CONVEXITY AND THE DIRICHLET PROBLEM FOR DIRECTED GRAPHS - THE DUAL CASE

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1. Introduction

Previously, I considered conductivity networks where current along each edge was given by a nonlinear (but monotone and continuous) function of voltage, and showed that the Dirichlet-to-Neumann map is well-defined. Here, I consider the dual situation, where voltage along each edge is given by a monotone, continuous function of current. I show that a very similar approach can be made. In particular, we have these results:

- Given boundary currents adding to 0, there is some solution to the von Neumann problem.
- The set of solutions is a convex set.
- The voltage drops in the solution are uniquely determined.

Assume we have a graph Γ with vertices V and edges E, and vertices partitioned into two sets: the boundary nodes ∂V and the interior nodes $\operatorname{int} V$.

Definition 1.1. *A* resistance function *on Γ is a collection of real-valued functions $f_{ij}(x)$ for each $v_iv_j \in E$, such that*

- *For every $v_iv_j \in E$, $f_{ij}(x)$ is a continuous and weakly increasing function of x.*
- $f_{ij}(0) = 0$
- $f_{ij}(-x) = -f_{ji}(x)$.

Definition 1.2. *A* current function *on V to be a function $I : V \times V \to \mathbb{R}$ such that $I_{ij} = -I_{ji}$, for all $v_j \in \operatorname{int} V = V \setminus \partial V$*

$$\sum_{v_k \text{ adjacent to } v_j} I_{jk} = 0.$$

and $I_{ij} = 0$ unless $v_iv_j \in E$.

Current functions form a vectors space W. Note that if I is any current function, then

$$\sum_{v_j \in \partial V} \sum_{v_kv_j \in E} I_{jk} = \sum_j \sum_{v_kv_j \in E} I_{jk} = \sum_{v_kv_j \in E} I_{jk} = \frac{1}{2}\left(\sum_{v_kv_j \in E} I_{jk} - I_{kj}\right) = 0.$$

Definition 1.3. *A current function I satisfies* Kirchhoff's Voltage Law *witih respect to the resistance functions f if there is a function $u : V \to \mathbb{R}$ such that for every $v_iv_j \in E$*

$$f_{ij}(I_{ij}) = u(v_i) - u(v_j)$$

Definition 1.4. *Given a function $\phi : \partial V \to \mathbb{R}$ with $\sum_{v_j \in \partial V} \phi_j = 0$, the* Neumann problem *is to find a current function I satisfying the Kirchhoff Voltage Law, and satisfying*

$$\sum_k I_{jk} = \phi_j$$

for $v_j \in \partial V$.

The ϕ_i are called boundary currents. Note that if $\sum_{v_j \in \partial V} \phi_j \neq 0$, then there could not be any solution. The boundary currents must add up to zero.

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2. Pseudopower (or its dual)

For each edge $v_iv_j \in E$, define the pseudopower function

$$p_{ij}(I_{ij}) = 2\int_0^{I_{ij}} f_{ij}(t)\,dt$$

Since $f_{ij}(x) = -f_{ji}(-x)$, we have $p_{ij}(x) = p_{ji}(-x)$. Also, because $f_{ij}(0) = 0$ and $f_{ij}(x)$ is weakly increasing, $p_{ij}(x) \geq 0$. Clearly, $p_{ij}(x)$ is a convex, continuously differentiable function of x. Define the total pseudopower $p(I) = \sum_{v_iv_j \in E} p_{ij}(I_{ij})$. Then p is a convex, continuously differentiable function on W, and is bounded below by *zero*.

Let W_ϕ be the set of all current functions having a given boundary current ϕ. As long as $\sum_{v_i \in \partial V} \phi_i = 0$, W_ϕ will be nonempty. (This should be straightforward to prove by induction on the size of the graph).

If p attains a minimum value at some $I \in W_\phi$, then I is a critical point. Take any cycle C. Create a new current function $I(\epsilon)$ by $I_{ij}(\epsilon) = I_{ij} + \epsilon$ if v_iv_j is in the cycle (in that order), $I_{ji}(\epsilon) = I_{ji} - \epsilon$ if v_iv_j is in the cycle (in that order), and $I_{ij} = I_{ij}$ otherwise. In other words, we increase the flow along around C by ϵ at each edge. $I(\epsilon)$ is certainly a current function in W_ϕ, and since I was a critical point, we must have

$$\left.\frac{\partial I(\epsilon)}{\partial \epsilon}\right|_{\epsilon=0} = 0 = \sum_{v_iv_j \in C} p'_{ij}(I_{ij}) = \sum_{v_iv_j \in C} f_{ij}(I_{ij}).$$

Since this is true for any cycle, we can choose $u : V \to \mathbb{R}$ such that $u_i - u_j = f_{ij}(I_{ij})$ for all $v_iv_j \in E$. Therefore, I satisfies the Kirchoff Current Laws. So a global minimum of p on W_ϕ is a solution to the Neumann problem. Conversely, any critical point of a convex function is a global minimum. Since the global minima of a convex function form a convex set, the solutions to the Neumann problem form a convex set. In summary, we have shown:

Theorem 2.1. *For boundary currents ϕ, $I \in W_\phi$ is a solution to the Neumann problem iff it is a global minimum of the total pseudopower p on W_ϕ. The solutions form a convex subset of W_ϕ.*

