On Multiple methods of counting spanning trees of Circulant graphs

Michael Dotzel

August 9, 2017

1 Introduction

Counting the number of spanning trees on arbitrary graphs is a topic of great interest in the mathematical community. Some applications include cluster analysis, information transmission requiring minimal connection, and modeling local particle activity in turbulent fluid flow. Holroyd et al. in their paper "Chip firing and rotor-routing on directed graphs" present, as a corollary to the matrix tree theorem, that the set of labeled spanning trees of a graph G is isomorphic to the number of recurrent configurations in the sandpile group of G. A question raised by David Jekel and Will Dana was what the explicit structure of the sandpile group looked like for C a Cayley graph of $\mathbb{Z}_n$ - they determined algebraically that for odd n, the sandpile group of $C = \mathbb{Z}_n^{(1,2)} \simeq \mathbb{Z}_n \times (\mathbb{Z}_{f_n})^2$, where f_n denotes the nth term in the Fibonacci sequence. We prove this identity by means of "unhooking" the circulant graph and studying the movement of related graph structures around $\mathbb{Z}_n^{(1,2)}$, a process which lends itself to extrapolation to higher numbers of generators. We also explore methods of counting labeled spanning trees on these Cayley graphs by means of the principle of deletion-contraction and labeling spanning trees from their non-isomorphic unlabeled spanning tree structures. We end our discussion with several other interpretations of counting these spanning trees, and possible ways of extrapolating methods to larger classes of graphs.

2 Method 1: unhooking the circulant graph

In this discussion, our main objective is to prove the

Theorem 1. *The number of spanning trees on $C = \mathbb{Z}_n^{(1,2)}$ is nf_n^2, where f_n denotes the nth Fibonacci number, with $f_1 = f_2 = 1$.*

Consider $\mathbb{Z}_n^{(1,2)}$, the circulant Cayley graph on the cyclic group $\mathbb{Z}_n$ generated by elements 1 and 2: (insert diagram of graph) Now, imagine drawing a line between two of the vertices and deleting the edges incident to it. This results

in a linearized, "non-modular" version of $\mathbb{Z}_n^{(1,2)}$, shown below: We will first determine the number of spanning trees on this unhooked circulant graph. We start by introducing an identity that will prove to be useful later:
Identity: $f_{2n-2} = f_n^2 - f_{n-2}^2$.

Lemma 1. *The number of spanning trees on the "line" graph on n vertices is* f_{2n-2}.

Proof. (insert diagram of line graph) Let $F(n)$ be the number of labeled spanning trees on this line graph, and let $G(n)$ be the number of labeled spanning trees on this line graph tat contain the edge $(n-1,n)$. If a spanning tree does not involve $(n-1,n)$ then it must contain the edge $(n-2,n)$, since the spanning tree is connected. But this is equivalent to finding the number of spanning trees on the remaining $(n-1)$-vertex line graph, which is $F(n-1)$. But the count of spanning trees with edge $(n-1,n)$ is $G(n)$. So $F(n) = F(n-1) + G(n)$. Now, consider the spanning tree with edge $(n-1,n)$. If we omit edge $(n-2,n)$ from the spanning tree, we obtain $F(n-1)$. If we involve the edge $(n-2,n)$ in the spanning tree as well, we cannot involve the edge $(n-2,n-1)$, otherwise we form a cycle. So involving both edges is equivalent to involving edge $(n-2,n-1)$ in a spanning tree on an $n-1$ vertex line graph, so we get $G(n-1)$. So $G(n) = F(n-1) + G(n-1)$. Arranging these functions as $F(n), G(n+1) = F(n)+G(n), F(n+1) = G(n+1)+F(n), \ldots$ yields the familiar Fibonacci sequence recursion $f_n = f_{n-1}+f_{n-2}$. By observation of smaller cases, we obtain that $G(n) = f_{2n-3}$ and $F(n) = f_{2n-2}$, with $F(1) = 1, G(0) = 0$. □

The standard line graph will be the graph as labeled above.

