Generalizing Boundary Edges and Boundary Spikes to Larger Subgraphs, and other partial recovery

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August 9, 2017

Introduction

Given a graph with boundary, and a conductance function on the edges, the response matrix is given by the Schur complement of the interior vertices in the graph Laplacian. We will denote the response matrix as ΛG. The inverse problem consists of reversing the Schur complement process, given knowledge of the graph topology. In this paper, we will give two situations in which we can undo a Schur complement given extra information about the graph.

The first situation attempts to extend the removal of boundary edges and boundary spikes to larger subgraphs. Once we know the conductivity of the edge (which has exactly the same information as the response matrix of the single edge subgraph), we can calculate the response matrix of the graph that remains when we remove the edge. We consider when, given the response matrix of a subgraph, we can perform an analogous removal process.

The second situation considers the hypothetical situation where, starting from one graph, we know two response matrices given by taking two disjoint sets of vertices to be interior, one at a time. The question is when can we recover the response matrix of the whole graph. We obtain some weak results in general, and some stronger results when we have only made a single node interior both times.

Recovery once subgraph response known

NOTE: This operation was previously studied under the name of splicing in [3] and [4]. A cursory examination suggests that Theorem 3.4, The Shell Stripping Formula in [3] is a slightly less general version of the formulas below, where we are restricted to have $I = \partial G_1 \cap \partial G_2$.

Consider two graphs G_1, G_2 with boundary ∂G_1 and ∂G_2. Consider graph G where we identify $\partial G_1 \cap \partial G_2$ in G_1 and G_2. Furthermore, take $I \subseteq \partial G_1 \cap \partial G_2$ and consider $G' = G/I$ obtained by making I into interior vertices. Then, splitting our response matrices into blocks ordered by $\partial G_1 - \partial G_2$, $\partial G_1 \cap \partial G_2 - I$, I, $\partial G_2 - \partial G_1$, we will write

$$\Lambda G_1 = \begin{pmatrix} A & B & C & 0 \\ B^T & D & E & 0 \\ C^T & E^T & F & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda G_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & c \\ 0 & b^T & d & e \\ 0 & c^T & e^T & f \end{pmatrix},$$

$$\Lambda G = \Lambda G_1 + \Lambda G_2 = \begin{pmatrix} A & B & C & 0 \\ B^T & D+a & E+b & c \\ C^T & (E+b)^T & F+d & e \\ 0 & c^T & e^T & f \end{pmatrix},$$

and $\Lambda G'$ is obtained from ΛG by taking the Schur complement of $F+d$. Letting $\alpha = (F+d)^{-1}$, we have

$$\begin{aligned} \Lambda G' &= \Lambda G/(F+d) \\ &= \begin{pmatrix} A - C\alpha C^T & B - C\alpha(E+b)^T & -C\alpha e \\ & D + a - (E+b)\alpha(E+b)^T & c - (E+b)\alpha e \\ & & f - e^T \alpha e \end{pmatrix} \\ &= \begin{pmatrix} p & q & r \\ & s & t \\ & & u \end{pmatrix}. \end{aligned}$$

(Where the lower triangle is the transpose of the upper triangle).

Now assume that C has a left inverse. This means that the columns of C are independent, which means that in G_1, manipulation of the potentials of vertices I must induce independent current vectors in the vertices $\partial G_1 - \partial G_2$. Denote the left inverse of C by $C^\dagger$. Note that $C^{\dagger T}$ is a right inverse for C^T.

We will obtain a formula for ΛG_2 (a through f) in terms of ΛG_1 (A through

F) and $\Lambda G'$ (p through u). We have

$$\begin{aligned}
\alpha =&(F+d)^{-1} = C^{\dagger}(A-p)C^{\dagger T},\\
d =&\alpha^{-1} - F,\\
b =&(\alpha^{-1}C^{\dagger}(B-q))^T - E,\\
e =&-\alpha^{-1}C^{\dagger}r,\\
a =&s + (E+b)\alpha(E+b)^T - D,\\
c =&t + (E+b)\alpha e,\\
f =&u + e^T\alpha e.
\end{aligned}$$

Thus we have computed ΛG_2 from ΛG_1 and $\Lambda G'$, assuming that C has a left inverse. One way of guaranteeing that C has a left inverse is to find a subset P of $\partial G_1 - \partial G_2$ such that $|P| = |I|$ and all the connections between P and I through G_1 have the same parity (and such a connection exists), since then the determinant-connection formula gives that the sub-matrix of C where we look at the rows corresponding to P is non-singular. One non-trivial example is the circle with six boundary nodes, considering every other node to be in I and the other half in P.

