The N-gon in N-gon Network Revisited

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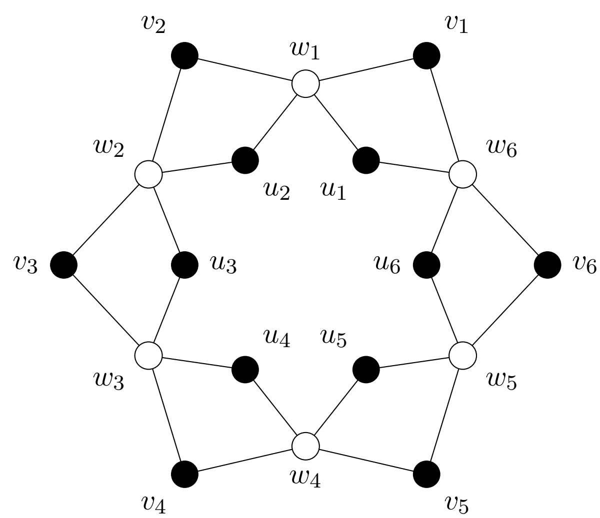
Abstract

I reexamine the 2-to-1 network in [2], giving a new and complete proof of their results. Using the determinant-connection formula, I show that there are always exactly one or two valid conductivity functions for any valid response matrix and give explicit formulas for the solutions in terms of one of the conductivity functions. The conditions for a unique solution and a symmetric characterization of the response matrix are included.

Contents

Figure 1: The network Γ with $N = 6$.



1 Introduction

For definitions of *graphs with boundary*, *electrical networks*, the *Kirchhoff matrix*, the *response matrix*, and *k-connections* see [1].

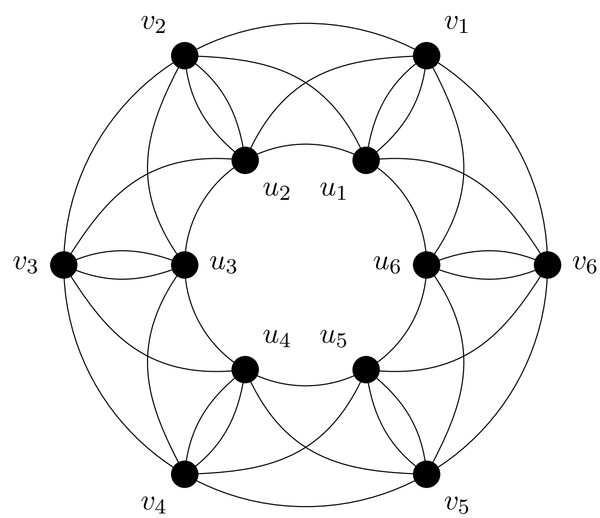
We consider a network Γ on a graph which is a loop of N four-stars. Γ has boundary vertices $u_1, \ldots, u_N$ and $v_1, \ldots, v_N$, and interior vertices $w_1, \ldots, w_N$. The nth four-star has central vertex w_n and boundary vertices u_n, v_n, u_{n+1}, and v_{n+1} with indices reduced modulo N. For pictoral purposes, assume Γ is embedded in an annulus with vertices $u_1, \ldots, u_N$ on the inner circle and $v_1 \ldots v_N$ on the outer circle, and that the indices proceed counterclockwise around the annulus.

Always assume $N \geq 3$ and that indices are reduced modulo N (for instance, v_{N+1} is the same as v_1). The conductances of the edges will be labeled

$$\begin{aligned} a_n &= \gamma(u_n w_n) \\ b_n &= \gamma(v_n w_n) \\ c_n &= \gamma(w_n u_{n+1}) \\ d_n &= \gamma(w_n v_{n+1}), \end{aligned}$$

and we let $\sigma_n = a_n + b_n + c_n + d_n$.

Figure 2: Γ with $N = 6$ after ★-$\mathcal{K}$ transformation.



