TORSION OF THE GRAPH LAPLACIAN: SANDPILES, ELECTRICAL NETWORKS, AND HOMOLOGICAL ALGEBRA

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Abstract. An important technique in the electrical inverse problem is to "layer-strip" a graph with boundary by iteratively contracting boundary spikes and deleting boundary edges. We construct an algebraic invariant Υ to test whether a graph can be completely layer-stripped. This invariant generalizes the sandpile group of a graph to networks with boundary with weights in an arbitrary commutative ring. We use harmonic functions and discrete complex analysis to understand algebraic properties of Υ. We compute Υ for a family of cylindrical lattice graphs arising in the electrical inverse problem.

1. Introduction

1.1. **Motivation: Electrical Networks and Layer-Stripping.** Any bicolored graph can be interpreted as a **graph with boundary**: one color (say white) represents interior vertices, and the other color (black) represents boundary vertices. An **electrical network** is an edge-weighted graph with boundary, where each weight $w(e)$ is strictly positive. An **electrical potential** is a function $u : V \to \mathbb{R}$ and the edge weights are called **conductances**. The **net current** at a vertex x is given by the **weighted Laplacian**

$$\Delta u(x) = \sum_{y \sim x} w(x, y)[u(x) - u(y)].$$

A potential function is **harmonic** if the net current vanishes at each interior vertex.

The discrete electrical inverse problem asks whether the conductances of a network can be recovered by performing boundary measurements of harmonic functions. In its discrete form, the problem was introduced and studied in [10] and [31], inspired by an analogous problem in partial differential equations. This theory forms the mathematical basis for electrical impedance tomography and has been studied in the context of numerical analysis.

The electrical inverse problem cannot be solved for all graphs, but many graphs can be recovered via **layer-stripping**, a technique in which the edge weights are recovered iteratively, working inwards from the boundary. One recovers the conductance of a near-boundary edge, then removes that edge by a **layer-stripping operation** of deletion or contraction, and thus reduces the problem to a smaller graph. Layer-stripping operations have been interpreted as a group action of the symplectic Lie group by [27] and are applied to the inverse problem for nonlinear

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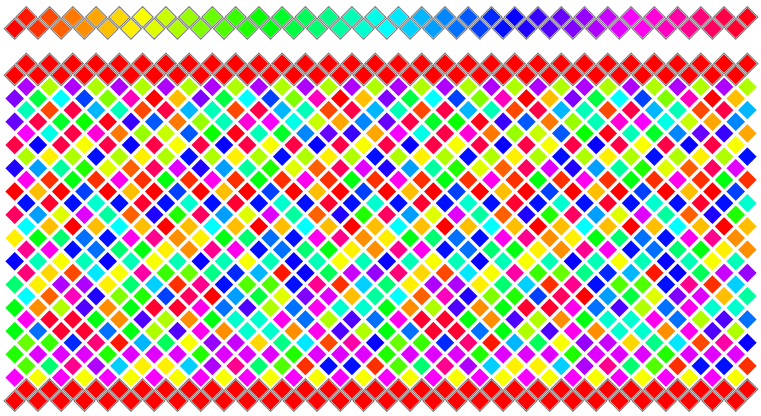

FIGURE 1. A $(\mathbb{Z}/64)$-valued harmonic function on a graph we will study in §9.

electrical networks in [21]. One of the authors explored the connections between layer-stripping and harmonic continuation and formalized the layer-stripping approach to the inverse problem in [20].

Our paper focuses on the layer-stripping operations themselves rather the inverse problem. We construct an algebraic graph invariant to test whether a graph can be completely reduced through layer-stripping operations.[1] We associate an R-module Υ to any graph with boundary such that Υ is free if and only if the graph is completely reducible. Torsion of Υ can thus be viewed as an algebraic description of the failure of complete reducibility.

The module Υ generalizes the sandpile group to graphs with boundary with edge weights in an arbitrary ring. Thus, our theory connects the sandpile group with homological algebra and $\mathbb{Q}/\mathbb{Z}$-valued harmonic functions.