This is a generalization of boundary edges and boundary spikes (once we found the response matrix / conductivity). For a boundary edge with conductivity ξ, $|\partial G_1| = 2$, $\partial G_1 - \partial G_2 = \varnothing$, $I = \varnothing$, so

$$\Lambda G = \Lambda G' = \begin{pmatrix} \begin{pmatrix} \xi & -\xi \\ -\xi & \xi \end{pmatrix} + a & c \\ c^T & f \end{pmatrix},$$

and we can recover ΛG_2 simply by subtracting out $\begin{pmatrix} \xi & -\xi \\ -\xi & \xi \end{pmatrix}$ (d, b, e are empty matrices). (Refer to [1] §6.1).

For a boundary spike with conductivity ξ, we have $|\partial G_1| = 2$, $|\partial G_1 - \partial G_2| = 1$, $\partial G_1 \cap \partial G_2 = I$, $|I| = 1$, so

$$\Lambda G' = \begin{pmatrix} \xi & -\xi & 0 \\ -\xi & \xi + d & e \\ 0 & e^T & f^T \end{pmatrix} / (\xi + d) = \begin{pmatrix} \xi - \frac{\xi^2}{\xi+d} & \frac{\xi}{\xi+d}e \\ \frac{\xi}{\xi+d}e^T & f - \frac{1}{\xi+d}e^Te \end{pmatrix}.$$

(Refer to [1] §6.3).

Recovery of subgraph responses

When you can just read it from the larger response matrix

Consider the edge spans of a graph (partition the edges, putting two edges in the same component if they share an interior endpoint). Continue merging partitions that share two or more boundary nodes. The response matrix for each of the final parts is simply the restriction of the response matrix to that subgraph, since all 1-connections through the graph lie in a single part.

This is another way of explaining the smallest recoverable flower [2], a tetrahedron with interior nodes on the points and boundary nodes on the edges. Another way to view it is as four Y graphs glued together with any two sharing at most one point. Thus the response of each Y can be recovered, and the Y graph is recoverable, so the whole flower is recoverable.

This process seems to correspond to boundary edge recovery where the broken connection is a 1-connection (pq). For boundary edges and spikes, we can recover the response matrix even when a larger connection is broken. Generalizing this process to larger subgraphs is an area that needs more investigation.

Recovery from two partial recoveries

Moving to a different approach to partial recovery, assume we have a graph G, and two disjoint sets of vertices I and J, such that we can recover the response matrix of the graphs G/I and G/J obtained by making I and J interior vertices respectively.

For simplicity, assume that G is connected (if not, we can handle each component separately), and that both I and J are non-empty (if either one is empty, then we were already able to recover ΛG). Also assume that $\partial G - I - J$ is non-empty. These conditions ensures that any proper principal sub-matrix of ΛG is strictly positive definite, and so invertible, and that all the sub-matrices we want to take the inverse of are of this form.

If we split our response matrix into blocks ordered by $\partial G - I - J$, I, J, then we have

$$\Lambda G = \begin{pmatrix} A & B_1 & B_2 \\ & C_{11} & C_{12} \\ & & C_{22} \end{pmatrix},$$

$$\Lambda G/I = \Lambda G/C_{11} = \begin{pmatrix} A - B_1 C_{11}^{-1} B_1^T & B_2 - B_1 C_{11}^{-1} C_{12} \\ & C_{22} - C_{12}^T C_{11}^{-1} C_{12} \end{pmatrix} = \begin{pmatrix} x_1 & y_1 \\ & z_1 \end{pmatrix},$$

$$\Lambda G/J = \Lambda G/C_{22} = \begin{pmatrix} A - B_2 C_{22}^{-1} B_2^T & B_1 - B_2 C_{22}^{-1} C_{12}^T \\ & C_{11} - C_{12} C_{22}^{-1} C_{12}^T \end{pmatrix} = \begin{pmatrix} x_2 & y_2 \\ & z_2 \end{pmatrix}.$$