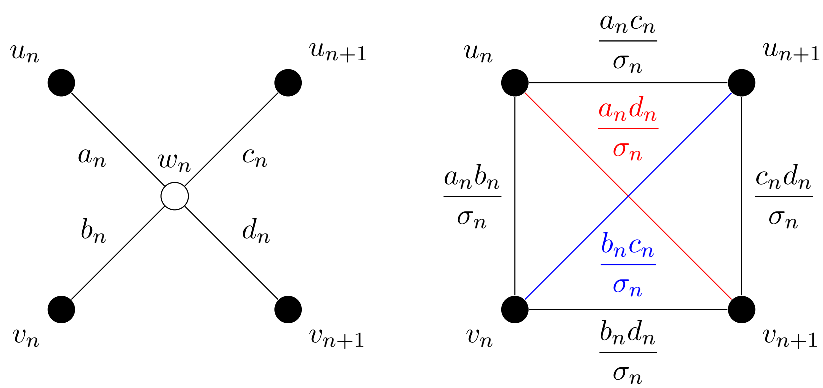
Following the sign conventions for N-to-1 graphs, we assume the off-diagonal entries of Λ are *nonnegative* and the diagonal entries are *negative.* To avoid nested subscripts, we will denote the entry of Λ for vertices u and v by $\lambda(uv)$ (for instance, $\lambda(u_1v_2)$).

By a *quasi-conductivity* we mean a function defined on edges that allows negative or complex "conductances." A conductivity or a quasi-conductivity is called consistent (with a given response matrix Λ) if it has Λ as its response matrix.

We apply the ★-$\mathcal{K}$ transformation described in [3] to the network, replacing each four-star with an electrically equivalent complete graph on four vertices. The new network has a double edge between u_n and v_n. We will denote the single edges in these pairs, $e_1, \ldots, e_N$ and $e'_1, \ldots, e'_N$. The index corresponds to the index of the four-star to which the edge belongs (or to the index of the w_n); e_n is the edge between u_n and v_n corresponding to w_n and e'_n is the edge between u_{n+1} and v_{n+1} corresponding to w_n. All other edges in the transformed graph will be named by the vertices at their endpoints.

The conductivity function on the transformed graph will be called μ,

Figure 3: Conductances on a four-star and the corresponding $\mathcal{K}_4$.



and is given by

$$\begin{aligned}
\mu(u_n u_{n+1}) &= \frac{a_n c_n}{\sigma_n} = \lambda(u_n u_{n+1}) \\
\mu(v_n v_{n+1}) &= \frac{b_n d_n}{\sigma_n} = \lambda(v_n v_{n+1}) \\
\mu(u_n v_{n+1}) &= \frac{a_n d_n}{\sigma_n} = \lambda(u_n v_{n+1}) \\
\mu(v_n u_{n+1}) &= \frac{b_n c_n}{\sigma_n} = \lambda(v_n u_{n+1}) \\
\mu(e_n) &= \frac{a_n b_n}{\sigma_n} \\
\mu(e'_n) &= \frac{c_n d_n}{\sigma_n} \\
\lambda(u_n v_n) &= \mu(e'_{n-1}) + \mu(e_n).
\end{aligned}$$

The conductances satisfy the *quadrilateral rule*:

$$\mu(e_n) \cdot \mu(e'_n) = \mu(u_n u_{n+1}) \cdot \mu(v_n v_{n+1}) = \mu(u_n v_{n+1}) \cdot \mu(v_n u_{n+1}).$$

In fact, as shown in [3], any conductivity function μ on the transformed graph which satisfies this condition corresponds to a conductivity function γ on the original graph. Thus, specifying such a μ is a valid way to define conductivities on Γ.

2 The Determinant-Connection Formula

Minors of Λ are related to connections in the graph by the *determinant-connection formula* (Lemma 3.12 of [1]). When considering a k-connection α, we will write $p \mapsto q$ if a path of α connects p and q. If $P = \{p_1, \ldots, p_k\}$ and $Q = \{q_1, \ldots, q_k\}$ are disjoint sets of boundary vertices, and α is a k-connection between P and Q, then τ_α is the permutation of the symmetric group S_k such that $p_n \mapsto q_{\tau(n)}$ for each n. We let I be the set of interior vertices, I_α be the set of interior vertices used in α, and J_α as the set of interior vertices *not* used in α and let $D_\alpha = \det K(J_\alpha; J_\alpha)$. The determinant-connection formula says that

$$\det \Lambda(P; Q) \det K(I; I) = (-1)^k \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \sum_{\substack{\alpha \\ \tau_\alpha = \tau}} D_\alpha \prod_{e \in E_\alpha} \gamma(e),$$

where the second sum is taken over all k-connections which exist between P and Q.