Lemma 2. *Given the line graph on n vertices, add k a positive integer to each of the vertex labels mod n. Then these graphs yield distinct spanning trees, and each have the same number of spanning trees.*

Proof. The second part is trivial, as they have isomorphic graph structures. It suffices to show that the sets of spanning trees on these distinct line graphs are distinct. WLOG we can pick the standard line graph and the line graph obtained by adding 1 to each vertex. The latter involves vertex 1 at the far right connected to vertices n and n-1, connections which don't exist in former graph. So spanning trees are distinct in these graphs, and thus are distinct for any k-addition to each vertex of the standard line graph. More generally, adding k to each vertex of the standard line graph will require that a spanning tree passes through either 1 and 2 to the left of them (moving from left to right), edges which are not present in the standard line graph. □

Since there are n of these distinct labelings, the number of spanning trees on these line graphs $S = nf_{2n-2} = n(f_n^2 - f_{n-2}^2)$. However, it would seem that we UNDERCOUNTED by nf_{n-2}^2, so what other spanning trees are missing from the count? We start by considering the standard line graph, and the spanning trees it contains. However, it does not contain spanning trees of the following type: **(include examples)** In order to include these (the 1-shifted spanning trees), we need to consider the 1-shifted standard line graph. In general, the spanning trees will be contained in k-shifted line graphs, for $k < \lfloor n/2 \rfloor$, counting the n different labelings of these line graphs. So we have the following

Lemma 3. *The spanning trees on each of these k-shifted graphs are distinct.*

Proof. The argument is the same as that provided in lemma 2. $\square$

So now we have the following

Lemma 4. *There is a 1-1 correspondence between the number of spanning trees on $\mathbb{Z}_n^{(1,2)}$ and the spanning trees on all labeled k-shifted graph structures.*

Proof. (sketch) For a given labeling, a 0-shifted graph (i.e. the standard line graph) determines all possible spanning trees on $\mathbb{Z}_n^{(1,2)}$ that "wrap around" $\mathbb{Z}_n^{(1,2)}$ exactly once. By this we mean that if our tree starts at vertex k and we trace the tree around to its opposite endpoint $k \pm 1 \pmod{n}$, then an axis drawn between these vertices cannot intersect an edge between vertices k and $k+1$ or $k+2$, or vertex $k-1$ with 1. Less formally, there is no edge that "traverses" (or crosses past) a vertex that is already included in the spanning tree. Similarly, a 1-shifted graph (with n labelings) determines all possible spanning trees on $\mathbb{Z}_n^{(1,2)}$ that "wrap around" $\mathbb{Z}_n^{(1,2)}$ more than once, but by one edge. In other words, for all spanning trees counted in these types of graphs there exists two edges of the spanning tree that intersect between vertices $k, k+1$, as in the diagram pictured below. (INSERT DIAGRAMS) Since any spanning tree of $\mathbb{Z}_n^{(1,2)}$ can cycle around $\mathbb{Z}_n^{(1,2)}$ at most twice, and the $\lfloor (n-1)/2 \rfloor$-shifted graph determines spanning trees of all ranges including that, we observe that the set of these labeled k-shifted graphs exhaustively determine the spanning trees of $\mathbb{Z}_n^{(1,2)}$. $\square$

Lemma 5. $\sum_{k=1}^{(n-1)/2} f_{2(n-2k)-2} = f_{n-2}^2$.

Proof. $f_{2(n-2k)-2} = f_{n-2k}^2 - f_{n-2k-2}^2$, so we have $\sum_{k=1}^{(n-1)/2} f_{2(n-2k)-2}$
$= \sum_{k=1}^{(n-1)/2} f_{n-2k}^2 - f_{n-2k-2}^2 = f_{n-1}^2 + f_{n-2}^2 - 2 = f_{n-2}^2$. $\square$

Now, we get to work. There are n labeled k-shift line graphs for any $0 \leq k \leq (n-1)/2$, and so for arbitrary k we observe that there are $nf_{2(n-k)-2}$ spanning trees. Summing these using lemma 5 gives us the number of spanning trees on our circulant graph, namely $nf_{n-2}^2 + n(f_n^2 - f_{n-2}^2) = nf_n^2$.

3 Further work

- vector field interpretation
- discuss Prüfer sequences and how they count the complete graph case; goal is to generalize counting of circulant graphs or otherwise graphs with symmetry