrical function for an inner boundary interval* I_j is defined similarly. It is the γ-harmonic function f with $c(I_j) = 1$ and all other voltages and all currents on the inner boundary equal to zero. We let y_A and y_B be the clockwise and counterclockwise endpoints of I_j. Counterintuitively, h_A is the clockwise outward pseudo-geodesic with endpoint at y_B and h_B is the counterclockwise outward pseudo-geodesic with endpoint at y_A.

If $P_{j,1}$ touches both boundaries, then h_A and h_B do not intersect, and we define the zone of no propagation as S and the zone of propagation as J_j. Otherwise, h_A and h_B intersect at $P_{j,1}P_{j,2}$. We let h'_A and h'_B be the arcs of h_A and h_B starting at $P_{j,1}P_{j,2}$ and ending at the outer boundary. Then U, the zone of no propagation, is the component of $S \setminus (h'_A \cup h'_B)$ which touches the inner boundary, and $V = S \setminus U$. (The picture is the same as Figure 19 except that h_A and h_B are crossed at their inner endpoints.)

Then we have the following theorem. The proof is essentially the same as for Theorem 5.6, so the details are left to the reader:

Theorem 5.7. *Let f be the principal electrical function for I_j. Suppose the zone of propagation is simple and that the rays are indexed with $Q_B = P_{1,K_1}$. Then*

- *If $P_{n,k}$ intersects U, then $v(P_{n,k}) = 0$.*
- *If $P_{n,k} \subset \overline{V}$, then $(-1)^{n+j}v(P_{n,k}) < 0$.*
- *If the juncture $P_{n,k}P_{n,k+1} \in U$, then $c(P_{n,k} \to P_{n,k+1}) = 0$.*
- *If $P_{n,k}P_{n,k+1} \in V$, then $(-1)^{n+j}c(P_{n,k} \to P_{n,k+1}) > 0$.*
- *If $J_n \subset \overline{U}$, then $c(J_n) = 0$.*
- *If J_n intersects V, then $(-1)^{n+j+1}c(J_n) > 0$.*

The last two theorems hold in more generality when N is even:

Corollary 5.8. *If N is even, Theorems 5.6 and 5.7 hold even if the zone of propagation is not simple.*

Proof. Examine the proof of Theorem 5.6. The only case that required the assumption that the region of propagation was simple was the last case where Γ_{m+1} was a boundary-juncture network with juncture $P_{N,k}P_{1,k'}$. If we remove that assumption, $P_{N,k}P_{1,k'}$ may be in the zone of propagation. If N is odd, $v(P_{N,k})$ and $v(P_{1,k'})$ have the same sign, so we do not know the sign of $c(P_{N,k} \to P_{1,k'})$. However, if N is even, the two voltages have opposite signs, and we can apply the argument given in subcase 2 of the case when Γ_{m+1} is a boundary-juncture network. □

For the odd case, the theorems do not hold in general, but there is a subregion of the network for which they do hold. We define the *zone of simple propagation* W as follows. If the zone of propagation is simple, $W = V$. Otherwise, if the zone of propagation is simply connected but not simple, then $Q_A = Q_B$ and W is defined to be V minus the outer boundary interval of Q_A. Otherwise, if the zone is for a plate, let h''_A and h''_B be the arcs h_A and h_B from their first point of intersection to the outer boundary; if the zone is for a boundary interval, let h''_A and h''_B be the arcs of h_A and h_B from their second point of intersect to the outer boundary. Let X be the component of $S \setminus (h''_A \cup h''_B)$ which touches the inner boundary, and let $W = X \setminus U$.