We denote by $\Lambda(u_K \ldots u_L; v_K \ldots v_L)$ the submatrix formed by rows u_K, u_{K+1}, …, u_K and v_J, v_{J+1}, …, v_K with indices reduced modulo N.

Theorem 2.1. *For $K \leq L < K + N - 1$,*

$$\det \Lambda(u_K \ldots u_L, v_K \ldots v_L) = \sum_{i=K-1}^{L+1} \prod_{j=K-1}^{i-1} \frac{c_j d_j}{\sigma_j} \prod_{j=i+1}^{L+1} \frac{a_j b_j}{\sigma_j}$$

Proof. We consider the original form of the graph and apply the determinant-connection formula.

First, we compute the possible permutations for a connection. Notice that for each k, either u_k is connected with either v_{k-1}, v_k, or v_{k+1}, and the same is true with u and v switched. I claim that $u_k \mapsto v_k$ for $k = K, \ldots, L$. First, consider u_K. Since v_{K-1} is not one of the vertices under consideration, either $u_K \mapsto v_K$ or $u_K \mapsto v_{K+1}$. If $u_K \mapsto v_{K+1}$, then $v_K \mapsto u_{K+1}$. But then the connections intersect. Thus, $u_K \mapsto v_K$ and repeating the argument shows that $u_k \mapsto v_k$ for all k. Thus, the only possible permutation is identity.

Now we consider the possible connections. Start with u_K and v_K. One possibility connects u_K to v_K through w_K, which will we write $u_K \to w_K \to v_K$ for short, and the other possibility is $u_K \to w_{K-1} \to v_K$. In the first case, we must have $u_{K+1} \to w_{K+1} \to v_{K+1}$ and continuing inductively $u_k \to w_k \to v_k$ for all k. In the case where $u_K \to w_{K-1} \to v_K$, we have a choice whether to connect $u_{K+1} \to w_{K+1} \to v_{K+1}$ or $u_{K+1} \to w_K \to v_{K+1}$, so we consider two subcases and repeat the process. In the end, the connections we have are exactly the connections of the form

- $u_k \to w_{k-1} \to v_k$ for all $k \leq i$,
- $u_k \to w_k \to v_k$ for all $k \geq i$,

for some i between $K-1$ and $L+1$.

Now we make some simplifications in the determinant-connection formula. In our case, $K(I;I)$ is a diagonal matrix with entries $-\sigma_1$, $-\sigma_2$, ..., $-\sigma_N$ because there are no interior-to-interior edges. Thus,

$$\det K(I;I) = (-1)^N \prod_{n=1}^{N} \sigma_n.$$

In each of the connections we computed, $|I_\alpha| = L - K + 1$, so $|J_\alpha| = N - (L - K + 1)$.

$$D_\alpha = (-1)^{L-K+1} \prod_{w_n \in J_\alpha} \sigma_n.$$

Hence,

$$\begin{aligned}\det \Lambda(u_J \dots u_K, v_J \dots v_K) &= \sum_\alpha \frac{\prod_{w_n \in J_\alpha} \sigma_n}{\prod_{n=1}^{N} \sigma_n} \prod_{e \in E_\alpha} \gamma(e) \\ &= \sum_\alpha \prod_{w_n \in I_\alpha} \frac{1}{\sigma_n} \prod_{e \in E_\alpha} \gamma(e).\end{aligned}$$

Evaluating the summands for each of the connections we computed gives the desired formula. □

Theorem 2.2.