1.2. **Main Ideas and Results.** Our strategy for constructing algebraic invariants is to consider electrical networks over rings. Instead of restricting the edge weights to lie in $\mathbb{R}^+$ or $\mathbb{C}$ or $\mathbb{Z}$ as is usually done, we allow them to reside in an arbitrary commutative ring R. We consider the weighted Laplacian as an operator on functions u from V to any R-module M. We denote the R-module of M-valued harmonic functions by $\mathcal{U}(\Gamma, M)$, where Γ is an R-weighted graph with boundary.

Our fundamental module Υ is constructed via 'homology theory' as a quotient of modules involving linear combinations of vertices and edges (§2.3), but has the key property that $\mathcal{U}(\Gamma, M) = \operatorname{Hom}(\Upsilon(\Gamma), M)$. In other words, Υ is the *representing object* for the functor $\mathcal{U}(\Gamma, -)$. We will also interpret $\operatorname{Tor}_1(\Upsilon(\Gamma), M)$ and $\operatorname{Ext}_1(\Upsilon(\Gamma), M)$ in terms of harmonic functions (§3).

The module-theoretic duality between Υ and $\mathcal{U}(\Gamma, M)$ captures the interplay between the structure of the graph and the behavior of harmonic functions, which is not unlike the relationship between topology and function theory on a Riemann surface.[2] Our combinatorial and algebraic theory thus draws much of its intuition from topology and complex analysis. Our first main result provides an algebraic characterization of the 'topological' condition of complete reducibility.

[1] We also allow boundary wedge-sum decomposition, see §5 for precise definitions.

[2] The analogy between graphs and Riemann surfaces was observed in [3, 30, 8].

Essence of Theorem 1. A graph is completely reducible if and only if Υ is free for all choices of edge weights in $R^\times$ for all commutative rings R.

The forward direction of the theorem is proved inductively by showing that if we add small extensions on to the boundary of a graph, then Υ does not change. The dual interpretation (and in fact motivation) for this result is that any harmonic function on the smaller graph extends uniquely to the larger graph. The layer-stripping process thus provides a geometric model for the process of harmonic continuation inspired by complex analysis (§7).

For the converse implication of Theorem 1, we must take a graph that is not completely reducible and concoct unit edge weights for which Υ is not free. This relies on the ability to assign 'exotic' unit edge weights by considering rings like the Laurent polynomial algebra $F[t_e^{\pm 1}]$ with generators indexed by the edges of the graph, rather than only rings like $\mathbb{Z}$ with very few units.

We use the characterization of $\mathrm{Tor}_1(\Upsilon, M)$ as the module $\mathcal{U}_0(\Gamma, M)$ of M-valued harmonic functions that have vanishing potential and net current on the boundary (§3.1). Figure 1 shows such a function. The existence of harmonic functions that are undetectable from boundary data would flagrantly violate the uniqueness principles of complex analysis and PDE. But in our discrete algebraic world, they merely measure torsion, and we complete the proof of the theorem by choosing unit edge weights that will allow harmonic functions with zero boundary data.

The precise statements of Theorem 1 and its corollaries give several other equivalent algebraic conditions that come out of the proof. Theorem 2 complements Theorem 1 by describing how far layer-stripping operations can simplify a finite graph in the general case.

Essence of Theorem 2. A finite graph with boundary can be layer-stripped until an impenetrable core remains. The resulting graph (known as a **flower**) is independent of the sequence of layer-stripping operations. Mapping a graph to its flower defines a functor on the category of ∂-graphs and unramified harmonic morphisms (see §4, §6, §6.2).

In §5, we generalize the main result in [9], showing that

Theorem 3. A connected circular planar $R^\times$-network and its dual have isomorphic fundamental modules.

Once again, the graph-theoretic proof involving free modules on the edges translates into a discrete-complex-analytic statement about harmonic functions: For any harmonic function u on Γ, there is a unique harmonic conjugate v on the dual network $\Gamma^\dagger$ satisfying the Cauchy-Riemann equation $w(e)du(e) = dv(e^\dagger)$. We apply harmonic conjugates and analytic continuation to understand $\mathbb{Q}/\mathbb{Z}$-valued harmonic functions on a family of wheel graphs studied by Biggs [6], and hence compute the sandpile group for these graphs.

In a similar vein, we use harmonic continuation to study a family of 'chain-link fence' graphs $\mathrm{CLF}(m, n)$ in §9, which were studied in [26] in the context of the electrical inverse problem on the cylinder. Using symmetry (rather than cumbersome Smith normal form computations), we establish the following result.

