# Doppelgangers: Revisiting Recurrences and the Ur-Operation

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## Contents



**Abstract**

In 1971, Kenneth Johnson introduced a recurrence relation on the order polynomial of finite partially ordered sets [2]. Two years later, Richard Stanley offered a version of the recurrence on what is now known as the strict order polynomial [3]. More recently, Hamaker, Patrias, Pechenik, and Williams proposed thinking of the order polynomial as an equivalence relation on posets, and call two posets doppelgangers if their order polynomials are equal [6]. We offer new recurrence relations in a similar spirit to Stanley and Johnson on labeled posets, as well as offering new applications of both old and new recurrences to Hamaker's doppelgangers such as classifying doppelgangers of large height. In addition, we note how our new recurrences may shed light on the related multivariate generating function of a poset studied recently by McNamara and Ward [8]. Finally we introduce a new generalized operation on posets which generalizes results offered by the recurrence relations.

# 1 Introduction

## 1.1 Background

Richard Stanley introduced the order polynomial $\Omega_{P,\omega}(m)$ of a partially ordered set (poset) in 1970 as an analog to chromatic polynomials [1]. Stanley and others continued to study the polynomial for years, but until recently, the subject had fallen out of favor. In 2012, Feray and Reiner published a series of two papers the order polynomial and related material viewed from a new angle, in a ring-theoretic manner by examining the ring of weak P-partitions which is the subring of $k[x_1, \ldots, x_n]$ spanned by $x^{f(1)} \ldots x^{f(n)}$ for each P-partition $f$ of $P$ [4] [5]. In 2014, McNamara and Ward set out to classify the equivalence classes of the multivariate generating function $K_{(P,\omega)}$, a function similar to one discussed in [4], and related to the order polynomial by

$$\Omega_{P,\omega}(m) = K_{(P,\omega)}(\underbrace{1, ..., 1}_{m}, 0, ...) \text{ [8].}$$

In their work they found a number of important invariants for the functions, and offered several conjectures and unexplained equivalences. Two years later, Hamaker et al. introduced doppelgangers. Hamaker et al. call two unlabeled posets *doppelgangers* if they have the same order polynomial and discusses the importance of such classification in its consequences to fields such as K-theory. In doing so, the paper focuses on an infinite family of grid-like doppelgangers, which raises the natural question of the existence and importance of other such families. We initially set out to create other such infinite families, and in our investigations, came across a number of recurrence relations on the order polynomial. The simplest of these relations were introduced in the early 1970's by Johnson [2] and expanded upon by Stanley [3].

Our work is focused in two main areas: applications of both new and old recurrence relations to $\Omega_{P,\omega}$ and $K_{P,\omega}$, and the introduction and analysis of a new operation on posets which generalizes these results. The potential of Johnson and Stanley's recurrence relations was never fully explored. For instance, using Stanley's recurrence and the method of induction on incomparable elements introduced in the same paper [3], one can provide an elegant proof of Stanley's poset reciprocity theorem [7]. Stanley's old recurrences can also be used to investigate doppelgangers, and may be used to find invariants, infinite families, and classifications for special families. In the simplest case, we use a single step of the recurrence to construct an infinite family of doppelgangers. We may use invariants on the order polynomial such as height and the fact that posets of size $n$ and of height greater than $k$ are closed under Johnson's recurrence to classify doppelgangers of heights at least $n-2$. In addition, we introduce improper recurrences and recurrences on labeled posets which may be used to clarify previously not understood examples of equivalence of the multivariate generating function on small posets, and may provide a springboard for further research in the area or in doppelgangers of labeled posets.

Lastly, we introduce the Ur-Operation: a new poset operation which generalizes the standard poset operations disjoint union, ordinal sum, and direct product. Using this operation we may decompose arbitrary posets to build doppelgangers. Further, just as this operation generalizes direct and ordinal sum, so it generalizes the results on these operations given by the aforementioned recurrences. In addition, we investigate the more nuanced structure behind the operation and its implications for doppelgangers, and provide the first steps towards a powerful conjecture for the construction and classification of doppelgangers.

## 1.2 Definitions and Notation

We begin with a brief review of basic poset definitions and notations. Let $C_k$ denote a *chain*, a totally ordered set, of size $k$. Likewise let $A_k$ denote an *anti-chain*, a set with no order relations, of size $k$. Note that as a result that $A_1 = C_1$, and $A_0 = C_0$. We use these interchangeably as may be convenient. The *height* of a poset $P$ is the size of the largest chain in $P$, denoted $h(P)$. The *width* of a poset $P$ is the size of the largest antichain in $P$, denoted $w(P)$. A *linear extension* of a poset $P$ is an order preserving bijection

$P \to [|P|]$. The number of linear extensions of $P$ is denoted by $l(P)$. We use $\mathbb{N} = \{1, 2, \cdots, \}$ to denote the natural numbers and $[m] = \{1, \cdots, m\}$.

This paper will generalize and provide results for three basic poset operations: ordinal sum, disjoint union, and ordinal product. The *ordinal sum* of two posets $P$ and $Q$ is denoted $P \oplus Q$ and is the poset on $P \cup Q$ such that for $r, s \in P \oplus Q$ $r \leq s$ when $r \in P, s \in Q$, or $r \in P, s \in P$ and $r \leq s$ in $P$, or $r \in Q, s \in Q$ and $r \leq s$ in $Q$. The *disjoint union* of $P$ and $Q$ is denoted $P + Q$, and is the poset $P \cup Q$ with the relations $r \leq s$ if $r, s \in P$ or $r, s \in Q$ and $r \leq s$ in $P$ or $Q$ respectively. Finally, the *direct sum* of $P$ and $Q$ is denoted $P \times Q$, and is the poset on the Cartesian product $P \times Q$ such that $(r, s) \leq (r', s')$ if $r \leq r'$ in $P$ and $s \leq s'$ in $Q$. [7]

A *P-partition* of height $m \in \mathbb{N}$ of a poset $P$ is an order preserving map $F \colon P \to [m]$. The set of such maps is denoted by $\mathscr{P}\mathscr{P}^{[m]}(P)$ and its cardinality is denoted by $F_P(m) = |\mathscr{P}\mathscr{P}^{[m]}(P)|$. $F_P(m)$ is a polynomial in $m$ of order $|P|$ (Section 3.12 in [7]), and is thus called the order polynomial. Posets $P$ and $Q$ are doppelgangers if their order polynomials are equal, which we write as $P \sim Q$. The dual of a poset $P$, $P^*$, is the poset where all relations are flipped. It is obvious that $P \sim P^*$. Lastly the multivariate generating polynomial is defined as

$$K_{(P,\omega)}(x) = \sum_{f \in (P,\omega)-\text{partitions}} x_1^{|f^{-1}(1)|} x_2^{|f^{-1}(2)|} \cdots$$

# 2 Order Polynomial Recurrence

We begin by presenting the several useful recurrence relations. These are by no means the only recurrence relations on order polynomials of posets; we leave out, for instance, the strict recurrence discussed by Stanley in [3]. However, the recurrences described below are enough to provide results on both doppelgangers and the multivariate generating function. In addition, we re-introduce the method of induction on incomparable elements, first used by Stanley to prove a theorem similar to Proposition 2 below. This method easily provides several results linking doppelgangers to the ordinal sum poset operation.