$$\begin{aligned}&\det \Lambda(u_1 \dots u_N; v_1 \dots v_N) \\ &\quad = \prod_{n=1}^{N} \frac{a_n b_n}{\sigma_n} + \prod_{n=1}^{N} \frac{c_n d_n}{\sigma_n} - (-1)^N \prod_{n=1}^{N} \frac{a_n d_n}{\sigma_n} - (-1)^N \prod_{n=1}^{N} \frac{b_n c_n}{\sigma_n}.\end{aligned}$$

Proof. By similar reasoning as in the previous theorem, there are four possible connections:

- $u_k \to w_k \to v_k$ for all k;
- $u_k \to w_{k-1} \to w_k$ for all k;
- $u_k \to w_k \to v_{k+1}$ for all k;
- $u_k \to w_{k-1} \to v_{k-1}$ for all k.

Applying the determinant-connection formula to these connections proves the theorem. □

3 Solving for Valid Quasi-Conductivities

In this section, we assume that Λ is a valid response matrix produced by conductivities $\{a_n, b_n, c_n, d_n\}$, and we determine whether there are any other conductivity functions with response matrix Λ.

We begin with the transformed graph, whose original conductivity function is μ. We construct a possibly different quasi-conductivity function μ_y by supposing that $\mu_y(e_1) = y$ and using the quadrilateral rule to determine μ_y such that it has Λ as its response matrix.

First off, we need

$$\begin{aligned}\mu_y(u_n u_{n+1}) &= \lambda(v_n v_{n+1})\\ \mu_y(v_n v_{n+1}) &= \lambda(v_n v_{n+1})\\ \mu_y(u_n v_{n+1}) &= \lambda(u_n v_{n+1})\\ \mu_y(v_n u_{n+1}) &= \lambda(v_n u_{n+1})\end{aligned}$$

because the entries of the response matrix come directly from these conductivities.

By the quadrilateral rule, we need

$$\mu_y(e_1)\mu_y(e_1') = \mu_y(u_1 v_2)\mu_y(v_1 u_2) = \lambda(u_1 v_2)\lambda(v_1 u_2),$$

and so

$$\mu_y(e_1') = \frac{\lambda(u_1 v_2)\lambda(v_1 u_2)}{y}.$$

Then since we want $\lambda(u_2 v_2) = \mu_y(e_1') + \mu_y(e_2)$, we need

$$\mu_y(e_2) = \lambda(u_2 v_2) - \frac{\lambda(u_1 v_2)\lambda(v_1 u_2)}{y} = \frac{\lambda(u_2 v_2)y - \lambda(u_1 v_2)\lambda(v_1 u_2)}{y}.$$

We will let $g_1(x)$ be the linear fractional transformation in the last equation, that is, the function mapping $\mu_y(e_1)$ to $\mu_y(e_2)$ by the quadrilateral rule and subtraction. Similarly, we let g_n be the function mapping $\mu_y(e_n)$ to $\mu_y(e_{n+1})$:

$$g_n(x) = \frac{\lambda(u_{n+1} v_{n+1})x - \lambda(u_n v_{n+1})\lambda(v_n u_{n+1})}{x},$$

so that

$$\mu_y(e_n) = g_{n-1} \circ g_{n-2} \circ \cdots \circ g_1(y).$$

If μ_y is a valid quasi-conductivity function, we need

$$\mu_y(e_1) = \mu_y(e_{N+1}) = g_N \circ g_{N-1} \circ \cdots \circ g_1(y).$$

That is, y must be a fixed point of the LFT $g(y) = g_N \circ g_{N-1} \circ \cdots \circ g_1(y)$.

Finding the fixed points means solving a quadratic equation (unless g turns out to be linear!). Thus, to begin with, we know that there are probably either one or two possible conductances for e_1, one of which is the original conductance because it satisfies all the quadrilateral conditions we used to construct g. Since all the other edges can be found in terms of e_1, we know there one or two valid quasi-conductivity functions, and we know that one of them must be the original conductivity function. To know more, we have to compute the coefficients for g.