Theorem 4. Consider $\mathbb{Z}$-networks $\textsc{clf}(m,n)$ with all edges of weight one.

$$\mathcal{U}_0(\textsc{clf}(m,n),\mathbb{Q}/\mathbb{Z}) \cong \begin{cases} (\mathbb{Z}/2)^n, & m \text{ odd} \\ (\mathbb{Z}/2)^{2n}, & m \equiv 2 \mod 4 \\ \bigoplus_{j=1}^{n} \left(\frac{\mathbb{Z}}{\gcd(4^j, 2m)}\right)^2, & m \equiv 0 \mod 4. \end{cases}$$

Figure 1 shows a $(\mathbb{Z}/64)$-valued harmonic function with vanishing boundary potential and net current on one of the CLF networks, chosen uniformly at random.

Both the wheel graphs and chain-link fence graphs illustrate the interaction between graph symmetry and the torsion of Υ. In §10, we formulate general symmetry principles and deduce number-theoretic constraints on the torsion primes of the sandpile group.

1.3. **Overview.** Sections 2 - 4 develop the definitions and basic properties of our objects of study. §2 defines the fundamental module Υ and explains its relationship with discrete differential geometry and the sandpile group. §3 connects homological-algebraic properties of Υ to harmonic functions over R-modules. §4 generalizes the harmonic morphisms studied in [30, 3] to R-networks.

Sections 5 - 7 develop the theory of layer-stripping. §5 proves Theorem 1 characterizing completely reducible ∂-graphs and gives an application to bipartite networks. §6 establishes functoriality of the layer-stripping operations on unramified harmonic morphisms, and proves Theorem 2. §7 interprets layer-stripping as a geometric model of harmonic continuation and gives several applications.

Sections 8 - 10 analyze the effect of symmetry and duality on Υ through theory and examples. §8 proves Theorem 3 on network duality, and applies the idea of harmonic conjugates in the example of the wheel graphs. §9 computes Υ for the family of symmetric chain-link fence graphs. §10 formulates general principles relating graph symmetry to the group structure of Υ.

Lastly, we present a selection of open problems in §11.

2. The Fundamental Module Υ

2.1. **Definitions and Conventions.** We assume some familiarity with category theory, commutative algebra, and homological algebra. For background refer to [1], [33, Chapters 1-3],[24, Chapters I, II, III, V], [32].

In this paper, a **graph** G is a countable, locally finite, undirected multi-graph with self-loops allowed. We write V for the vertex set and E for the set of **oriented** edges. If e is an oriented edge, e_+ and e_- refer to its starting and ending vertices, and $\overline{e}$ refers to its reverse orientation. The **degree** of a vertex x is the number of oriented edges with $e_+ = x$.

A **graph with boundary** (abbreviated to **∂-graph**) is a graph with a specified partition of V into two sets V° and ∂V, called the **interior** and **boundary vertices** respectively.

If R is a commutative ring, then an **R-network** Γ is a ∂-graph together with a weight function $w : E \to R$ such that $w(e) = w(\overline{e})$. An R-network is called an **$R^\times$-network** if $w : E \to R^\times$.

If Γ is an R-network and M is an R-module, then the **weighted Laplacian** operator on functions $u : V \to M$ is given by

$$\Delta u(x) = \sum_{e:e_+=x} w(e)(u(x) - u(e_-)).$$

We say u is **harmonic** if $\Delta u(x) = 0$ for all *interior* vertices x. We denote the space of harmonic functions by $\mathcal{U}(\Gamma, M)$. This is an R-module and moreover, $\mathcal{U}(\Gamma, -)$ is a functor on R-modules.

2.2. **Construction of Υ.** Let Γ be an R-network. Let RV be the free R-module on the vertices of G and let RV° be the submodule generated by the interior vertices. We define the **chain Laplacian** $\Delta : RV \to RV$ by

$$\Delta x = \sum_{e:e_+=x} w(e)(x - e_-).$$

If we view RV as the submodule of finitely supported elements of R^V, then the chain Laplacian agrees with the Laplacian on functions by a standard symmetry computation, so we use the same notation for both.

