Electrical Networks: Open Questions and Brainstorms

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1) Consider a graph with $V = \{1A, 2A, 3A, 4A, 1B, 2B, 3B, 4B\}$; we have $iA \sim jB$ if and only if $i \neq j$, and there are no other edges; the interior vertices are $4A$ and $4B$. (Draw a picture.)[1] Note: If the sum of the conductances on the edges around one of the interior vertices is NOT zero, we can do a Y-Δ transformation, and from there it is easy to recover the network. However, I don't know what happens if both interior vertices have the conductances summing to zero.

2) Finish determining for what conductances the annular network $G(2, 4)$ is recoverable. It is recoverable for positive linear conductances and strictly increasing nonlinear ones. However, it's not recoverable for some signed conductances (or so I claim).[2] Characterize the response matrix/boundary relationship of the graph. What about $G(n, 2n)$?

3) Look at the networks in David Jekel's Recoverable Annular Network with Nonrecoverable Dual. For what conductances are they recoverable? What happens with signed conductances? Nonlinear?

4) Signed conductances: Take a graph G with signed conductances. What can you say about the dimension of the space of homogeneous solutions to the Dirichlet problem, based on the structure of the graph? What

[1] This graph is the bipartite cover of the graph $\mathcal{K}_4$ with one vertex interiorized, which is a non-recoverable circular planar graph for which the inverse problem generally has a three-parameter family of solutions.

[2] To see this, consider the subgraph G' obtained by contracting the spikes at north outer boundary, south outer boundary, east inner boundary, and west inner boundary; if the subgraph is not recoverable, then neither is the whole graph. In the subgraph, apply the star-K transformation to each interior vertex; choose conductances (depending on some parameters) which satisfy the quadrilateral rule and yield the same response matrix regardless of the values of the parameters. Then reverse the star-K transformation.

about the Neumann problem? (See Konrad Schroder's paper.) For what graphs is it possible to find conductances such that there are γ-harmonic functions with voltage and current zero on the boundary?[3]

5) Pick your favorite surface with boundary (Torus? Mobius strip?) What can you say about electrical networks on graphs embedded on the surface? What conditions on the medial graph or on the graph itself guarantee recoverability?

6) What happens with nonlinear conductances on the triangle-in-triangle graph?[4]

7) Study infinite electrical networks with nonlinear and signed conductances. How far can Ian Zemke's methods be generalized? What other approaches will work for signed conductances? (See also #23 below)

8) Under what conditions is an infinite tree recoverable? (What does recoverable mean?)

9) When is the universal cover of a graph (an infinite tree) recoverable?

10) What can you say about covering graphs in general? Can you construct finite covering graphs which are "nicer" than the original graph? (See footnote to 1) for example.)

11) I say a graph is layerable if there is a way to reduce it to the empty graph by contracting spikes and deleting boundary edges (or more precisely, reduce it to a set of disconnected boundary vertices). True or false: A layerable graph cannot be N-to-1.

12) Are "most" graphs layerable? Are "most" graphs recoverable?

13) Global electrical equivalences: Suppose that two graphs produce the exact same set of response matrices. If they are not necessarily circular planar, do they have to be Y-Δ equivalent?[5]

14) Suppose that two graph produce the same set of boundary relationships over all bijective nonlinear conductances.[6] Are the two graphs the same?

[3]For this to occur, I claim the graph must contain a flower; however, there exist flowers for which this is not possible.

[4]You could look at the Jacobian of the Dirichlet-to-Neumann map and try to reduce to the linear case as in my paper on nonlinear conductances.

[5]Excluding silly things like interior dead branches.

[6]Or your favorite class of conductances.

15) For linear conductances, electrical networks have a probabilistic interpretation in terms of random walks. Is this true for nonlinear conductances?

16) Suppose that for a certain graph, any C^1 conductances with strictly positive derivative are recoverable. Are all strictly increasing conductances recoverable?

17) Can we recover networks with nonlinear conductances which are not strictly increasing near 0?

18) See David Jekel's paper on nonlinear conductances/the heat equation. Consider a network with nonlinear weakly increasing conductances. How smoothly does the solution depend on the initial data?

19) Does the wave equation ($u''(t) = -K(u(t)) + \theta(t)$ on the interior vertices) have a unique solution for nonconstant (continuous) boundary conditions?

20) See Lam and Pylyavsky's paper on "cylindrical electrical networks." They prove many results using the universal response matrix.[7] When do their results hold for ordinary response matrix and ordinary graph on the cylinder?