4 Computing the Coefficients

The coefficients for g can be computed by matrix multiplication:

$$g(y) = \frac{Ay + B}{Cy + D},$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \prod_{n=N}^{1} \begin{pmatrix} \lambda(u_{n+1}v_{n+1}) & -\lambda(u_n v_{n+1})\lambda(v_n u_{n+1} \\ 1 & 0 \end{pmatrix}.$$

In my convention, the indices of the product proceed from left to right (for instance, $\prod_{n=N}^{1} M_n = M_N M_{N-1} \ldots M_1$). We can think of this formula as specifying the recursion relation described in [2].

We will compute the coefficients A, B, C, D in terms of certain subdeterminants of Λ. To do this, we need a basic fact about tridiagonal matrices:

Lemma 4.1. *Let T be an $m \times m$ tridiagonal matrix. Then*

$$\det T = T_{m,m} \det T(1, \ldots, m-1; 1, \ldots m-1) \\ - T_{m,m-1} T_{m-1,m} \det T(1, \ldots, m-2; 1, \ldots m-2).$$

Proof. Use cofactor expansion on the last column of T. There are only two nonzero entries. The entry $T_{m,m}$ yields the first term. For the entry $T_{m-1,m}$, apply cofactor expansion again to $T(1, \ldots, m-1; 1, \ldots, m-2, m)$, noticing that only one entry in the bottom row is nonzero. □

Now we can prove

Theorem 4.2.

$$\begin{aligned} A &= \det \Lambda(u_1 \dots u_N; v_1 \dots v_N) + (-1)^N \left(\prod_{n=1}^{N} \lambda(u_n v_{n+1}) + \prod_{n=1}^{N} \lambda(v_n u_{n+1}) \right) \\ &\quad + \lambda(u_1 v_2)\lambda(v_1 u_2) \det \Lambda(u_3 \dots u_N; v_3 \dots v_N) \\ B &= -\lambda(u_1 v_2)\lambda(v_1 u_2) \det \Lambda(u_3 \dots u_{N+1}; v_3 \dots v_{N+1}) \\ C &= \det \Lambda(u_2 \dots u_N; v_2 \dots v_N) \\ D &= -\lambda(u_1 v_2)\lambda(v_1 u_2) \det \Lambda(u_3 \dots u_N; v_3 \dots v_N). \end{aligned}$$

Proof. Let

$$\begin{pmatrix} A_n & B_n \\ A_{n-1} & B_{n-1} \end{pmatrix} = \prod_{j=n}^{1} \begin{pmatrix} \lambda(u_{j+1} v_{j+1}) & -\lambda(u_j u_{j+1}) \cdot \lambda(v_j v_{j+1}) \\ 1 & 0 \end{pmatrix}$$

We know this definition is consistent over various indices because the bottom row of each matrix is $(1, 0)$. We want to compute $A = A_N$, $B = B_N$, $C = A_{N-1}$, and $D = B_{N-1}$.

First, we argue by induction that $A_n = \det \Lambda(u_2 \dots u_{n+1}; v_2 \dots v_{n+1})$ for $n = 1, \dots, N-1$. For the base case, we notice $A_0 = 1$ (which is the "determinant of the 0×0 matrix") and $A_1 = \lambda(u_2 v_2) = \det \Lambda(u_2; v_2)$. For the induction step, notice that for $n = 1, \dots N-1$, $\Lambda(u_2 \dots u_{n+1}; v_2 \dots v_{n+1})$ is tridiagonal; the other entries of the matrix are zero because there is no connection through the graph from u_j to $v_{j+\ell}$ or v_j to $u_{j+\ell}$ if $1 < \ell < N-1$. Then write out part of the 2×2 matrix multiplication and apply the preceding lemma on tridiagonal matrices:

$$\begin{aligned} A_n &= \lambda(u_{n+1} v_{n+1}) A_{n-1} - \lambda(u_n v_{n+1}) \lambda(v_n u_{n+1}) A_{n-2} \\ &= \lambda(u_{n+1} v_{n+1}) \det \Lambda(u_2 \dots u_n; v_2 \dots v_n) \\ &\quad - \lambda(u_n v_{n+1}) \lambda(v_n u_{n+1}) \det \Lambda(u_2 \dots u_{n-1}; v_2 \dots v_{n-1}) \\ &= \det \Lambda(u_2 \dots u_{n+1}; v_2 \dots v_{n+1}). \end{aligned}$$

This formula is not true for $n = N$, however, because $\Lambda(u_2 \dots u_{N+1}; v_2 \dots v_{N+1})$ is not tridiagonal; it has nonzero entries $\lambda(u_2 v_1)$ in the last row, first column and $\lambda(u_1 v_2)$ in the first row, last column. By the preceding argument, the value A_N is the determinant of the matrix we would get by removing the non-tridiagonal entries.