We define the **fundamental module** Υ by

$$\boxed{\Upsilon = RV/\Delta(RV^\circ).}$$

Υ has the key property that for any R-module M,

$$\boxed{\mathcal{U}(\Gamma, M) = \mathrm{Hom}(\Upsilon(\Gamma), M).}$$

To see this, observe that a function $u : V \to M$ is a equivalent to an R-module morphism $RV \to M$, and u is harmonic if and only if

$$\Delta u(x) = u(\Delta x) = 0 \text{ for each } x \in V^\circ.$$

Thus, u is harmonic if and only if it vanishes on $\Delta(RV^\circ)$, so that harmonic functions are equivalent to R-module morphisms $RV/\Delta(RV^\circ) \to M$.

For technical convenience, we will sometimes work with "reduced" (co)homology by setting

$$\widetilde{\mathcal{U}}(\Gamma, M) = \mathcal{U}(\Gamma, M)/\{\text{constant functions}\}.$$

Let $\epsilon\colon RV \to R$ be the augmentation homomorphism that sums the coefficients of the vertices. Then $\widetilde{\mathcal{U}}(\Gamma, M) = \mathrm{Hom}(\widetilde{\Upsilon}, M)$ where $\widetilde{\Upsilon} = \ker \epsilon / \Delta RV^\circ$. Note that $\Upsilon \cong \widetilde{\Upsilon} \oplus R$. Thus, the "torsion" properties of Υ and $\widetilde{\Upsilon}$ are the same.

2.3. **Υ and Discrete Differential Geometry.** The fundamental module Υ directly generalizes constructions from discrete differential geometry on simplicial complexes [16] that are motivated by analogy with de Rham cohomology. In light of Hodge theory, harmonic differential forms on a manifold represent elements of the de Rham cohomology groups. On the other hand, these groups are characterized as $\mathrm{Hom}_{\mathbb{R}}(H^n, \mathbb{R})$ by the de Rham Theorem, where H^n is the homology of a chain complex defined using formal linear combinations of simplices. In a similar way, the module $\mathcal{U}(\Gamma, M)$ of harmonic functions on an R-network can be represented as $\mathrm{Hom}(\Upsilon, M)$, where Υ is obtained by considering linear combinations of vertices.

This analogy can be made quite precise. Consider the chain groups of Γ,

$$C_0 := RV, \qquad C_1 := RE/\{-e = \bar{e}\}_{e \in E},$$

that is, the free R-modules on the vertex and edge sets respectively, after identifying the negative of an oriented edge with its reverse orientation. Dual to chains, we have modules $\Omega^j(\Gamma, M)$ consisting of M-valued j-forms:

$$\Omega^j(\Gamma, M) := \mathrm{Hom}(C_j, M), \qquad j = 0, 1.$$

The boundary map $\partial : C_1 \to C_0$ given by $\partial e = e_+ - e_-$ induces the discrete gradient $d : \Omega^1 \to \Omega^0$ given by $df(e) = f(e_+) - f(e_-)$. The coboundary map $\partial^* : C_0 \to C_1$ given by $x \mapsto \sum_{e:e_+=x}$ induces the discrete divergence $d^* : \Omega^1 \to \Omega^0$ given by $d^*\omega(x) = \sum_{e:e_+=x} \omega(e)$. The weighted chain Laplacian $\partial w \partial^* : C_0 \to C_0$ induces the weighted Laplacian on functions $d^* w d : \Omega^0 \to \Omega^0$.

For a graph without boundary, the module $\mathcal{U}(\Gamma, M)$ arises from 'cohomology theory' as the kernel of $d^* w d : \Omega^0 \to \Omega^0$. On the other hand, Υ arises from 'homology theory' as the cokernel of $\partial w \partial^* : C_0 \to C_0$.

2.4. **Υ and the Sandpile Group.** The fundamental module Υ is a generalization of the sandpile group that has received significant attention from physicists, combinatorialists, probabilists, algebraic geometers and number theorists

The Abelian sandpile model was introduced in statistical physics by Dhar [13], who was motivated by the study of self-organized criticality. Grains of sand are placed on the vertices of a graph and are allowed to topple: when this occurs, a vertex sends one grain of sand to each of its neighbors. The elements of the sandpile group are the critical configurations of sand [18], [6]. Extending work of [29] on the balancing game, [7] introduced the chip firing game and uncovered its connection to greedoids. The dollar game appeared in [5] and was analyzed extensively using the methods of algebraic potential theory.