(where we are always working with symmetric matrices, so the lower triangle is assumed to contain the transpose of the upper triangle). We then have

$$B_1 = (y_2 + y_1 C_{22}^{-1} C_{12}^T) z_2^{-1} C_{11},$$

$$B_2 = (y_1 + y_2 C_{11}^{-1} C_{12}) z_1^{-1} C_{22},$$

$$A = x_1 + B_1 C_{11}^{-1} B_1^T = x_2 + B_2 C_{22}^{-1} B_2^T.$$

Therefore, if we know C_{11}, C_{12}, C_{22}, then we can recover the rest of ΛG (note that this approach recomputes the known values of z_1 and z_2).

If we know both C_{11} and C_{12}, then $C_{22} = z_1 + C_{12}^T C_{11}^{-1} C_{12}$. Symmetrically $C_{11} = z_2 + C_{12} C_{22}^{-1} C_{12}^T$.

To make further use of the information given by z_1 and z_2, let us restrict our attention to the case where $|I| = |J| = 1$. Then $C_{11}, C_{12}, C_{22}, z_1, z_2$ are real numbers, with $z_1, z_2, C_{11}, C_{22} > 0$, and $C_{12} \leq 0$.

If we know both C_{11} and C_{22}, then

$$C_{12} = -\sqrt{C_{11}(C_{22} - z_1)} = -\sqrt{C_{22}(C_{11} - z_2)}.$$

Now assume that we only know C_{12}.

Note that in the general case,

$$z_2 C_{11}^{-1} C_{12} = C_{12} C_{22}^{-1} z_1 = C_{12} - C_{12} C_{22}^{-1} C_{12}^T C_{11}^{-1} C_{12}.$$

In the singleton case we have commutativity of multiplication, so we can instead use

$$t = \frac{z_2}{C_{11}} = \frac{z_1}{C_{22}} = 1 - \frac{C_{12}^2}{C_{11} C_{22}} > 0.$$

Then

$$t = 1 - \frac{C_{12}^2}{z_1 z_2} t^2,$$

$$\frac{C_{12}^2}{z_1 z_2} t^2 + t - 1 = 0.$$

If $C_{12} = 0$, then $t = 1$. Otherwise, note that the left side is a quadratic that opens upwards, and has a point ($t = 0$) where it takes negative values. Therefore there are two real solutions for t, one positive and one negative. However, the sign conditions given by restricting to positive edge weights mean that we are only looking for $t > 0$. Therefore, we have

$$t = \frac{-1 + \sqrt{1 + 4\frac{C_{12}^2}{z_1 z_2}}}{2\frac{C_{12}^2}{z_1 z_2}},$$

$$C_{22} = \frac{z_1}{t}, \qquad C_{11} = \frac{z_2}{t}.$$

Remaining questions: Can we solve if we only know C_{11}? Can we extend the derivations from C_{11} and C_{22} or from C_{12} to the general case where I and/or J are not singletons?

Even more ambitiously, can we use the equation given by deriving A from x_1 and from x_2 to derive all of C_{11}, C_{12}, C_{22}? This was the original goal, since it would mean that there is a maximal subset of the interior vertices that can be recovered.

It may also be interesting to consider what happens when I and J are not disjoint, or when we have knowledge of more than two responses to start with.

References

[1] Curtis, Edward B. and James Morrow. *Inverse Problems for Electrical Networks.* World Scientific. 2000.

[2] Nicholas Reichert. "The Smallest Recoverable Flower" (2004).

[3] Ryan Card and Brandon Muranaka, *Using Network Amalgamation and Separation to Solve the Inverse Problem* (2003)

[4] Brian Lehmann, *Recoverability of Spliced Networks* (2002)