Corollary 5.9. *Theorems 5.6 and 5.7 hold in the zone of no propagation and the zone of simple propagation.*

Proof. If the whole zone of propagation is simple, then $W = V$, so we are done. Otherwise, consider a subnetwork of Γ which intersects all the plates and contains all the junctures in the zone of no propagation and the zone of simple propagation, but no other plates and junctures. Apply the theorems to this subnetwork. □

5.4 Factorization of the Interboundary Map

For each $\mathbf{y} \in \mathbb{R}^{2N}$ representing data on one boundary, let $\tilde{\mathbf{y}}$ be the vector with the signs of the current entries changed. That is, if $\mathbf{y} = y_1, \dots y_N, y_{N+1}, \dots, y_{2N}$, then $\tilde{\mathbf{y}} = y_1, \dots, y_N, -y_{N+1}, \dots, -y_{2N}$. The current entries for $\mathbf{y}$ represent current *entering* the network; the current entries for $\tilde{\mathbf{y}}$ represent current *exiting* the network.

Let $\mathbf{x} \in \mathbb{R}^{2N}$ be a vector representing voltage and current data on the inner boundary of a radial network Γ with no removable lenses. There is a unique $\mathbf{y}$ representing data on the outer boundary which is compatible with $\mathbf{x}$. There is an invertible linear transformation Ξ mapping $\mathbf{x}$ to $\tilde{\mathbf{y}}$, which we will call the *interboundary map*. By direct computation,

Proposition 5.10. *In a radial network with no removable lenses, if no plate touches both boundaries, then*

$$\Xi = \begin{pmatrix} -\Lambda_{IO}^{-1}\Lambda_{II} & \Lambda_{IO}^{-1} \\ -\Lambda_{OI} + \Lambda_{OO}\Lambda_{IO}^{-1}\Lambda_{ii} & -\Lambda_{OO}\Lambda_{IO}^{-1} \end{pmatrix}.$$

Each column of Ξ represents outer boundary data for one of the principal electrical functions. Thus, the theorems of the previous sections provide sign conditions on the entries of Ξ. Ξ behaves nicely with respect to partitions:

Theorem 5.11. *Let $\Gamma_1, \dots, \Gamma_M$ be a partition of Γ into elementary boundary-juncture and spike networks, with boundary plates indexed according to the rays of Γ. Then $\Xi = \Xi_M \Xi_{M-1} \dots \Xi_1$.*

Proof. Let $\Gamma_1, \dots, \Gamma_M$ be a partition of Γ into elementary boundary-juncture and spike networks. Let Σ_m consist of $\Gamma_0, \dots, \Gamma_m$. We prove the theorem for each Σ_m by induction. The base case Σ_1 is trivial.

Suppose the claim is true for Σ_m, and I will show it is true for Σ_{m+1}. Let $Q_{n,m}$ and $J_{n,m}$ be the outer boundary plate and interval for Σ_m on the nth ray, and let $R_{n,m}$ and $I_{n,m}$ be the inner boundary plate and interval for Γ_m.

Let $\mathbf{x}$ be a vector of inner boundary data for Σ_{m+1}, which determines a unique γ-harmonic function f. Then $\Xi_m \Xi_{m-1} \dots \Xi_1 \mathbf{x}$ gives the data on the outer boundary of Σ_m: voltages for $Q_{1,m}, \dots, Q_{N,m}$ and minus the current on $I_{1,m}, \dots, I_{N,m}$. Since $Q_{n,m}$ and $R_{n,m+1}$ are subplates of the same plate in Σ_{m+1}, we need $v(R_{n,m+1}) = v(Q_{n,m})$ and $c(I_{n,m+1}) = -c(J_{n,m})$. Thus, the data on the inner boundary of Γ_{m+1} is $\Xi_m \Xi_{m-1} \dots \Xi_1 \mathbf{x}$, so the data on the outer boundary of Σ_{m+1} is $\Xi_{m+1} \Xi_m \dots \Xi_1 \mathbf{x}$. □

We can think of Ξ as a matrix with rows and columns indexed by integers $1, \dots, 2N$. Alternatively, we can index the rows and columns by the boundary plates and the boundary intervals, where the plates correspond to the voltage data and the intervals to the current data. That is, if the inner and outer boundary plates are $R_n = P_{n,1}$ and $Q_n = P_{n,K_n}$, then for

