# UW REU 15: Nick Reichert's Talk

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**Definition 1** (Chromatic Number)**.** *The chromatic number, $k(G)$ of a graph $G$, is the minimal number of colors required so that each vertex of the graph can be given a color and no two adjacent vertices have the same color.*

In this discussion, we will assume that there are NO loops, parallel edges or spikes in graphs. Finding the chromatic number of a graph is similar to finding a coloring of a map with all adjacent countries having distinct colors. We will use the following fact from Topology: *every compact, connected orientable surface is a finite connected sum of tori.* i.e. we can think of surfaces as spheres with finitely many handles $g$ also known as the genus of the surface.

**Definition 2** (Embedding)**.** *A graph $G$ is embedded to a surface $S_g$ if the graph is drawn on a surface without edges crossing. That is, for each edge $E_j$ of $G$ there exists an embedding: $e_j : [0,1] \to S_g$ such that:* (1) *the images of edges intersect iff edges share a vertex and* (2) *images of edges should only intersect at endpoints.*

We'll also assume our emeddings are such that the components of $S_g$ are homeomorphic to disks.

**Definition 3** (Chromatic Number of Surface)**.** *The chromatic number, $K(S_g)$ of a surface $S_g$ is the largest chromatic number of any graph $G$ embedded in $S_g$.*

The Four color theorem states that $K(S_0) = 4$. The Heawood Conjecture (1890) states that:

$$K(S_g) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor = \left\lfloor \frac{7 + \sqrt{49 - 24\chi}}{2} \right\rfloor$$

where $\lfloor \rfloor$ denotes the floor function and $\chi = 2 - 2g$ is the Euler Characteristic.

**Definition 4** (Criticality)**.** *A graph is critical if no subgraph has the same chromatic number.*

**Lemma 1.** *If $G$ is a critical graph with chromatic number $K$, then the degree of each vertex is $\geq k - 1$.*

*Proof.* Suppose not. Then, there would exist a vertex $p \in G$ with degree $< k - 1$. Now, remove $p$ and its adjacent edges. This produces the subgraph $G'$ where $k(G') = k - 1$. This implies that vertices adjacent to $p$ can be colored with $k - 2$ colors (last remaining color is used for $p$). By criticality, we have that $G$ is $k - 1$ colored, a contradiction. □

**Lemma 2.** *If $G$ is critical with $\alpha_0$, number of vertices, $\alpha_1$, number of edges and $\chi(G) = K$, then $(K-1)\alpha_0 \leq 2\alpha_1$. For a polyhedron $P$, define the Euler characteristic $\chi(P) = V - E + F$ where $V$ is the number of vertices, $E$ is the number of edges and $F$ is the number of faces.*

Think of our embeddings of graphs into surfaces to have the following properties: vertices of graph $\leftrightarrow$ vertices of polyhedron, edges of graph $\leftrightarrow$ edges of polyhedron, and faces of an embedding will refer to the faces of the polyhedron. If $P(G)$ is an embedding of $G$ into $S_g$, we say that $\chi(P(G)) = V - E + F = 2 - 2g$.

**Lemma 3.** *If $G$ has vertices of degree $\leq 2$ and $P(G)$ is an embedding of $G$, then $|E| \leq 3|V| - 3\chi(P(G))$.*

**Theorem 1** (Heawood)**.** *If $\chi(S) \leq 0$, then*

$$K(S) \leq \left\lfloor \frac{7 + \sqrt{49 - 24\chi}}{2} \right\rfloor.$$

*Proof.* Suppose $G$ is embedded in $S$ with $K(G) = K(S)$. Without loss of generality, suppose $G$ is critical. Then from the lemmas, we see that:

$$\begin{aligned}(K-1)V &\leq 2E \\ &\leq 6V - 6\chi\end{aligned}$$

Since $K \leq V$, we see that $K-1 \leq 6-\frac{6\chi}{V} \leq 6-\frac{6\chi}{K}$. This translates to the quadratic polynomial $K^2-7K+6\chi \leq 0$. Since $\chi \leq 0$ and $K > 1$, we have that $K - \frac{7+\sqrt{49-24\chi}}{2} < 0$. □

We can check that for a graph $G$,

$$g(G) \geq \left\lfloor \frac{E-3V+6}{6} \right\rfloor$$

where $g(G)$ is the smallest genus in which $G$ can be embedded. For complete graphs $K_n$, $g(K_n) \geq \left\lfloor \frac{(n-3)(n-4)}{12} \right\rfloor$.

**Theorem 2.** *If*

$$g(K_n) = \left\lfloor \frac{(n-3)(n-4)}{12} \right\rfloor,$$

*then the Heawood conjecture is true.*

**Definition 5** (Triangular Embedding)**.** *An embedding $P(G)$ is triangular if all faces have 3 edges. If $P(G)$ is triangular then $g(P(G)) = g(G)$.*

Note that if we can find a triangular embedding for $K_n$ (or any graph), then we've determined the genus of a graph. However, triangular embeddings do not always exist. For $K_n$ to have an embedding, we need $3F = 2E$. We can deduce that triangular embeddings exist for $K_n$ when $n \equiv 0, 3, 4, 7 \mod 12$.

**Definition 6** (Rotation System)**.** *A rotation system on a graph gives an embedding where a rotation system is a cyclic ordering of vertices adjacent to each vertex to a graph.*

If each row $i$ of the rotaional system is obtained by taking the row $i-1$, adding 1 to each entry and moding out by $n$, then the embedding is triangular. Therefore to see if a triangular embedding is possible, we only need the first row of the rotation system.