## 2.1 The Recurrence Relations

Given a poset $P$ with incomparable elements $x$ and $y$, we can define the poset $P|x \leq y$ to be the result of adding the cover relation $x \leq y$ and all other relations implied by transitivity. We can define the poset $P|x = y$ to be the result of identifying $x$ and $y$. Stanley considers these constructions in his paper [3] where he mentions the following recurrence relation

$$F_P = F_{P|x \leq y} + F_{P|y \leq x} - F_{P|x=y}.$$

In this relation, an order-preserving map $f \colon P \to [n]$ either has $f(x) < f(y)$ in which case it is counted by the first term, $f(x) > f(y)$ in which case it is counted by the second term, or $f(x) = f(y)$ in which case it is counted by all three terms. This recurrence easily produces the following infinite family of doppelgangers. The significance of this example is that it demonstrates how this recurrence relation can be applied to product nontrivial infinite families of doppelgangers without having to compute order polynomials or give explicit bijections.

**Proposition 1.** *For each $n \geq 2$, the posets in Figure 1a and Figure 1b are doppelgangers.*

*Proof.* After applying the recurrence relation $F_P = F_{P|x \leq y} + F_{P|y \leq x} - F_{P|x=y}$ to the two posets, the three resulting posets are equal. □

Labeled posets admit a similar recurrence. Recall that a labeled poset $(P, \omega)$ is a poset $P$ equipped with a bijective labeling $\omega \colon P \to [|P|]$. In this case, an map $f \colon P \to [n]$ is order-preserving when $f(x) \leq f(y)$ whenever $x \leq y$, and $f(x) < f(y)$ whenever $x < y$ and $\omega(x) > \omega(y)$. The *order polynomial of a labeled*

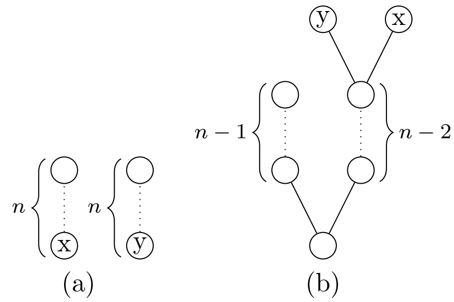



Figure 1: An infinite family of doppelgangers following from the unlabeled recurrence

*poset* is then defined similarly as $\Omega_{P,\omega}(m) = \#\{(P,\omega)\text{-partitions of height } m\}$ Then we have the following recurrence:

$$\Omega_{P,\omega} = \Omega_{P|x\leq y,\omega} + \Omega_{P|y\leq x,\omega}.$$

To see this, suppose without loss of generality that $\omega(x) < \omega(y)$. Let $f\colon P \to [m]$ be order preserving. If $f(x) \leq f(y)$, then $f$ is only counted by $\Omega_{P|x\leq y,\omega}$. If $f(y) < f(x)$, then $f$ is only counted by $\Omega_{P|y\leq x,\omega}$. Note that labeled posets can be viewed as an assignment of strict and weak edges. For incomparable $x, y \in P$, let $P|x < y$ be the poset with the added relation that $x < y$ and all other relations implied by transitivity. This restriction might not result in a valid labeled poset. However, plane partitions, and thus the order polynomial and multivariate generating functions are still well-defined on these improper posets. Recall that the *multivariate generating function* of $(P,\omega)$ is defined as

$$K_{(P,\omega)}(x) = \sum_{f\in(P,\omega)-\text{partitions}} x_1^{|f^{-1}(1)|} x_2^{|f^{-1}(2)|} \ldots$$

Then similarly to above, $K_{(P,\omega)}$ admits the following improper recurrence:

$$K_{P,\omega} = K_{P|x<y,\omega} + K_{P|y\leq x,\omega}.$$

In [8], McNamara and Ward offer four unexplained equivalences of size 5 as a spring board for further exploration. This improper recurrence explains their first example as demonstrated in Figure 2.

## 2.2 Induction on Incomparable Elements

The recurrences described above can be used to prove results involving order polynomials by strong induction on the number of incomparable pairs of elements. In particular, each of the terms of the recurrences have fewer pairs of incomparable elements than the original poset. In [3], Stanley uses this technique to prove the well-known expression for the strict order polynomial in the $\binom{m}{k}$ basis. We will apply this technique to provide a novel and short proof of Stanley's poset reciprocity theorem and an expression for the order polynomial of an ordinal sum.

For a labeling $\omega$ of a poset $P$, let $\overline{\omega}$ be the dual labeling to $\omega$ given by $\overline{\omega}(x) = |P| + 1 - \omega(x)$. In what follows, we will use the binomial reciprocity theorem which states that

$$\binom{-n}{p} = \frac{(-n)(-n-1)\ldots(-n-(p-1))}{p!} = (-1)^p \frac{n(n+1)\ldots(n+p-1)}{p!} = (-1)^p \binom{n+p-1}{p}.$$

**Theorem 2** (Poset Reciprocity)**.** *For all labeled posets, $(P,\omega)$,*

$$\Omega_{P,\overline{\omega}}(m) = (-1)^{|P|}\Omega_{P,\omega}(-m).$$

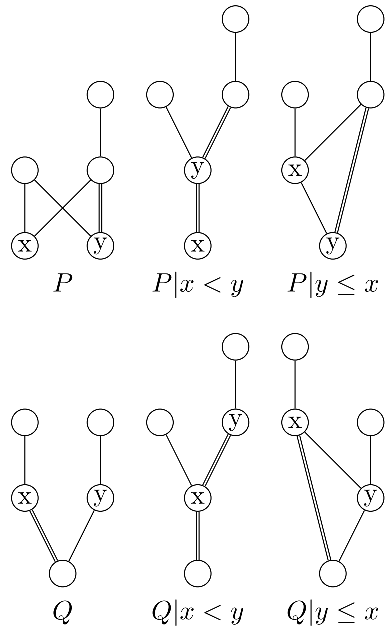



Figure 2: Equivalence of $K_{P,\omega}$ and $K_{Q,\omega}$

*Proof.* We shall proceed by strong induction on the number of pairs of incomparable elements in $P$. For the base case where $P$ has no pairs of incomparable elements, $P$ is a chain. Then $(P, \omega)$ can be thought to be a chain with $i$ strict edges and $j$ non-strict edges where $i + j = |P| - 1$. Using a modified stars and bars technique, we get that $\Omega_{P,\omega}(m) = \binom{m+j}{|P|}$. Since $(P, \overline{\omega})$ is a chain with $j$ strict edges and $i$ non-strict edges, $\Omega_{P,\overline{\omega}}(m) = \binom{m+i}{|P|}$. Then by the binomial reciprocity theorem,

$$\Omega_{P,\overline{\omega}}(m) = \binom{m+i}{|P|} = (-1)^{|P|}\binom{-(m+i)+|P|-1}{|P|} = (-1)^{|P|}\binom{-m+j}{|P|} = (-1)^{|P|}\Omega_{P,\omega}(-m)$$

which shows the base case. Now suppose that the result holds for all posets with fewer than $n$ pairs of incomparable elements and suppose that $P$ has $n$ pairs of incomparable elements. Then let $x, y \in P$ be incomparable. By our inductive assumption,

$$\begin{aligned}
\Omega_{P,\overline{\omega}}(m) &= \Omega_{P|x \leq y,\overline{\omega}}(m) + \Omega_{P|y \leq x,\overline{\omega}}(m) \\
&= (-1)^{|P|}\Omega_{P|x \leq y,\omega}(-m) + (-1)^{|P|}\Omega_{P|y \leq x,\omega}(-m) \\
&= (-1)^{|P|}\Omega_{P,\omega}(-m)
\end{aligned}$$

which shows the inductive step and completes the proof. $\square$

As a consequence of this result, we can easily derive the expression for the strict order polynomial in the $\binom{m}{k}$ basis which we shall use in investigating the families of large height.