The provenance of the sandpile group has since expanded into other areas of combinatorics, graph theory, and even algebraic geometry. Graph theorists study the sandpile group in the guise of the quotient of the chain group by the submodule generated by cycles and bonds [5, §26-29]. Probabilists study the abelian sandpile model due to its intimate connections with generating uniformly random spanning trees [19, 22]; the sandpile group acts freely and transitively on the set of spanning trees of the graph [5]. Viewing sand configurations as divisors on the graph, [23, 4] interpreted the sandpile group as the Jacobian variety of a degenerate curve and proved a Riemann-Roch theorem for graphs. The sandpile group is indeed a fruitful object with surprisingly deep and diverse connections.

The precise relationship between Υ and the sandpile group is as follows. Let G be a connected graph without boundary, considered as a $\mathbb{Z}$-network Γ with edge weights 1 and with no boundary vertices. The sandpile group is known to be isomorphic to $\ker \epsilon / \operatorname{im} \Delta = \widetilde{\Upsilon}(\Gamma)$ [16]. In the sandpile literature, one often designates a special vertex to be a 'sink.' Similarly, when it is convenient, we can designate one vertex x of Γ to be a boundary vertex. This does not change $\Delta(\mathbb{Z}V^\circ)$ since $\Delta u = \Delta(u - u(x))$ and $u - u(x) \in \mathbb{Z}V^\circ$. Thus, the sandpile group is still isomorphic to $\widetilde{\Upsilon}(\Gamma)$.

By Pontryagin duality, the sandpile group is also isomorphic to $\mathrm{Hom}(\widetilde{\Upsilon}, \mathbb{Q}/\mathbb{Z})$, the space of $\mathbb{Q}/\mathbb{Z}$-valued functions modulo constants [19, p. 11]. This perspective makes the computation of the sandpile group accessible to the powerful technique of harmonic continuation, which has been exploited explicitly or implicitly by various authors. For relatively sparse networks, harmonic continuation can provide sharp bounds on the number of invariant factors and reduce the Smith normal form computation to a smaller matrix than the Laplacian. We present a systematic

approach to harmonic continuation in §7 and apply it more informally in §8.2 and §9.

3. Harmonic Functions and Homological Algebra

3.1. **Torsion and Degeneracy.** A standard way to measure the torsion of an R-module N is to use the functors $\mathrm{Tor}_j(N,-)$, which are the left-derived functors of $N\otimes -$. An R-module N is called **flat** if $N\otimes -$ exact, which is equivalent to $\mathrm{Tor}_j(N,-)=0$ for $j>0$ (see [1, Exercise 2.25]). If R is a principal ideal domain (PID), then N is flat if and only if it is torsion-free (see [15, Exercise 10.4.26]).

The functor $\mathrm{Tor}_1(\Upsilon,-)$ turns out to have an easy description in terms of harmonic functions. Let $\mathcal{U}_0(\Gamma,M)$ be the module of finitely-supported M-valued harmonic functions with $u|_{\partial V}=0$ and $\Delta u|_{\partial V}=0$.

Proposition 1. *Suppose* $\mathcal{U}_0(\Gamma,R)=0$. *Then* $\mathrm{Tor}_1(\Upsilon(\Gamma),M)=\mathcal{U}_0(\Gamma,M)$ *and* $\mathrm{Tor}_j(\Upsilon(\Gamma),M)=0$ *for* $j>1$.

Proof. Note that RV° can be interpreted as the module of finitely-supported R-valued functions that vanish on ∂V. Thus, $\mathcal{U}_0(\Gamma,R)=0$ if and only if $\Delta: RV^\circ\to RV$ is injective. If this occurs, then

$$\cdots\to 0\to RV^\circ\xrightarrow{\Delta} RV\to\Upsilon\to 0$$

is a free resolution of Υ. Thus, $\mathrm{Tor}_j(\Upsilon,M)$ is the homology of the sequence

$$\cdots\to 0\to RV^\circ\otimes M\xrightarrow{\Delta\otimes\mathrm{id}} RV\otimes M\to 0.$$

