A Class of PAs with Efficient Contraction

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Abstract

Optimal permutation arrays (PAs) have a sharply transitive group structure. A contraction operation is defined that constructs new permutation arrays from old ones. We characterize the effect of contraction on all sharply transitive group PAs.

1 Introduction

In section 2, we define m-contraction and show that $m \leq 3$ for all PAs. Next in section 3, we restrict our attention to group PAs and prove equivalent conditions for $m = 3$. The main result is in section 4, where we consider sharply transitive group PAs. Theorem 4.1 classifies the contraction of all sharply transitive group PAs.

In this paper, e denotes the identity permutation. PA stands for "permutationa array". When σ, τ are permutations, $d(\sigma, \tau)$ denotes the Hamming distance between σ, τ; it is invariant under permutation composition [put citation].

2 m-Contraction

Definition 2.1. *The contraction [put citation] of σ is*

$$\sigma' = \big(n\ \sigma^{-1}(n)\big)\,\sigma$$

Definition 2.2. *The $PA(n,d)$ is said to m-*contract *if the contractions of the elements of the $PA(n,d)$ form a $PA(n,d-m)$.*

Let σ, τ be permutations on $\{1, 2, \cdots, n\}$.

Lemma 2.1. $d(\sigma', \tau') \geq d(\sigma, \tau) - 3$
When equality holds, $\pi^3(n) = n, \pi(n) \neq n$ where $\pi = \sigma\tau^{-1}$.

Proof. Let $s = \sigma^{-1}(n), t = \tau^{-1}(n)$

$$\begin{aligned} d(\sigma', \tau') &= d((n\ s)\sigma, (n\ t)\tau) \\ &= d(\pi, (n\ s\ t)) \\ (*) &\geq d(\pi, e) - d(e, (n\ s\ t)) \\ &= d(\sigma, \tau) - d(e, (n\ s\ t)) \\ (**) &\geq d(\sigma, \tau) - 3 \end{aligned}$$

Now, we examine the equality case. Step $(**)$ implies n, s, t are distinct. Step $(*)$ follows from the triangle inequality, which states that $d(a, b) + d(b, c) \geq d(a, c)$.

$$d(a, b) + d(b, c) = d(a, c) \iff \left(a(i) \neq b(i) \implies b(i) = c(i)\right)$$

Applied to $(*)$

$$d\left(e, (n\ s\ t)\right) + d\left((n\ s\ t), \pi\right) = d\left(e, \pi\right) \iff \pi : (n\ s\ t) \to (s\ t\ n)$$

Hence $\pi^3(n) = n, \pi(n) \neq n$. □

As a consequence, this shows that $m \leq 3$ in m-contraction.

3 Conditions for 3-Contraction

In this section, we prove equivalent conditions for 3-contraction of groups.

Definition 3.1. *A $PA(n, d)$ is called a $G(n, d)$ if it is also a group.*

Theorem 3.1. *A $G(n, d)$ 3-contracts iff G contains a permutation π such that*

1. $\pi^3(n) = n$

2. $\pi(n) \neq n$

3. $d(e, \pi) = d$

Proof. Suppose G contains such an element π. Define s, t such that $(n\ s\ t) = (n\ \pi(n)\ \pi^2(n))$ Then the contractions of π, π^2 have distance $d - 3$. Indeed,

$$\begin{aligned} d(\pi', (\pi^2)') &= d\left((n\ t)\pi, (n\ s)\pi^2\right) \\ &= d\left((n\ s\ t), \pi\right) \\ (*) &= d(e, \pi) - 3 \\ &= d - 3 \end{aligned}$$

Step $(*)$ requires explanation. In all locations besides n, s, t, permutations e, π differ iff $(n\ s\ t), \pi$ differ. At locations n, s, t, e, π differ but $(n\ s\ t), \pi$ match. Thus the number of mismatches decreases by 3. Since we have found a pair of contracted permutations with Hamming distance $d - 3$, and Lemma 2.1 implies that this is the minimal distance, this implies that $G(n, d)$ 3-contracts.

For the other direction, suppose that the $G(n, d)$ 3-contracts. Then there exist permutations $\sigma, \tau \in G$ for which the equality case of Lemma 2.1 holds. Thus, $\pi^3(n) = n$ and $\pi(n) \neq n$. Furthermore, $d(\sigma, \tau) - 3 = d - 3 \implies d(e, \pi) = d$. Taking $g = \pi \in G$, we have constructed a g satisfying the conditions of this theorem. □

4 Classification

Using Theorem 3.1, we classify contractions of all sharply-transitive $G(n, d)$.

Theorem 4.1. *Let G be a sharply-transitive $G(n, d)$.*

Condition	Contracts to
$d \equiv 0 \mod 3$	$PA(n-1, d-3)$
$d \not\equiv 0 \mod 3$	$PA(n-1, d-2)$

Proof. Let the $G(n, d)$ undergo m-contraction. We've shown generally that $m \leq 3$. Now suppose that $m < 2$. If this was the case, after contraction there would be $\frac{n!}{(d-1)!}$ permutations of length $n-1$, with pairwise Hamming distance at most $d-1$. This would imply

$$M(n-1, d-1) \geq \frac{n}{d-1} \frac{(n-1)!}{(d-2)!} > \frac{(n-1)!}{(d-2)!} \geq M(n-1, d-1)$$

This contradiction follows from the maximality of the sharply-transitive group PAs [put citation here]. We conclude $m \in \{2, 3\}$.

The rest of the classification involves the following two cases:

- $d \equiv 0 \mod 3$

  In this case, we will show that 3-contraction occurs by finding an element that satisfies the conditions of Theorem 3.1. Consider the set

  $$S = \{\pi \in G | 1 \leq i \leq n - d \implies \pi(i) = i\}$$

  It is straightforward to verify that S is a subgroup of G. Moreover, since G is sharply $n - d + 1$-transitive, there is a unique element in S for every value of $\pi(n-d+1)$. Since $\pi(n-d+1)$ takes on each of the d values from $n-d+1$ to n inclusive, there are precisely d elements in S.

  By Cauchy's Theorem, $3 | d = |S| \implies S$ has an element of order 3 [put citation here]. Call this element π. Then $\pi^3(n) = n$. Now consider $d(e, \pi)$. The two permutations match for positions $i \leq n - d$, by construction. By $n-d+1$-transitivity, they can not match anywhere else. Thus $d(e, \pi) = d$. As a consequence, $\pi(n) \neq n$. Thus by Theorem 3.1, the $G(n, d)$ undergoes 3-contraction.

- $d \not\equiv 0 \mod 3$

  We proceed by assuming for contradiction that $G(n, d)$ 3-contracts. By Theorem 3.1, there exists an element π with $\pi^3(n) = n$ such that $\pi(n) \neq n$. This implies that n is contained in a 3-cycle. Thus π contains a 3-cycle, so its order is a multiple of 3 [put citation here].

  Now we construct a group S' that mimics the construction of S above, such that $\pi \in S'$. Let I be the set of fixed points of π. By $(n-d+1)$-transitivity, $|I| = n - d$. Then define

  $$S' = \{\sigma \in G | i \in I \implies \sigma(i) = i\}$$

Note that $\pi \in S'$. As before, S' is a group. By sharp transitivity, $|S| = d$. Thus $3|\text{ord}(\pi)|d$, which is a contradiction. Thus $G(n, d)$ undergoes 2-contraction.

□

5 Conclusions/Results/Citations

Pending. Will report new lower bounds as a consequence of this theorem with data from our table.