$k, \ell = 1, \ldots, N$, we write

$$\begin{aligned}
\xi_{\ell,k} &= \xi(Q_\ell, R_k) \\
\xi_{\ell,N+k} &= \xi(Q_\ell, I_k) \\
\xi_{N+\ell,k} &= \xi(J_\ell, R_k) \\
\xi_{N+\ell,N+k} &= \xi(J_\ell, I_k).
\end{aligned}$$

Partition into elementary networks corresponds to factorization of Ξ. Each factor is a nearly-elementary matrix. For instance, consider an elementary spike network with inner boundary plates where $R_n = Q_n$ for $n \neq 1$ and R_1 and Q_1 are connected by a juncture with conductance γ. For $n \neq 1$, we need $v(Q_n) = v(R_n)$. Since the currents on R_n must add up to zero, $c(J_n) = -c(I_n)$ for $n \neq 1$. Also, $c(I_1) = c(R_1 \to Q_1) = -c(J_1)$ and, since $c(R_1 \to Q_1) = \gamma(v(R_1) - v(Q_1))$,

$$v(Q_1) = v(R_1) - \frac{1}{\gamma} c(I_1).$$

Thus, $\xi(Q_n, R_n) = 1$ and $\xi(J_n, I_n) = 1$ for all n, $\xi(Q_1, I_1) = -1/\gamma$, and all other entries of Ξ are zero. For example, when $N = 3$,

$$\Xi = \begin{pmatrix} 1 & 0 & 0 & -1/\gamma & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Now consider an elementary boundary-juncture network where the juncture connectes P_1 and P_2. Since $R_n = Q_n$ for all n, we need $v(R_n) = v(Q_n)$. For $n \neq 1, 2$, $c(J_n) = -c(I_n)$. Since $c(I_1) + c(R_2 \to R_1) + c(J_1) = 0$ and $c(R_2 \to R_1) = \gamma(v(R_2) - v(R_1))$ for any γ-harmonic function, we have

$$-c(J_1) = c(I_1) + \gamma v(R_2) - \gamma v(R_1).$$

Similarly,

$$-c(J_2) = c(I_2) + \gamma v(R_1) - \gamma v(R_2).$$

Thus, $\xi(Q_n, R_n) = 1$ and $\xi(J_n, I_n) = 1$ for all n, $\xi(J_1, R_2) = \xi(J_2.R_1) = \gamma$, $\xi(J_1, R_1) = \xi(J_2, R_2) = -\gamma$, and all other entries of ξ are zero. For example, when $N = 3$,

$$\Xi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\gamma & \gamma & 0 & 1 & 0 & 0 \\ \gamma & -\gamma & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

If the juncture is at a different place, the matrix has a similar form, only the rows and columns are permuted by shifting the indices some amount modulo N.

Each "elementary" matrix has determinant 1, so $\det \Xi = 1$.

When we modify Γ by contracting a spike or deleting a boundary juncture, Ξ is easy to update. Suppose that Γ is partitioned into elementary networks $\Gamma_1, \dots, \Gamma_M$ where Γ_M is an elementary spike network. Suppose that Γ' is obtained from Γ by contracting the spike. Then $\Xi = \Xi_M \Xi_{M-1} \dots \Xi_1$, and Ξ_M. Contracting the spike, or transforming Γ into Γ', is equivalent to replacing Γ with the subnetwork consisting of $\Gamma_1, \dots, \Gamma_{M-1}$. Hence, the interboundary map for Γ' is

$$\Xi' = \Xi_{M-1} \Xi_{M-2} \dots \Xi_1 = \Xi_M^{-1} \Xi.$$

If Q is the spike, PQ is its juncture, and I is the boundary interval, then Ξ_M is an elementary matrix which corresponds to the row operation of subtracting $1/\gamma(PQ)$ times row I to row Q. Ξ_M^{-1} corresponds to the reverse row operation. Contracting a spike on the *inner* boundary is similar, except that we *right*-multiply by the appropriate matrix (perform a *column* operation).

Deleting a boundary edge is similar. If Γ_M is an elementary boundary-juncture network, then Ξ_M has four off-diagonal entries, which correspond to four row operations. Thus, the inverse matrix also corresponds to four row operations.