**Proposition 3.** *For all posets $P$, there exist $c_k \in \mathbb{N}$ such that*

$$F_P(m) = (-1)^{|P|} \sum_{k=h(P)}^{|P|} (-1)^k c_k \binom{m+k-1}{k}.$$

*Proof.* Let $\omega$ be a natural labeling for $P$. By the poset and binomial reciprocity theorems, it suffices to show that there exist $c_k \in \mathbb{N}$ such that $\Omega_{P,\overline{\omega}}(m) = \sum_{k=h(P)}^{|P|} c_k \binom{m}{k}$. It is straightforward to verify that we can let $c_k$ be the number of surjective strict order-preserving maps $f \colon P \to [k]$. $\square$

We can generalize $\oplus$ to labeled posets in the following way: Given labeled posets $(P, \omega)$ and $(Q, \psi)$, let $\omega \oplus \psi$ be a labeling on $P \oplus Q$ given by

$$(\omega \oplus \psi)(x) = \begin{cases} \omega(x) & x \in P \\ |P| + \psi(x) & x \in Q \end{cases}.$$

Then $(P \oplus Q, \omega \oplus \psi)$ is the labeled poset where every element of $P$ is weakly less than every element of $Q$. The following result gives a formula for the order polynomial of an ordinal sum. The proof requires Vandermonde's identity which states that

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}.$$

**Lemma 4.** *For all labeled posets* $(P, \omega), (Q, \psi)$,

$$L(\Omega_{P \oplus Q, \omega \oplus \psi}) = L(\Omega_{P,\omega}) L(\Omega_{Q,\psi})$$

*for any linear transformation $L$ on the polynomials in $m$ such that*

$$L\left(\binom{m+c+d-1}{c+d}\right) = L\left(\binom{m+c-1}{c}\right) L\left(\binom{m+d-1}{d}\right)$$

*for all integer $c, d \geq 0$.*

*Proof.* We first show the result in the case where $P$ and $Q$ are chains. Suppose that $(P, \omega)$ has $i$ strict edges and $j$ non-strict edges and suppose that $(Q, \psi)$ has $k$ strict edges and $l$ non-strict edges. Then $(P \oplus Q, \omega \oplus \psi)$ has $i + k + 1$ strict edges and $j + l$ non-strict edges. Then it suffices to show that

$$L\left(\binom{m+j}{|P|}\right) L\left(\binom{m+l}{|Q|}\right) = L\left(\binom{m+j+l+1}{|P|+|Q|}\right) \text{ where } j \leq |P|-1,\ l \leq |Q|-1.$$

We shall proceed by induction on $|P| - 1 - j + |Q| - 1 - l$. For the base case of $|P| - 1 - j + |Q| - 1 - l$, $j = |P| - 1$ and $l = |Q| - 1$ in which case the result reduces to the hypothesis on $L$. Now suppose that $|P| - 1 - j + |Q| - 1 - l > 0$ and that the result holds for smaller values of $|P| - 1 - j + |Q| - 1 - l$. Then without loss of generality, $j < |P| - 1$ and

$$\begin{aligned}
L\left(\binom{m+j}{|P|}\right) L\left(\binom{m+l}{|Q|}\right) &= L\left(\binom{m+j+1}{|P|} - \binom{m+j}{|P|-1}\right) L\left(\binom{m+l}{|Q|}\right) \\
&= L\left(\binom{m+j+1}{|P|}\right) L\left(\binom{m+l}{|Q|}\right) - L\left(\binom{m+j}{|P|-1}\right) L\left(\binom{m+l}{|Q|}\right) \\
&= L\left(\binom{m+j+l+2}{|P|+|Q|}\right) - L\left(\binom{m+j+l+1}{|P|+|Q|-1}\right) \\
&= L\left(\binom{m+j+l+1}{|P|+|Q|}\right)
\end{aligned}$$

which shows the inductive step and completes the proof of the case where $P$ and $Q$ are chains. For the general result, shall proceed by strong induction on the number of pairs of incomparable elements in $P \oplus Q$. For the base case where $P \oplus Q$ has no pairs of incomparable elements, $P$ and $Q$ are chains which was dealt with above. Now suppose that the result holds for all posets where $P \oplus Q$ has fewer than $n$ pairs of incomparable elements and suppose that $P \oplus Q$ has $n$ pairs of incomparable elements. Then without loss of generality, $P$ has an incomparable pair of elements, $x$ and $y$. Then by the linearity of $L$ and our inductive assumption,

$$\begin{aligned} L(\Omega_{P\oplus Q,\omega\oplus\psi}) &= L(\Omega_{(P\oplus Q)|x\le y,\omega\oplus\psi} + \Omega_{(P\oplus Q)|y\le x,\omega\oplus\psi}) \\ &= L(\Omega_{(P|x\le y)\oplus Q,\omega\oplus\psi}) + L(\Omega_{(P|y\le x)\oplus Q,\omega\oplus\psi}) \\ &= L(\Omega_{P|x\le y,\omega})L(\Omega_{Q,\psi}) + L(\Omega_{P|y\le x,\omega})L(\Omega_{Q,\psi}) \\ &= L(\Omega_{P|x\le y,\omega} + \Omega_{P|y\le x,\omega})L(\Omega_{Q,\psi}) \\ &= L(\Omega_{P,\omega})L(\Omega_{Q,\psi}) \end{aligned}$$

which shows the inductive step and completes the proof. □

**Theorem 5.** *For labeled posets $(P,\omega),(P',\omega'),(Q,\psi),(Q',\psi')$, any two conditions imply the third:*
*1) $(P,\omega) \sim (P',\omega')$*
*2) $(Q,\psi) \sim (Q',\psi')$*
*3) $(P \oplus Q,\omega \oplus \psi) \sim (P' \oplus Q',\omega' \oplus \psi')$*

**Corollary 6.** *For all labeled posets $(P,\omega),(Q,\psi)$,*

$$(P \oplus Q, \omega \oplus \psi) \sim (Q \oplus P, \psi \oplus \omega).$$

Note that thanks to the poset reciprocity theorem, both of these corollaries still hold if $(P \oplus Q, \omega \oplus \psi)$ is taken to be the labeled poset where every element of $P$ is weakly less than every element of $Q$. In particular, both of these corollaries hold for unlabeled posets.

# 3 Closed Families

We have already seen how the recurrences can provide elegant proofs of otherwise difficult problems with implications for doppelgangers, but its use does not end there. Consider a family of posets which is closed under one of our recurrence relations. If we can build a systematic way to decompose the more complicated posets, we would be able to easily calculate the order polynomials of the family and equivalently find and classify the doppelganger equivalence classes. Below we will consider a very simple family that is closed under the $P|x \le y$ and $P|x = y$ operations. Let $\mathscr{P}_n$ be the set of all posets of size $n$. For any k, the set $\mathscr{H}_k = \{P \in \mathscr{P}_n \mid h(P) > n - k\}$ is a closed family under our recurrence relations. In fact, picking $k = 2$, we can build such a systematic system for any n, thus circumventing traditional calculation of the order polynomial which is assymptotically hard (#P-hard) in the size of the poset.

## 3.1 Invariants

The previous section introduces a highly restrictive invariant on doppleganger classes, height. Invariants that can be easily calculated allow for classification of doppelgangers of certain families of posets. Lemma 7 presents four such invariants that have simple recursive formulas over the operations of direct and ordinal sum.