Let

- P_1 be the set of permutations τ with $\tau(u_{N+1}) \neq v_2$ and $\tau(u_2) \neq v_{N+1}$,

- P_2 be the set with $\tau(u_{N+1}) \neq v_2$ and $\tau(u_2) = v_{N+1}$,
- P_3 be the set with $\tau(u_{N+1}) = v_2$ and $\tau(u_2) \neq v_{N+1}$,
- P_4 be the set with $\tau(u_{N+1}) = v_2$ and $\tau(u_2) = v_{N+1}$.

These are a partition of S_N which we can use to group the terms in the determinant. The terms in the group for P_1 add up to A_N. For P_2, there is only one nonzero term, $\prod_{n=1}^{N} \lambda(v_n u_{n+1})$. For P_3, the only term is $\prod_{n=1}^{N} \lambda(u_n v_{n+1})$. For P_4, the terms add up to

$$\lambda(u_1 v_2)\lambda(v_1 u_2) \det \Lambda(u_3 \ldots u_N; v_3 \ldots v_N).$$

To complete the proof of the formula for A, simply notice that

$$\det \Lambda(u_1 \ldots u_N; v_1 \ldots v_N) = \det \Lambda(u_2 \ldots u_{N+1}; v_2 \ldots v_{N+1})$$

by permutation of the rows and columns. The proof for C is also complete. For B and D, the induction argument is similar, and there are no difficulties with the final step. □

5 Evaluating the Quadratic Formula

Knowing the formulas for the coefficients of $g(y)$, we are ready to solve for its fixed points, the solutions of

$$\frac{Ay + B}{Cy + D} = y,$$

which are given by the quadratic formula:

$$\frac{A - D \pm \sqrt{(A - D)^2 + 4BC}}{2C}.$$

We will evaluate this formula in terms of the original conductances. We start by rewriting the discriminant:

$$(A - D)^2 + 4BC = (A + D)^2 - 4(AD - BC).$$

By the matrix formula for g,

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \prod_{n=N}^{1} \det\begin{pmatrix} \lambda(u_{n+1}v_{n+1}) & -\lambda(u_n v_{n+1})\lambda(v_n u_{n+1}) \\ 1 & 0 \end{pmatrix}$$
$$= \prod_{n=1}^{N} \lambda(u_n v_{n+1})\lambda(v_n u_{n+1})$$
$$= \prod_{n=1}^{N} \frac{a_n b_n c_n d_n}{\sigma_n}.$$

By Theorem 4.2, $A + D$ equals

$$\det \Lambda(u_1 \dots u_N; v_1 \dots v_N) + (-1)^N \left(\prod_{n=1}^{N} \lambda(u_n v_{n+1}) + \prod_{n=1}^{N} \lambda(v_n u_{n+1}) \right).$$

Notice that the last two terms are

$$(-1)^N \prod_{n=1}^{N} \frac{a_n d_n}{\sigma_n} + (-1)^N \prod_{n=1}^{N} \frac{b_n c_n}{\sigma_n^2}.$$