5.5 Recovery

We can recover certain boundary edges and spikes using the interboundary map and the principal electrical functions. Consider a boundary plate $P_{j,1}$. Let h_A and h_B be the corresponding pseudo-geodesics, and let Q_A and Q_B be as in §5.3, and let f be the principal electrical function for $P_{j,1}$. Suppose that $Q_B = P_{i,K_i}$ is a boundary spike. If the zone of propagation is simple, then P_{i,K_i-1} intersects the zone of no propagation, so its voltage is zero. But we know by Theorem 5.6 that the voltage and current of P_{i,K_i} are nonzero. Thus, we can determine the conductance of the spike.

The outer boundary data for f is represented by the jth column of Ξ. Thus,

$$\gamma(P_{i,K_i-1}P_{i,K_i}) = \frac{c(P_{i,K_i-1} \to P_{i,K_i})}{v(P_{i,K_i-1}) - v(P_{i,K_i})} = \frac{-c(J_i)}{-v(P_{i,K_i})} = -\frac{\xi_{N+i,j}}{\xi_{i,j}}.$$

The conductance of the spike is the quotient of two entries of Ξ.

The same reasoning applies if f is the principal electrical function for I_j and $Q_B = P_{i,K_i}$ is a boundary spike. If the zone of propagation is simple, then

$$\gamma(P_{i,K_i-1}P_{i,K_i}) = -\frac{\xi_{N+i,N+j}}{\xi_{i,N+j}}.$$

In fact, this method will work in a slightly more general case. If the zone is propagation is not simple, but if P_{i,K_i} is the *only* plate not in the zone of simple propagation, then P_{i,K_i-1} will still have voltage zero. If P_{i,K_i} has nonzero voltage (which will happen if N is even), then the above formulae are still valid.

Now consider the case of a boundary juncture. Let f be the principal electrical function for $P_{j,1}$. Suppose $P_{i,K_i}P_{i+1,K_{i+1}}$ is a boundary juncture and $Q_B = P_{i,K_i}$, and that the zone of propagation is simple. Then P_{i,K_i} and all junctures of P_{i,K_i} except $P_{i,K_i}P_{i+1,K_{i+1}}$ are in the zone of no propagation. Thus, $v(P_{i,K_i}) = 0$ and $c(J_i) = c(P_{i,K_I} \to P_{i+1,K_{i+1}}) \neq 0$. Thus,

$$\gamma(P_{i,K_i}P_{i+1,K_{i+1}}) = \frac{c(J_i)}{-v(P_{i+1,K_{i+1}})} = \frac{\xi_{N+i,j}}{\xi_{i+1,j}}.$$

Similarly, if f is the principal electrical function for an interval I_j instead of a plate $P_{j,1}$, then

$$\gamma(P_{i,K_i}P_{i+1,K_{i+1}}) = \frac{\xi_{N+i,N+j}}{\xi_{i+1,N+j}}.$$

Unfortunately, this method does not work if the zone of propagation is not simple. Even the “simplicity” requirement is stronger for the boundary juncture than for the spike. If Q_B is on the clockwise side of h_B as in the case with the spike, then it merely requires that the outer endpoint of h_B is clockwise from the outer endpoint of h_A (that is, clockwise as we pass through the zone of no propagation). However, if Q_B is on the counterclockwise side of h_B, the endpoint of h_B must be clockwise from the outer endpoint of h_A and there must be at least one geodesic endpoint in between them.

To discuss recoverability globally, we use the lemmas of §5.2.

Theorem 5.12. *Let Γ be a radial network with no removable lenses. Suppose that for each inner boundary plate and boundary interval, the zone of propagation is simple. Then Γ is recoverable.*

Proof. To determine the conductances of Γ, it is sufficient to determine the conductances of a Y-Δ equivalent network. We recover conductances in the following way:

1. Use Y-Δ transformations to change Γ into a network with an empty boundary triangle at which one of the geodesics, g, is counterclockwise-dominant. This is possible by Lemma 5.5. By Corollary 5.3, the endpoints of pseudo-geodesics are not affected by Y-Δ transformations, so the zones of propagation are still simple.