**Lemma 7.** *If $P \sim Q$ then $|P| = |Q|$, $F_P(2) = F_Q(2)$, $h(P) = h(Q)$, $l(P) = l(Q)$. Additionally,*

$$|P + Q| = |P| + |Q| \tag{1}$$
$$|P \oplus Q| = |P| + |Q| \tag{2}$$
$$F_{P+Q}(2) = F_P(2)F_Q(2) \tag{3}$$
$$F_{P\oplus Q}(2) = F_P(2) + F_Q(2) - 1 \tag{4}$$
$$h(P + Q) = \max(h(P), h(Q)) \tag{5}$$
$$h(P \oplus Q) = h(P) + h(Q) \tag{6}$$
$$l(P + Q) = \binom{|P| + |Q|}{|P|} l(P)l(Q) \tag{7}$$
$$l(P \oplus Q) = l(P)l(Q) \tag{8}$$

*for all posets $P, Q$.*

*Proof.* By Proposition 3, $|P| = \deg F_P$, $h(P)$ is the index of the first nonzero term of $F_P$ in the $\binom{x+k-1}{k}$ basis, and $l(P)$ is $(\deg F_P)!$ times the leading coefficient of $F_P$ as shown in Section 3.12 of [7]. Then all four invariants depend only on $F_P$ which shows the first part of the lemma. The recursive formulas follow from elementary combinatorial arguments. □

By Proposition 3, $F_P$ has roots at $0, -1, \ldots, -h(P) + 1$. This allows for the following necessary and sufficient condition for two posets to be doppelgangers.

**Proposition 8.** *$P \sim Q$ if and only if $|P| = |Q|$, $h(P) = h(Q)$, $l(P) = l(Q)$, and $F_P$ and $F_Q$ agree at $|P| - h(P) - 1$ distinct points (not counting the trivial roots at $0, -1, \ldots, -h(P)$).*

*Proof.* Lemma 7 shows the forward direction. For the converse, dividing $F_P$ and $F_Q$ by $\frac{l(P)}{|P|!}x(x+1)\ldots(x+h(P)-1)$ results in two monic polynomials of degree $|P|-h(P)$ which agree at $|P|-h(P)-1$ distinct points. Subtracting these two polynomials results in a polynomial of degree at most $|P| - h(P) - 1$ which vanishes at $|P| - h(P) - 1$ distinct points which is necessarily identically 0. Thus, $F_P = F_Q$ and $P \sim Q$. □

**Corollary 9.** *If $h(P) = |P| - 1$ then $P \sim Q$ if and only if $|P| = |Q|$, $h(P) = h(Q)$, and $l(P) = l(Q)$.*

**Corollary 10.** *If $h(P) = |P| - 2$ then $P \sim Q$ iff $|P| = |Q|$, $F_P(2) = F_Q(2)$, $h(P) = h(Q)$, and $l(P) = l(Q)$.*

### 3.2 $\mathscr{H}_1$ and $\mathscr{H}_2$

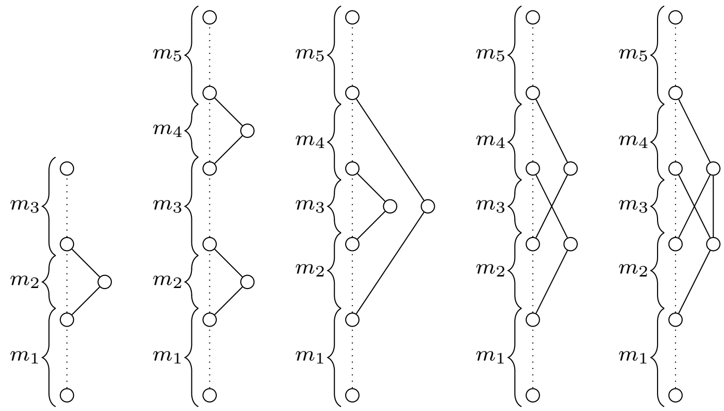
These results suggest a new way to classify all doppelgangers of posets with larger height. In particular, if we can compute $l(P)$ and $F_P(2)$ for all posets with $|P| = n$, $h(P) = n - 2$, we would be able to find all doppelgangers of such posets. In the next couple results, we prove that such posets fall into only a few infinite families.

**Lemma 11.** *If $x_1 \le \cdots \le x_n$ is a chain in $P$ and $x$ is some other element of $P$, then there exist nonnegative integers $m_1+m_2+m_3 = n$ such that $x$ is greater than $x_1, \cdots, x_{m_1}$, $x$ is incomparable to $x_{m_1+1}, \cdots, x_{m_1+m_2}$, and $x$ is less than $x_{m_1+m_2+1}, \cdots, x_{m_1+m_2+m_3}$.*

*Proof.* Let $m_1$ be maximal such that $x_{m_1} \le x$, let $m_2$ be minimal such that $x \le x_{m_1+m_2+1}$, and let $m_3 = n-m_1-m_2$. Then by transitivity, $x$ is greater than $x_1, \cdots, x_{m_1}$ and $x$ is less than $x_{m_1+m_2+1}, \cdots, x_{m_1+m_2+m_3}$. Additionally, $x$ is neither less than nor greater than $x_{m_1+1}, \cdots, x_{m_1+m_2}$. □

**Proposition 12.** *All posets $P$ with $|P| - h(P) = 1$ are isomorphic to a poset depicted by Figure 3(a).*

*Proof.* Let $C$ be a maximal chain in $P$ and let $x$ be the remaining element of $P$. Let $m_1, m_2, m_3$ be the result of applying Lemma 11 to $C$ and $x$. Then $P \cong Tri(m_1, m_2, m_3)$. □




(a) $Tri$ (b) $Dtri$ (c) $Ntri$ (d) $Xdis$ (e) $Xcon$

Figure 3: Five infinite families of posets.

| Infinite Family | $l(P)$ | $F_P(2)$ |
|---|---|---|
| $Tri(m_1, m_2, m_3)$ | $m_2 + 1$ | $\lvert P\rvert + m_2 + 1$ |
| $Dtri(m_1, m_2, m_3, m_4, m_5)$ | $(m_2 + 1)(m_4 + 1)$ | $\lvert P\rvert + m_2 + m_4 + 1$ |
| $Ntri(m_1, m_2, m_3, m_4, m_5)$ | $(m_2 + m_3 + m_4 + 2)(m_3 + 1)$ | $\lvert P\rvert + m_2 + 3m_3 + m_4 + 2$ |
| $Xdis(m_1, m_2, m_3, m_4, m_5)$ | $(m_2 + m_3 + 1)(m_3 + m_4 + 1) + m_3 + 1$ | $\lvert P\rvert + m_2 + 3m_3 + m_4 + 2$ |
| $Xcon(m_1, m_2, m_3, m_4, m_5)$ | $(m_2 + m_3 + 1)(m_3 + m_4 + 1) - \frac{1}{2}m_3(m_3 + 1)$ | $\lvert P\rvert + m_2 + 2m_3 + m_4 + 1$ |

Table 1: Values of $l(P)$ and $F_P(2)$ for the five infinite families in Figure 1.