21) Probability: Suppose that $\gamma = \{\gamma_{pq}\}$ where the γ_{pq}'s are positive-valued independent random variables on a probability space $(\Omega, \mathcal{F}, P)$. Then since expectations commute with sums and products of independent variables, the expectation of the Kirchhoff matrix $E(K(\gamma))$ is simply $K(E\gamma)$, and the same is true of determinants of any submatrix of K. What can you say about $E(\Lambda(\gamma))$?

22) More probability: With the setup above, fix a response matrix Λ_0 and let $U \subset M_{n\times n}$ be a neighborhood of Λ_0. What is $P(\Lambda(\gamma) \in U)$? Consider the expected value of γ given that $\Lambda(\gamma) \in U$, which is

$$\frac{1}{P(\Lambda(\gamma) \in U)} \int_{\Lambda(\gamma)\in U} \gamma \, dP$$

(guess what this means and verify it makes sense).

[7] This is essentially the Dirichlet-to-Neumann map of the network on the infinite strip which is the universal cover of the cylinder. However, since they didn't have Ian Zemke's machinery for dealing with infinite graphs, they phrased it differently. By the way, beware of their opposite sign conventions.

This may give a more numerically stable answer to the inverse problem. Rather than finding γ that will map to Λ_0, we find γ's that map into a neighborhood of Λ_0 and average them. We can also control the conductances under consideration by adjusting the distribution of the γ_{pq}'s. That way, we can make it less likely that we'll get unreasonably large conductances for the answer to the inverse problem.

23) Functional analysis, anyone? Consider the following generalization of infinite electrical networks: Let (V, E, G) be an infinite graph (with possibly uncountably many vertices and vertices with infinite valence). Suppose

- μ is a σ-finite complete measure on V with some σ-algebra $\mathcal{M}$.
- For each $p \in V$, $\{q : q \sim p\}$ is $\mathcal{M}$-measurable.
- $E \subset V \times V$ is $\mathcal{M} \otimes \mathcal{M}$-measurable.
- $\gamma(p, q)$ is a function in $L^2(E)$ and for each p, $\int_{q\sim p} |\gamma(p, q)|\, dq$ exists and is bounded by some $M < \infty$ independent of p.
- $\gamma(p, q) = \gamma(q, p)$.

Define $K : L^2(V) \to L^2(V)$ by

$$Ku(p) = \int_{q\sim p} \gamma(p, q)(u(p)-u(q))\, dq = u(p) \int_{q\sim p} \gamma(p, q)\, dq - \int_{q\sim p} \gamma(p, q)u(q)\, dq.$$

The first term makes sense because $\int_{q\sim p} |\gamma(p, q)|\, dq \leq M$, and the second term defines a Hilbert-Schmidt operator with norm $\leq \|\gamma\|_{L^2(E)}$. Note $\|K\| \leq M + \|\gamma\|$. You could

- For $\gamma \geq 0$, define the Dirichlet inner product and norm on $L^2(V)$ (modulo the constant functions), and solve the Dirichlet problem using orthogonal projection.
- For signed γ, investigate the behavior of the kernel of K.
- If $\int_{q\sim p} \gamma(p, q)\, dq = 1$ for all p, then K is of the form $I - T$, where T is a compact symmetric operator.[8] By the Fredholm alternative, K is injective if and only if it is surjective. By the spectral theorem, T is diagonalizable; thus, $I - T$ is diagonalizable. What can we do with this?

This notion of an electrical network is very flexible–some examples:

[8]Symmetric because $\gamma(p, q) = \gamma(q, p)$ and compact because it is Hilbert-Schmidt.

- Infinite graphs with countably many vertices and $\sum_{pq \in E} \gamma_{pq}^2 < \infty$.
- Continuous problem: Let $V = \mathbb{R}^d$ and $p \sim q$ if $|p - q| < R$ with $\gamma(p, q) = \phi(|p - q|)$, where ϕ is some smooth function supported in $[0, R]$.
- Mix of continuous and discrete problems. Let $V = [0, 1]$ and $\mu = m + \delta_0 + \delta_{1/2} + \delta_1$ (Lebesgue measure plus some point measures). Define the edges however you want.
- V is the rational numbers with counting measure and $p = a/b$ and $q = c/d$ are adjacent iff $\gcd(b, d) > 1$. A number theorist could come up with something even more interesting here.
- $V = [0, 1]$ with Lebesgue measure and $p \sim q$ iff $|p - q|$ is not in the Cantor set. Is this kind of electrical network useful for geometry?