2. Let $P_{j,1}$ be the plate at the inner endpoint of g. If $P_{j,1}$ is on the clockwise side of g, we can use the principal electrical function of $P_{j,1}$ to recover the conductance of spike or boundary juncture at the empty boundary triangle. If $P_{j,1}$ is on the counterclockwise side of g, then we can use the principal electrical function of I_j.

3. Uncross the empty boundary triangle. Perform one or four row operations on Ξ to find the interboundary map of the modified network. In the modified network the zones of propagation are still simple. This is because uncrossing an empty boundary triangle can only *decrease* the slant of counterclockwise outward pseudo-geodesics and *increase* the slant of clockwise outward pseudo-geodesics.

4. Repeat the process. The modified network still satisfies the hypotheses of the theorem.

Eventually, all junctures have been removed and all conductances recovered. Γ has been transformed into a trivial network, and Ξ has been row-reduced to the identity matrix. □

Of course, there is an analogous theorem with the roles of the inner and outer boundaries reversed.

6 Open Problems

6.1 Recoverability

The recoverability results of this paper are incomplete in several ways. First and most importantly, I have not dealt with networks which have type 0 geodesics. For some results on a particular class of networks with type 0 geodesics, see [4]. For networks with type 0 geodesics, the inverse problem may have *finitely many solutions*, something which does not happen in circular planar networks. In other words, the network is *discretely unrecoverable.*

I conjecture that

- If a type 0 geodesic intersects itself, then the inverse problem has infinitely many solutions.
- If the network is discretely unrecoverable, then there are type 0 geodesics.

Second, Theorem 4.7 and 5.12 are incomplete as recoverability criteria. There are recoverable networks in which type 1 geodesics form lenses. There are recoverable radial networks which cannot be recovered by the method described in this paper. However, I believe that networks which "flagrantly" violate the hypotheses of these theorems are not recoverable.

Third, the recovery methods of this paper relied heavily on Y-Δ transformations. More numerically efficient methods should be explored.

6.2 Nonlinear Conductances

We should consider a generalization to nonlinear conductance functions of the type described by Will Johnson in [3]. Like Johnson's geodesic closures, partition into subnetworks provides a way to analyze how boundary-value information propagates through the network. And partition into subnetworks does not rely on covoltages (which are problematic for non-simply-connected regions).

Solution spaces are well-defined for nonlinear conductances. Certain theorems of this paper state that any data is feasible and the only zero-compatible data is $\mathbf{0}$ for a certain boundary curve or cut (such as Lemmas 3.9 and 4.2). I believe these theorems will hold for nonlinear conductances with essentially the same proof. In other cases (Theorems 3.10 and 4.1), it is difficult to define the dimension of a solution space, although we can at least say that it has a continuous injective parametrization in a specified number of variables.

I believe Theorem 4.7 (partial recovery by removal of type 1 geodesics) will hold with essentially the same proof. However, widespread application of the theorem requires the ability to perform Y-Δ transformations while computing conductances which make the new network electrically equivalent. To my knowledge, this has not yet been done for nonlinear conductances. The same issue may interfere with the proof of Theorem 5.12 (recoverability of certain radial networks).

The theorems on the principal electrical functions for radial networks will still hold. However, "factorization" of the interboundary map will require care.

It is anyone's guess what might happen when we put nonlinear conductances on discretely unrecoverable networks.

6.3 More Complicated Regions of Embedding

Consider a generalization to planar regions with an arbitrary number of boundary curves. Such networks will require all the lens removal techniques for annular networks and possibly more, although I suspect the most dramatic contrast is between networks in simply connected regions and networks in non-simply-connected regions. There are probably analogues of Theorem 2.20 (layered form) and Theorem 4.7.

We might also consider networks on a surface. [5] has considered infinite networks in a half-plane, but perhaps annular techniques can be applied to networks in an infinite, non-simply-connected region.

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