**Proposition 13.** *All posets $P$ with $|P| - h(P) = 2$ are isomorphic to poset depicted by Figures 3(b-e).*

*Proof.* Let $C$ be a maximal chain in $P$ and let $x, y$ be the two remaining elements of $P$. Let $m_1, m_2, m_3$ be the result of applying Lemma 11 to $C$ and $x$ and let $n_1, n_2, n_3$ be the result of applying Lemma 11 to $C$ and $y$. Then

$$P \cong \begin{cases} Ntri(m_1, n_1 - m_1, n_2, n_3 - m_3, m_3) & m_1 \leq n_1,\ m_3 \leq n_3,\ x \text{ and } y \text{ incomparable} \\ Ntri(n_1, m_1 - n_1, m_2, m_3 - n_3, n_3) & m_1 \geq n_1, m_3 \geq n_3,\ x \text{ and } y \text{ incomparable} \\ Xdis(m_1, n_1 - m_1, m_1 + m_2 - n_1, m_3 - n_3, n_3) & m_1 \leq n_1, m_3 \geq n_3,\ x \text{ and } y \text{ incomparable} \\ Xdis(n_1, m_1 - n_1, n_1 + n_2 - m_1, n_3 - m_3, m_3) & m_1 \geq n_1, m_3 \leq n_3,\ x \text{ and } y \text{ incomparable} \\ Xcon(m_1, n_1 - m_1, m_1 + m_2 - n_1, m_3 - n_3, n_3) & m_1 + m_2 - n_1 \geq 0,\ x \leq y \\ Xcon(n_1, m_1 - n_1, n_1 + n_2 - m_1, n_3 - m_3, m_3) & n_1 + n_2 - m_1 \geq 0,\ y \leq x \\ Dtri(m_1, m_2, n_1 - m_1 - m_2, n_2, n_3) & n_1 - m_1 - m_2 \geq 0,\ x \leq y \\ Dtri(n_1, n_2, m_1 - n_1 - n_2, m_2, m_3) & m_1 - n_1 - n_2 \geq 0,\ y \leq x \end{cases}.$$

□

The values of the invariants for the Posets in Figure 3 are given in Table 1 and the computation of these values can be found in the appendix. The result of this table is that we can compute all doppelgangers among posets of height at most $|P| - 2$ by solving various pairs of Diophantine equations. For example, this leads to the infinite families of posets depicted in Figure 4.

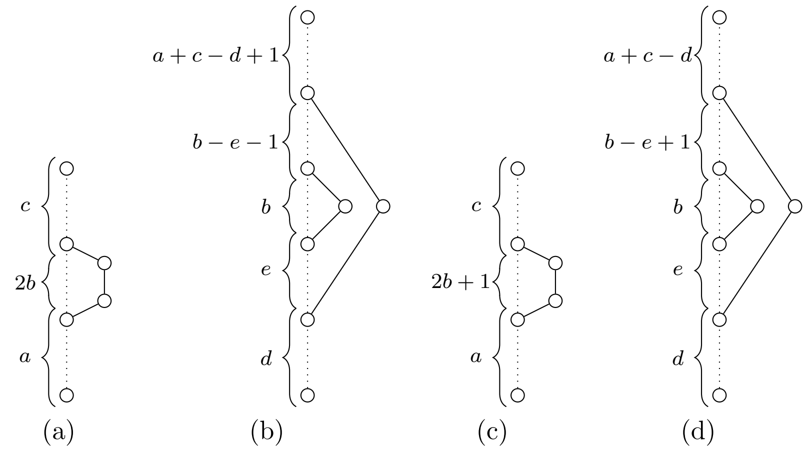



Figure 4: Two Infinite Families of Doppelgangers which follow from Table 1: (a)∼(b) and (c)∼(d)

# 4 The Ur-Operation

Much of our work above relies or provides results on the interactions of standard poset operations with the order polynomial. In fact, by considering a generalization of these operations, it is possible in turn to extend our results on the disjoint and ordinal sum operations. The operation itself is simple: consider replacing some subset of points in a poset $\mathscr{P}$ by a corresponding set of posets $\{P_1, \cdots, P_k\}$.

**Definition 14.** *For a poset $\mathscr{P} = \{x_1, \cdots, x_n\}$ and a sequence of posets $\{P_1, \cdots, P_n\}$, let $\mathscr{P}[x_k \to P_k]_{k=1}^n$ be the poset on $\bigcup_k P_k$ with the following operation:*

$$\text{For } p \in P_j, q \in P_k,\ p \leq q \text{ when } \begin{cases} p \leq q & j = k \\ x_j \leq x_k & j \neq k \end{cases}.$$

*We denote this as the Ur-operation on $\mathscr{P}$ by $\{P_1, \cdots, P_n\}$. All $P_k$ are assumed to be $C_1$ if not specified.*

Note that the disjoint sum operation denoted by $P_1 + P_2$ can be expressed as $A_2[x_k \to P_k]_{k=1}^2$, the ordinal sum operation denoted by $P_1 \oplus P_2$ can be expressed as $C_2[x_k \to P_k]_{k=1}^2$, and the ordinal product operation denoted by $P \otimes Q$ can be expressed as $P[x_k \to Q]_{k=1}^n$. In these ways, the Ur-operation generalizes the disjoint sum, ordinal sum, and ordinal product operations. The order polynomial of the Ur-operation relies heavily on the structure $\mathscr{P}$. Therefore it is convenient throughout the rest of this section to have the following definition

**Definition 15.** *For a poset $P$ and $x \in P$, define $g_x^P(n, m)$ to be the number of order-preserving maps $f : \mathscr{P}[x \to \varnothing] \to [m]$ such that $1 + \min_{x \leq y} f(y) - \max_{y \leq x} f(y) = n$, where the min and max are taken to be $m$ and $1$ respectively if not well defined.*

## 4.1 The Order Polynomial

With this in hand, we offer a simple formula for the order polynomial of a single substitution. The polynomial for the general operation may be given by repeated application

**Proposition 16.** *For a poset $\mathscr{P}$ with $x \in \mathscr{P}$, a poset $Q$, and $m \geq 1$,*

$$F_{\mathscr{P}[x \to Q]}(m) = \sum_{n=1}^{m} g_x^{\mathscr{P}}(n, m) F_Q(n).$$

*Proof.* The result follows from summing over all plane partitions $f : \mathscr{P}[x \to \varnothing] \to [m]$ and counting at each step the possible plane partitions of Q that satisfy the arrangement. $\square$

As expected, the formulae for direct and ordinal sum follow immediately from our generalization:

**Corollary 17.** *$F_{P+Q}(m) = F_P(m)F_Q(m)$, and $F_{P\oplus Q} = \sum_{i=1}^{m} F_Q(m+1-i)(F_P(i) - F_P(i-1))$ where $F_P(0)$ is defined to be 0.*

*Proof.* By Proposition 16,

$$\begin{aligned} F_{P+Q}(m) &= F_{(P+e)[e\to Q]}(m) \\ &= \sum_{n=1}^{m} g_e^{P+e}(n,m)F_Q(n) \\ &= F_P(m)F_Q(m). \end{aligned}$$

where we leverage $g_e^{P+e}(n,m) = \begin{cases} F_P(m) & n = m \\ 0 & n \neq m \end{cases}$

Similarly,

$$\begin{aligned} F_{P\oplus Q}(m) &= F_{(P\oplus e)[e\to Q]}(m) \\ &= \sum_{n=1}^{m} g_e^{P\oplus e}(n,m)F_Q(n) \\ &= \sum_{n=1}^{m} (F_P(m+1-n) - F_P(m-n))F_Q(n) \end{aligned}$$

where the result follows by replacing $n$ by $i = 1 + m + 1 - n$. $\square$

Moreover, the following result shows that the Ur-operation generalizes the nice relation between the standard operations and doppelgangers.