3. Existence

We still need to show that any solutions exist. As noted above, it is not hard to show that W_ϕ is nonempty (assuming $\sum_k \phi_k = 0$). Let's say that a current function is *acyclic* if there does not exist a cycle in the graph along which all the currents are positive. For example, it is not the case that for some i, j, k, $I_{ij}, I_{jk}, I_{ki} > 0$. Let $K_\phi \subseteq W_\phi$ be the set of all acyclic current functions. Clearly, K_ϕ is closed. It is also bounded:

Lemma 3.1. *If $I \in K_\phi$, then for all $v_iv_j \in E$,*

$$I_{ij} \leq \sum_k |\phi_k|$$

Proof. Let S be the set of vertices upstream from v_i. This is meaningful, because I is acyclic. To be more precise, $v \in S$ iff there is a chain of vertices $v = v_{c_1}, v_{c_2}, \ldots, v_{c_r} = v_i$ such that $v_{c_l}v_{c_{l+1}} \in E$ and $I_{c_lc_{l+1}} > 0$. Then S does not contain v_j, since I is acyclic. The total current flowing from S to $V \setminus S$ is equal to the total boundary current along the boundary of S, so it is at most $\sum_k |\phi_k|$. Also, if $v_pv_q \in E$, $v_p \in S$, but $v_q \notin S$, then $I_{pq} \geq 0$, since otherwise the chain from v_p to v_i could be extended to be from v_q to v_i, contradicting $v_q \notin S$. Therefore, I_{ij} is bounded from above by the total amount of current flowing from S to $V \setminus S$, which as already noted is bounded by $\sum_k |\phi_k|$. □

It turns out that the minimum, if it occurs, must occur on K_ϕ.

Lemma 3.2. *For $I \in W_\phi$, there is some $I' \in K_\phi$, such that $p(I') \leq p(I)$*

Proof. If I is acyclic then we are done, so suppose there is some cycle C such that I has positive currents along C. Let μ be the smallest current of I along C, and let I_1 be the current function obtained by decreasing every current along C by μ. This is still a current function, in W_ϕ. Now decreasing the magnitude of a current along an edge does not increase the pseudopower there, so $p(I_1) \leq p(I)$. If $I_1 \in K_\phi$ then we are done. Otherwise, perform the same operation on I_1, and continue until we have a current function in K_ϕ. This process always terminates, because at each step we strictly increase the number of edges on which there is no current. □

Theorem 3.3. *The von Neumann problem has a solution if* $\sum_k \phi_k = 0$.

Proof. By Lemma 3.2(!) K_ϕ is nonempty. It is also bounded (by Lemma 3.1) and closed, so p attains a minimum on it, at some point I_0. Then for any $I \in W_\phi$, $p(I) \geq p(I') \geq p(I_0)$ for some $I' \in K_\phi$, by Lemma 3.2. Therefore, I_0 is a global minimum of p, so by Theorem 2.1 I_0 is a solution to the Neumann problem with boundary currents ϕ. □

4. Voltages Fixed

Now, we show that for a given boundary current ϕ, the voltage differences across each edge are uniquely determined. Suppose we have two solutions, I and J. Define an auxiliary directed graph on V, in which there is an edge from v_i to v_j if $f_{ij}(I_{ij}) < f_{ij}(J_{ij})$. If S is any set of vertices in the graph, it cannot be the case that every edge from S to $V \setminus S$ is in the auxiliary graph, since this would indicate that the current flowing from S to $V \setminus S$ increased, but that should be fixed by the boundary currents.

Suppose that $v_i v_j$ is an edge in the auxiliary graph. Let S be the set of all vertices that can reach i by traveling along the edges of the original graph, without ever going against the direction of the auxiliary graph. If $x \in S$ and $y \notin S$, and there is an edge from x to y, there must be an edge in the auxiliary graph from x to y, or else we could travel from y to v_i via x. So every edge from S to $V \setminus S$ is in the auxiliary graph. By the previous comment, this can only happen if $V \setminus S = \emptyset$. Then $v_j \in S$. So we can travel from v_j to v_i without traveling against the arrows. This produces a cycle C such that as we travel around C, we never travel against the arrow, and at at least one point (where we move from v_i to v_j), we travel with an arrow. Therefore, the sum of the voltage drops along C never decreased, and increased in at least one point, as we switch from I to J. But this is impossible, since the sum of the voltage drops along a cycle should be 0 in both I and J.

Therefore, there are no edges in the auxiliary graph. So for every edge $v_i v_j$, $f_{ij}(I_{ij}) \geq f_{ij}(J_{ij})$. Switching i and j, we also have

$$-f_{ij}(I_{ij}) = f_{ji}(I_{ji}) \geq f_{ji}(J_{ji}) = -f_{ij}(J_{ij})$$

So $f_{ij}(I_{ij}) = f_{ij}(J_{ij})$ for all edges.

Theorem 4.1. *For fixed boundary currents ϕ, all solutions to the Neumann problem have the same voltage drop across any given edge.*

By combining this with Theorem 3.3, we see that the Neumann-to-Dirichlet map is well defined. In the case where all the resistance functions are bijections, this combines with my prior result to show that there is essentially a one to one correspondence between boundary currents (summing to zero) and boundary voltages (summing to zero).