By Theorem 2.2, the determinant is

$$\prod_{n=1}^{N} \frac{a_n b_n}{\sigma_n} + \prod_{n=1}^{N} \frac{c_n d_n}{\sigma_n} - (-1)^N \left(\prod_{n=1}^{N} \frac{a_n d_n}{\sigma_n} + \prod_{n=1}^{N} \frac{b_n c_n}{\sigma_n} \right),$$

so $A + D$ is

$$\prod_{n=1}^{N} \frac{a_n b_n}{\sigma_n} + \prod_{n=1}^{N} \frac{c_n d_n}{\sigma_n},$$

and the whole discriminant is

$$\left(\prod_{n=1}^{N} \frac{a_n b_n}{\sigma_n} + \prod_{n=1}^{N} \frac{c_n d_n}{\sigma_n} \right)^2 - 4 \prod_{n=1}^{N} \frac{a_n b_n c_n d_n}{\sigma_n^2} = \left(\prod_{n=1}^{N} \frac{a_n b_n}{\sigma_n} - \prod_{n=1}^{N} \frac{c_n d_n}{\sigma_n} \right)^2.$$

Applying the theorems again, we have $A - D$ equal to

$$\prod_{n=1}^{N} \frac{a_n b_n}{\sigma_n} + \prod_{n=1}^{N} \frac{c_n d_n}{\sigma_n} + 2\lambda(u_1 v_2)\lambda(v_1 u_2) \det \Lambda(u_3 \dots u_N; v_3 \dots v_N),$$

so the two values for the numerator are

$$2\prod_{n=1}^{N}\frac{a_nb_n}{\sigma_n}+2\lambda(u_1v_2)\lambda(v_1u_2)\det\Lambda(u_3\ldots u_N;v_3\ldots v_N),$$
$$2\prod_{n=1}^{N}\frac{c_nd_n}{\sigma_n}+2\lambda(u_1v_2)\lambda(v_1u_2)\det\Lambda(u_3\ldots u_N;v_3\ldots v_N).$$

At this point, we know that both solutions for the conductance on e_1 are positive because the denominator is positive by Theorem 2.1.

The first value corresponds to the original conductance a_1b_1/σ_1. To see this, use Theorem 2.1 to rewrite $\frac{1}{2}$ the numerator as

$$\prod_{n=1}^{N}\frac{a_nb_n}{\sigma_n}+\frac{a_1b_1c_1d_1}{\sigma_1^2}\left(\sum_{i=2}^{N+1}\prod_{j=2}^{i-1}\frac{c_jd_j}{\sigma_j}\prod_{j=i+1}^{N+1}\frac{a_jb_j}{\sigma_j}\right)$$
$$=\frac{a_1b_1}{\sigma_1}\left(\sum_{i=1}^{N+1}\prod_{j=1}^{i-1}\frac{c_jd_j}{\sigma_j}\prod_{j=i+1}^{N+1}\frac{a_jb_j}{\sigma_j}\right),$$

which is a_1b_1/σ_1 times $\frac{1}{2}$ the denominator.

6 Two Valid Conductivity Functions

In light of the preceding argument,

Theorem 6.1. *The conductivities on Γ that produce a response matrix Λ are unique if and only if*

$$\prod_{n=1}^{N}\frac{a_nb_n}{\sigma_n}=\prod_{n=1}^{N}\frac{c_nd_n}{\sigma_n}$$

if and only if

$$\det\Lambda(u_1\ldots u_N;v_1\ldots v_N)$$
$$=(-1)^{N+1}\left(\prod_{n=1}^{N}\sqrt{\lambda(u_nv_{n+1})}+(-1)^{N+1}\prod_{n=1}^{N}\sqrt{\lambda(v_nu_{n+1})}\right)^2.$$

Proof. Both formulas are equivalent to setting the discriminant equal to zero. The first is obvious. For the second, we note that $A+D = \prod_{n=1}^{N} a_n b_n/\sigma_n + \prod c_n d_n \sigma_n$ is positive and rewrite

$$(A+D)^2 - 4(AD - BC) = 0.$$

as

$$A + D = 2\sqrt{AD - BC},$$

or

$$\det \Lambda(u_1 \dots u_N; v_1 \dots v_N) + (-1)^N \left(\prod_{n=1}^{N} \lambda(u_n v_{n+1}) + \prod_{n=1}^{N} \lambda(v_n u_{n+1}) \right) = 2 \prod_{n=1}^{N} \sqrt{\lambda(u_n v_{n+1}) \lambda(v_n u_{n+1})}.$$

Then we subtract the second term on the left hand side, and rewrite the new right hand side as a square. □

