Iterated Contraction of Permutation Arrays

Avi Levy

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1 Chains

1.1 Definitions

A *chain* $\Sigma = \sigma_0 \to \sigma_1 \to \cdots \to \sigma_m$ is a sequence of permutations. $|\Sigma| = m$ denotes the number of transitions. $d(\Sigma) = |fp(\sigma_m)| - |fp(\sigma_0)|$ where fp denotes the set of fixed points. Σ_0 denotes σ_0.

Every permutation σ can be decomposed into a product of disjoint cycles, which we call the *cycle decomposition* of σ. If a cycle is a singleton, then it is called *trivial*.

1.2 Types of Chains

Fix a chain Σ made of permutations σ_i.

Σ is called

- *decreasing* if $i < j \implies fp(\sigma_i) \subset fp(\sigma_j)$. Note that if Σ is decreasing then $d(\Sigma) \geq 0$.
- *K-bounded* if for all transitions $\sigma_i \to \sigma_{i+1}$, we have $d(\sigma_i, \sigma_{i+1}) \leq K$. This time, d denotes the Hamming distance.

2 Main Result

Theorem 2.1. *If Σ is a decreasing $(K+1)$-bounded chain, then the cycle decomposition of Σ_0 has at least $d(\Sigma) - K|\Sigma|$ non-trivial $(1 \mod K)$-cycles.*

Corollary 2.2. *If Σ is a decreasing $(K+1)$-bounded chain and*

$$\frac{d(\Sigma)}{|\Sigma|} > K,$$

then Σ_0 contains a j-cycle such that

- $1 < j \leq K|\Sigma|$
- $j \equiv 1 \mod K$

3 Iterated Contractions

Sudborough et. al. introduced a contraction operation for permutation arrays. For every σ, the contraction is defined to be

$$\sigma' = \sigma(n\ \sigma \cdot n)$$

where n is the symbol to be deleted from σ. $\sigma^{(m)}$ denotes a permutation that is obtained by performing m contractions on σ.

Lemma 3.1. *If σ and τ are permutations such that*

$$d(\sigma, \tau) - d(\sigma^{(m)}, \tau^{(m)}) > 2m,$$

then the cycle decomposition of $\sigma\tau^{-1}$ contains a j-cycle where $1 < j < 2m$ and j is odd.

Theorem 3.2. *Let $M(n,d)$ be a permutation array. Suppose that no element $\sigma \in M(n,d)$ contains a j-cycle in its cycle decomposition (where $1 < j < 2m$ and j is odd). Then $M^{(m)}$ is a $PA(n-m, d-2m)$.*

4 Application to Permutation Groups

Theorem 4.1. *Let $G(n,d)$ be a sharply transitive group. Then $G^{(m)}$ is a $PA(n-m, d-2m)$ if and only if d has no odd divisor j where $1 < j < 2m$.*