**Theorem 18.** *For a poset $\mathscr{P} = \{x_1, \cdots, x_n\}$ and two sequences of posets $\{P_1, \ldots, P_n\}$ and $\{Q_1, \ldots, Q_n\}$ such that $P_i \sim Q_i$, we have that $\mathscr{P}[x_k \to P_k]_{k=1}^n \sim \mathscr{P}[x_k \to Q_k]_{k=1}^n$.*

*Proof.* For a poset $P$, let $S_P(n)$ denote the number of strict surjective order preserving maps $f\colon P \to [n]$. By the proof of Proposition 3 it suffices to show that $S_{\mathscr{P}[x_k\to P_k]_{k=1}^n} = S_{\mathscr{P}[x_k\to Q_k]_{k=1}^n}$. We call an collection of intervals $\{[a_k, b_k]\}_{k=1}^n$ nice if they cover $[n]$ and if $b_j < a_k$ whenever $x_j < x_k$. Let $\mathscr{A}$ denote the set of nice collections of intervals. Then

$$S_{\mathscr{P}[x_k\to P_k]_{k=1}^n} = \sum_{\mathscr{A}} \prod_{k=1}^{n} S_{P_k}(b_k - a_k + 1) = \sum_{\mathscr{A}} \prod_{k=1}^{n} S_{Q_k}(b_k - a_k + 1) = S_{\mathscr{P}[x_k\to Q_k]_{k=1}^n}.$$

where the middle equality uses the fact that $S_{P_k} = S_{Q_k}$ which follows from the proof of Proposition 3 and the fact that a representation in the $\binom{x}{m}$ basis is unique. Also note that if any $b_k - a_k + 1 < h(P_k) = h(Q_k)$ or $b_k - a_k + 1 > |P_k| = |Q_k|$ then the corresponding term in the product will be 0 so, in fact, we only need to consider nice intervals where $h(P_k) = h(Q_k) \leq b_k - a_k + 1 \leq |P_k| = |Q_k|$. $\square$

Theorem 18 allows us to build new doppelgangers out of an arbitrary poset by iteratively replacing points with corresponding doppelgangers. We know as well, however, that one can construct doppelgangers by replacing different points of some poset P with corresponding doppelgangers, such as in $C_k$ or $A_k$. It is natural then to ask about a generalization of this occurrence. For posets $P$ and $Q$ with $x \in P$ and $y \in Q$, when do we have $P[x \to R] \sim Q[y \to S]$ for all doppelgangers $R \sim S$?

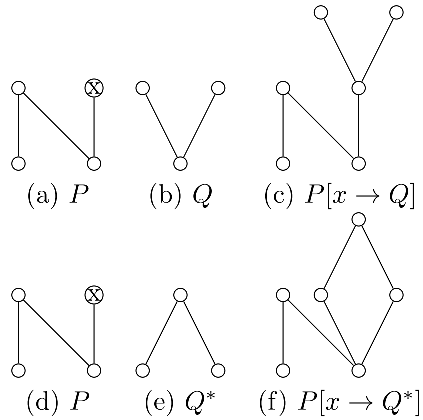



Figure 5: An example of dopplegangers due to the Ur-operation.

## 4.2 Ur-Equivalence

**Definition 19.** *We say $x \in P,\ y \in Q$ are Ur-equivalent when $P[x \to R] \sim Q[y \to S]$ for all posets $R \sim S$.*

In fact, Ur-equivalence relies on exactly the same structure the order polynomial does: on the values $g_x^P(n, m)$ and $g_y^Q(n, m)$.

**Proposition 20.** *$x \in P,\ y \in Q$ are Ur-equivalent if and only if $g_x^P = g_y^Q$*

*Proof.* The backward direction is immediate from proposition 18. If $x \in P, y \in Q$ are Ur-equivalent then we have $\sum_{i=1}^{m+1} g_x^P(i, m) * F_R(i) = \sum_{i=1}^{m+1} g_x^P(i, m) * F_R(i)$ for all posets $Q$. For any $m$, consider applying this equation to any set of posets $S_1, \ldots, S_m$ such that $|P_i| = i$. Let $g_x^P(i, m) - g_y^Q(i, m) = c(i, m)$. This gives the system of equations

$$\begin{bmatrix} F_{S_1}(1) & \cdots & F_{S_1}(n) \\ \vdots & \ddots & \vdots \\ F_{S_n}(1) & \cdots & F_{S_n}(n) \end{bmatrix} \begin{bmatrix} c(1, m) \\ \vdots \\ c(m, m) \end{bmatrix} = 0$$

This matrix is invertible due to the fact that the $F_{S_i}$ are linearly independent, thus the $c(i, m)$ are 0 and for $n \leq m$ $g_x^P(n, m) = g_y^Q(n, m)$, and by definition for $n > m$ both values are 0. $\square$

**Corollary 21.** *$x \in P, y \in Q, |P| = |Q| = n$ are Ur-equivalent if and only if there exist posets $\{S_1, \cdots, S_n\}$ with $|S_i| = i$ such that $P[x \to S_i] \sim Q[y \to S_i],\ 1 \leq i \leq n$*

Unfortunately, while $g_x^P$ reveals the structure behind Ur-equivalence, in general it is too difficult to calculate to be of practical use. However, one may note that $g_x^P$ is totally determined by the structure of $P[x \to \varnothing]$ and its relation to $P$. This suggests that we may be able to strengthen the above result:

**Conjecture 22.** *$x \in P$ and $y \in Q$ are Ur-equivalent if and only if $P[x \to A_k] \sim Q[y \to A_k]$ for $0 \leq k \leq 1$.*

While this conjecture may seem unlikely with the above information alone, like the order polynomial, $g_x^P$ has significant extra structure that is not understood. In fact, the conjecture holds for small posets, and further is equivalent to a number of simpler statements. For instance,

**Proposition 23.** *Given $P, Q$ and elements $x \in P, y \in Q$, if $P[x \to A_k] \sim Q[y \to A_k]$ for $0 \leq k \leq 1$ implies that $P[x \to A_2] \sim Q[y \to A_2]$, then Conjecture 22 holds.*

*Proof.* We will prove that $P[x \to A_k] \sim Q[y \to A_k]$ for $\forall\ k \in \mathbb{N}$ by induction on $k$. The RHS of Conjecture 22 is exactly the base case of our induction. Assume $P[x \to A_i] \sim Q[y \to A_i]$ for $0 \le i \le k$, and let $P' = P[x \to A_k] \sim Q' = Q[y \to A_k]$. $P'[x \to \varnothing] = P[x \to A_{k-1}] \sim Q[y \to A_{k-1}] = Q'[x \to \varnothing]$. Then our initial assumption gives that $P'[x \to A_2] = P'[x \to A_{k+1}] \sim Q'[y \to A_2] = Q[y \to A_{k+1}]$. This concludes our induction and the result then follows from Proposition 20. $\square$

While we do not know whether the assumption in Proposition 23 is true for all posets $P, Q$, we do know it is true for certain families of posets, such as $C_k$ or $A_k$, and it gives a simpler formulation of Conjecture 22.

# 5 Further directions

## 5.1 The Multivariate Generating Function

Perhaps the most obvious extension of this paper would be to more carefully investigate the implications of both the proper and improper recurrence relations on the multivariate generating function. Induction on incomparable elements led to a number of nice results over order polynomials, and such a tool could potentially be used to approach some of the questions offered in the end of McNamara and Ward's paper [8].