Theorem 6.2. *If this condition is not satisfied, then there are two valid conductivity functions.*

Proof. We know there are exactly two valid quasi-conductivity functions. We only have to show that the both have all positive conductances. The preceding computations verified that this was true for e_1. But the argument did not rely on the specific ordering of the vertices. We can rotate all the indices counterclockwise by 1, 2, …, $N-1$ and use the same argument to show that the two values on e_2, …, e_N are positive. Then, since $\mu_y(e_n)\mu_y(e'_n) = \lambda(u_n v_{n+1})\lambda(v_n u_{n+1})$, we know the two values for e'_n are positive. □

Theorem 6.3. *If* $\{a_n, b_n, c_n, d_n\}$ *are the original conductances of the network, then the other valid conductance for* e_1 *is*

$$\frac{a_1 b_1}{\sigma_1} + \frac{\displaystyle\prod_{n=1}^{N} \frac{c_n d_n}{\sigma_n} - \prod_{n=1}^{N} \frac{a_n b_n}{\sigma_n}}{\displaystyle\sum_{i=1}^{N+1} \prod_{j=1}^{i-1} \frac{c_j d_j}{\sigma_j} \prod_{j=i+1}^{N+1} \frac{a_j b_j}{\sigma_j}},$$

and similar formulas hold for the other edges.

Proof. Computation. □

7 Characterization of the Response Matrix

Theorem 7.1. *A response matrix Λ is a valid response matrix for Γ if and only if*

1. *Λ is symmetric and has row sums zero;*
2. *The diagonal entries are negative and the off-diagonal entries are nonnegative;*
3. *$\lambda(u_n u_m)$, $\lambda(v_n v_m)$, and $\lambda(u_n v_m)$ are nonzero if and only if $n - m \bmod N$ is 0, 1, or -1;*
4. *$\lambda(u_n v_{n+1})\lambda(v_n u_{n+1}) = \lambda(u_n u_{n+1})\lambda(v_n v_{n+1})$ for all n;*
5. *$\det \Lambda(u_K \ldots u_L; v_K \ldots v_L) > 0$ for all $K \leq L < K + N - 1$;*
6. *$\det \Lambda(u_1 \ldots u_N; v_1 \ldots v_N)$*

$$\geq (-1)^{N+1} \left(\prod_{n=1}^{N} \sqrt{\lambda(u_n v_{n+1})} + (-1)^{N+1} \prod_{n=1}^{N} \sqrt{\lambda(v_n u_{n+1})} \right)^2 .$$

Proof. Suppose that Λ is valid. The first three statements are obvious. The fourth is a straightforward application of the quadrilateral rule. The fifth statment follows from Theorem 2.1. The sixth says that $A + D$ is positive and the discriminant is nonnegative.

Suppose conversely that Λ satisfies all these conditions, and I will show that there are two positive conductivity functions counting multiplicity which produce the response matrix Λ. The first four conditions show that the entries have the correct sign and satisfy one of the three quadrilateral conditions for each four-star. We only have to solve for $\mu(e_n)$ and $\mu(e'_n)$ and show that the resulting values are positive.

The computations of Theorem 4.2 do not rely on the assumption that the response matrix is valid, so we can compute the coefficients A, B, C, and D for $g(y)$ using that theorem. Condition 5 guarantees that B and D are negative and C is positive. The sixth condition guarantees that $A + D$ is positive and the discriminant is nonnegative. Since D is negative, we know $A - D$ is also positive. Since the discriminant

$$(A - D)^2 + 4BC \geq 0,$$

and $BC < 0$, we know $A - D > \sqrt{(A - D)^2 + 4BC}$, and hence that there are two solutions which yield positive values at e_1. The same argument will work for all the other edges because the conditions imposed on Λ were symmetric, so the two conductivity functions are positive everywhere. $\square$

References

[1] Edward B. Curtis and James A. Morrow. *Inverse Problems for Electrical Networks.* World Scientific. 2000.

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[3] Jeff Russell. "$\bigstar$ and $\mathcal{K}$ Solve the Inverse Problem."