## 5.2 Single Step Chain Decomposition

We briefly touched on how our recurrences may be useful, even in only a single application, both in explaining one of McNamara's posets as well as easily constructing an infinite family of doppelgangers for $C_n + C_n$. What other dopplegangers can be explained through a single set of chain decomposition? $C_n + C_n$ has high structural symmetry and simplicity. In a similar vein, Stanley suggested classifying doppelgangers which cannot be shown in a single step of a recurrence.

## 5.3 Further Closed Families

Using height as a factor in our closed family is useful due to its invariant nature on the order polynomial, but many other closed families exist. For instance, series-parallel posets are closed under recurrence. This, however, is less interesting due to the fact that the order polynomial of series parallel posets is already simple to calculate due to the existence of simple formulas for the ordinal polynomial of a disjoint sum or an ordinal sum which are given in Lemma 4 and Corollary 17. Similar to the set of height families, for any $k$ the set $\{P \in \mathscr{P}_n \mid w(P) \le k\}$ is a closed family. Here the decomposition is not so simple as above, and width is not a invariant on the order polynomial. However, the idea could carry over to the multivariate generating function where matters of width and height create invariants. Further, similar to the k=n-2 case for height above, there is a systematic decomposition for width k=2, but without an invariant the system is too complicated to be particularly fruitful for doppelgangers.

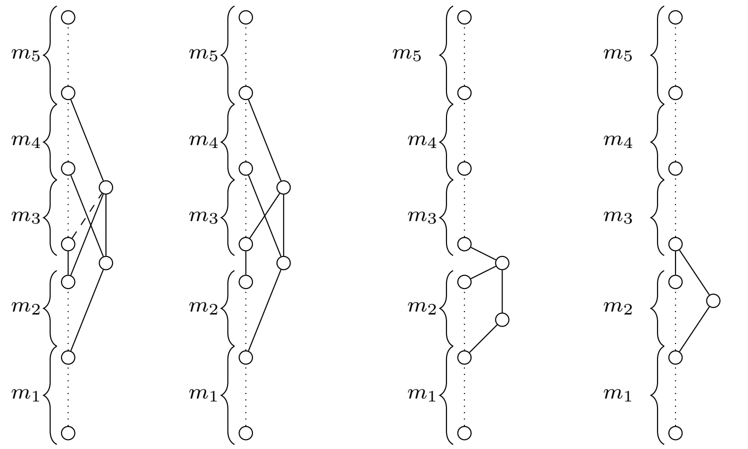
## 5.4 Explaining Non-Series Parallel Doppelgangers with the Ur-Decomposition

Since the Ur-Decomposition exists for all posets, it might be able to be used to recover some of the non-series parallel doppelgangers produced in [6].

# 6 Appendix

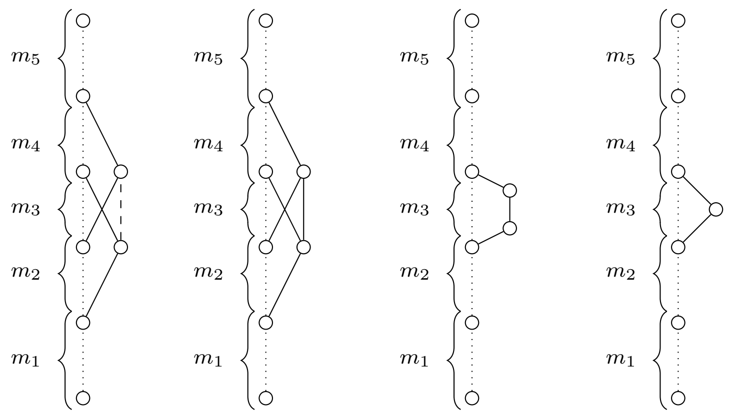
## 6.1 Computation of Invariants for Posets of large height

**Proposition 24.** *The values of $l(P)$ and $F_P(2)$ of the posets depicted in Figure 1 are given by Table 1.*




(a) $Xcon$ (b) $Xcon|x \le y$ (c) $Xcon|y \le x$ (d) $Xcon|x = y$

Figure 6: Chain decomposition of $Xcon$




(a) $Xdis$ (b) $Xdis|x \le y$ (c) $Xdis|y \le x$ (d) $Xdis|x = y$

Figure 7: Chain decomposition of $Xdis$

*Proof.* Note that

$$\begin{aligned} Tri(m_1, m_2, m_3) &\cong C_{m_1} \oplus (C_{m_2} + C_1) \oplus C_{m_3} \\ Dtri(m_1, m_2, m_3, m_4, m_5) &\cong C_{m_1} \oplus (C_{m_2} + C_1) \oplus C_{m_4} \oplus (C_{m_3} + C_1) \oplus C_{m_5} \\ Ntri(m_1, m_2, m_3, m_4, m_5) &\cong C_{m_1} \oplus ((C_{m_2} \oplus (C_{m_3} + C_1) \oplus C_{m_4}) + C_1) \oplus C_{m_5}. \end{aligned}$$

Then the formulas for $Tri$, $Dtri$, and $Ntri$ follow from Lemma 3. For $Xcon$, note that by chain decomposition and the formula for $l(Dtri)$,

$$\begin{aligned} l(Xcon(m_1, m_2, m_3, m_4, m_5)) &= l(Xcon(m_1, m_2 + 1, m_3 - 1, m_4, m_5)) + l(Tri(m_1, m_2, m_3 + m_4 + m_5 + 1)) \\ &= l(Xcon(m_1, m_2 + 1, m_3 - 1, m_4, m_5)) + m_2 + 1 \end{aligned}$$

and by induction,

$$\begin{aligned}l(Xcon(m_1,m_2,m_3,m_4,m_5)) &= l(Xcon(m_1,m_2+m_3,0,m_4,m_5)) + \sum_{k=1}^{m_3}(m_2+k)\\&= l(Dtri(m_1,m_2+m_3,0,m_4,m_5)) + m_2m_3 + \frac{1}{2}m_3(m_3+1)\\&= (m_2+m_3+1)(m_3+m_4+1) - \frac{1}{2}m_3(m_3+1).\end{aligned}$$

Again, by chain decomposition and the formula for $F_{Dtri}(2)$,

$$\begin{aligned}F_{Xcon(m_1,m_2,m_3,m_4,m_5)}(2) =& F_{Xcon(m_1,m_2+1,m_3-1,m_4,m_5)}(2) + F_{Tri(m_1,m_2,m_3+m_4+m_5+1)}(2)\\&- F_{Tri(m_1,m_2,m_3+m_4+m_5)}\\=& F_{Xcon(m_1,m_2+1,m_3-1,m_4,m_5)}(2) + 1\end{aligned}$$

and by induction,

$$\begin{aligned}F_{Xcon(m_1,m_2,m_3,m_4,m_5)}(2) &= F_{Xcon(m_1,m_2+m_3,0,m_4,m_5)}(2) + m_3\\&= F_{Dtri(m_1,m_2+m_3,0,m_4,m_5)}(2) + m_3\\&= |P| + m_2 + 2m_3 + m_4 + 1.\end{aligned}$$

For $Xdis$, note that by chain decomposition and the formula for $l(Xcon)$,

$$\begin{aligned}l(Xdis(m_1,m_2,m_3,m_4,m_5)) &= l(Xcon(m_1,m_2,m_3,m_4,m_5)) + l(Xcon(m_1+m_2,0,m_3,0,m_4+m_5))\\&= (m_2+m_3+1)(m_3+m_4+1) + (m_3+1)(m_3+1) - m_3(m_3+1)\\&= (m_2+m_3+1)(m_3+m_4+1) + m_3 + 1.\end{aligned}$$

Again, by chain decomposition and the formulas for $F_{Xcon}(2)$ and $F_{Tri}(2)$,

$$\begin{aligned}F_{Xdis(m_1,m_2,m_3,m_4,m_5)}(2) =& F_{Xcon(m_1,m_2,m_3,m_4,m_5)} + F_{Xcon(m_1+m_2,0,m_3,0,m_4+m_5)}\\&- F_{Tri(m_1+m_2,m_3,m_4+m_5)}\\&= |P| + m_2 + 2m_3 + m_4 + 1 + |P| + 2m_3 + 1 - (|P| - 1 + m_3 + 1)\\&= |P| + m_2 + 3m_3 + m_4 + 2.\end{aligned}$$

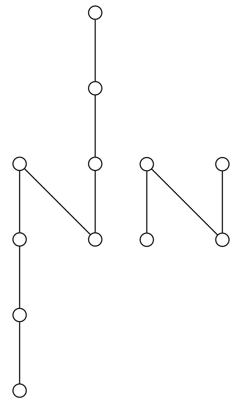
□

## 6.2 Ur-Decomposition

Every poset has a unique decomposition in terms of the Ur-operation, a generalization of the series-parallel decomposition, but the statement and proof of this decomposition requires slightly more machinery than our paper presents. To begin, we must define a notion analogous to prime posets. Recall that a poset $P$, $|P| > 1$, is called prime if it cannot be expressed as the ordinal sum or disjoint union of two posets. The decomposition of posets into primes by these two operations is known as the series-parallel decomposition. Similarly, a poset $P$, $|P| > 2$, is a strong prime if it cannot be expressed as a result of a non-trivial Ur-Operation. Note that a poset is prime if it is a strong prime, but the converse does not hold.

Consider when some poset $P$ could have been created via the Ur-Operation. This can only be the case if we can find some subposet which is reducible to a point, more formally

**Definition 25.** *A subset of a poset $\{x_k\} \subset P$ is reducible to a point (an RAP) when for every $y \in P - \{x_k\}$, either $y \leq \{x_k\}$, $\{x_k\} \leq y$, or $\{x_k\}$ and $y$ are incomparable. An RAP $\{x_k\}$ of $P$ is maximal when it is neither $P$ nor a subset of any other RAPs other than $P$.*



(a) (b)

Figure 8: A prime and its corresponding strong prime

Notably, an Ur-operation is an expression of the form $\mathscr{P}[x_k \to P_k]_{k=1}^n$ where each $P_k$ is an RAP. RAPs should be considerd under two circumstances. If there does not exist Non-trivially intersecting RAPs such that their union is the entire poset, then RAPs are closed under union and intersection. This allows for the result that in this non-degenerate case, RAPs partition the poset. The degenerate case corresponds to when a poset can be expressed as an ordinal sum or disjoint union. Then RAPs may be used to further decompose prime posets, and give a more geometric description of strongly prime posets.

**Proposition 26.** *Any poset $P$, $|P| > 2$ is a strong prime if and only if it does not contain any non-trivial RAPs.*

*Proof.* We will show that a poset is not a strong prime if and only if it contains an RAP. For the forward direction, suppose that $P$ is not a strong prime. By definition, $P = \mathscr{P}[x_k \to P_k]$. Then $P_1$ is an RAP of $P$. For the reverse direction, suppose that $L$ is an RAP of $P$. Let $\mathscr{P}$ be the poset given by reducing $L$ to a point $x$. Then $P = \mathscr{P}[x \to L]$ and $P$ is not a strong prime. $\square$

**Lemma 27.** *If $x$ and $y$ are RAPs of $P$ with $x \cap y \neq \varnothing$, then $x \cap y$ and $x \cup y$ are RAPs of $P$.*

*Proof.* Suppose that $z \in P - x \cup y$. Then $xRz$ and $yQz$ for some relations $R, Q$. Since $x \cap y \neq \varnothing$, $R = Q$ and $(x \cup y)Rz$ which shows that $x \cup y$ is an RAP. Since $x \cap y \subseteq x$, $x \cap y$ is RAP to $P - x$. Since $x \cap y \subseteq y$, $x \cap y$ is RAP to $P - y$. Then $x \cap y$ is RAP to $(P - x) \cup (P - y) = P - (x \cap y)$. $\square$

**Proposition 28.** *For any prime poset $P$, the maximal RAPs of $P$ partition $P$.*

*Proof.* Since every point is in an RAP, every point is in a maximal RAP. Suppose that $x$ and $y$ are two maximal RAPs with $x \cap y \neq \varnothing$. If $x \cup y \neq P$ then by Lemma 27 $x \cup y$ would be an RAP which contradicts the maximality of $x$ and $y$. Then $x \cup y = P$ and either $x \leq y$, $y \leq x$, or $x$ and $y$ are incomparable. In these cases, $P = x \oplus y$, $P = y \oplus x$, and $P = x + y$ respectively. Since these contradict the primality of $P$, such an $x$ and $y$ don't exist and all maximal RAPs are disjoint. Assume there exist two such partitions, then there must exist distinct RAPs $S, T$ where $S \cap T \neq \emptyset$, but this violates maximality by the same argument as above. $\square$

Finally we are ready to introduce the Ur-decomposition. We say a poset $P$ is Ur-decomposable if $|P| = 1$ or if $P$ can be expressed as $P = \mathscr{P}[x_k \to P_k]_{k=1}^n$, where $\mathscr{P}$ is a strong-prime, chain, or antichain and where each $P_k$ is Ur-decomposable. In the case that $\mathscr{P}$ is a chain or antichain, we additionally insist that each $P_k$ is maximal, in that it cannot be expressed as the result of an ordinal or direct sum respectively. Such a decomposition is called an Ur-decomposition.

**Theorem 29.** *All posets have a unique Ur-decomposition.*

*Proof.* It suffices to show that each nontrivial poset $P$ can be uniquely expressed as $P = \mathscr{P}[x_k \to P_k]_{k=1}^n$ where $\mathscr{P}$ is a strong-prime, chain, or antichain.

Assume $P$ is prime, let $\{P_k\}$ be the set of maximal RAPs which partition $P$, and let $\mathscr{P}$ be the poset defined on the RAPs. Assume $\mathscr{P}$ is not a strong prime, then by Proposition 2 $\mathscr{P}$ contains a non-trivial RAP $S$. Then the RAPs associated with $S$ from an RAP in $P$, which violates the maximality of RAPs in our partition. Furthermore, $P$ is prime, and thus cannot be decomposed into a chain or anti-chain, and the RAP partition of $P$ is unique.

Assume $P$ is not prime, then $P$ is expressible as the result of a direct or ordinal sum, and existence of an Ur-decomposition as a chain or anti-chain is immediate. Furthermore, these options are exclusive, and the insistance on maximal chains gives uniqueness within each class. Then it must only be shown that such a $P$ cannot be decomposed into a strong prime $\mathscr{P}$.

Let $P = \mathscr{P}[x_k \to P_k]_{k=1}^n$, because $P$ is not prime, there exists a subposet $S$ s.t. $S$ has the same relation to every element in $P - S$. $S$ must be contained at least partially in some $P_i$, but since $P_i$ is a RAP it must have the same relation to every element in $P - P_i$. Furthermore, this implies $x_i \in \mathscr{P}$ has the same relation to $\mathscr{P} - x_i$. Then either $\mathscr{P} = A_2$ or $C_2$, or $|\mathscr{P}| > 2$ and must contain a non-trivial RAP. In either case, $\mathscr{P}$ cannot be a strong prime. $\square$

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