Thus, $\mathrm{Tor}_j(\Upsilon,M)=0$ for $j>1$, and $\mathrm{Tor}_1(\Upsilon,M)$ is the kernel of the map $RV^\circ\otimes M\to RV\otimes M$. We can identify $RV^\circ\otimes M$ with the module of finitely supported functions $V\to M$ that vanish on ∂V, and then $\Delta\otimes\mathrm{id}$ is simply the Laplacian. This implies $\mathrm{Tor}_1(\Upsilon,M)=\mathcal{U}_0(\Gamma,M)$. □

We will call a network **non-degenerate** if $\mathcal{U}_0(\Gamma,R)=0$, and **degenerate** if $\mathcal{U}_0(\Gamma,R)\neq 0$. Non-degeneracy holds in many standard cases as a consequence of the maximum principle. An **ordered ring** [28, Chapter 6] is a ring R together with a (transitive) total order $<$ given on R such that, for all elements $a,b,c\in R$,

$$a<b\implies a+c<b+c,$$
$$0<a \text{ and } 0<b\implies 0<ab.$$

Proposition 2 ([11], [12])**.** *Suppose* R *is an ordered commutative ring and* Γ *is an* R*-network with* $w(e)>0$. *Assume the* ∂*-graph* G *has at least one boundary vertex in each connected component. Then for any finitely supported harmonic function* $u:V\to R$,

1. *The maximum of* u *is achieved on the boundary.*
2. *If* $u=0$ *on* ∂V, *then* u *must be identically zero.*
3. *If* $\Delta u=0$ *on all of* V, *then* u *must be constant.*

Proof. The proof is standard and not difficult (see [11] and [12]). □

Corollary 3. *If* G *has at least one boundary vertex in each connected component,* R *is totally ordered, and* $w(e)>0$, *then* Γ *is non-degenerate. In particular, this holds in the case where* $R=\mathbb{Z}$ *and* $w(e)=1$.

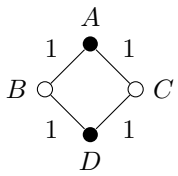

FIGURE 2. A $\mathbb{Z}$-network with 2-torsion; $\bullet \in \partial V$, $\circ \in V^\circ$, and $w(e) = 1$. An element of $\mathrm{Tor}_1(\Upsilon(\Gamma), \mathbb{Z}/2) = \mathcal{U}_0(\Gamma, \mathbb{Z}/2)$ is given by the function u with $u(B) = u(C) = 1$ and $u(A) = u(D) = 0$.

One intuitive interpretation of Proposition 1 is that torsion of Υ corresponds to a failure of the maximum principle for functions taking values in a torsion module. Alternatively, $\mathrm{Tor}_1(\Upsilon, M) \neq 0$ if and only if there are harmonic M-valued functions that are not detectable from boundary measurements of potential and current.

Since torsion and degeneracy are both measured by conditions of the form $\mathcal{U}_0(\Gamma, M) \neq 0$, it is not surprising that they are related. As a corollary of Proposition 1, we can show that *torsion* of Υ for a non-degenerate network over R is equivalent to *degeneracy* of networks over quotient rings of R. Given an R-network Γ and an ideal $\mathfrak{a} \subset R$, define $\Gamma/\mathfrak{a}$ as the $R/\mathfrak{a}$-network obtained by reducing the edge weights mod $\mathfrak{a}$.

Corollary 4. *Let Γ be a non-degenerate R-network. Then $\Upsilon(\Gamma)$ is flat if and only if $\Gamma/\mathfrak{a}$ is non-degenerate for any ideal $\mathfrak{a}$.*

Proof. An R-module N is flat if and only if $\mathrm{Tor}_1(N, R/\mathfrak{a}) = 0$ for all ideals $\mathfrak{a} \subset R$ (see [1, Exercise 2.26]). We know $\mathrm{Tor}_1(\Upsilon(\Gamma), R/\mathfrak{a}) = \mathcal{U}_0(\Gamma, R/\mathfrak{a})$. However, $\mathcal{U}_0(\Gamma, R/\mathfrak{a}) = \mathcal{U}_0(\Gamma/\mathfrak{a}, R/\mathfrak{a})$ because for any function $u : V \to R/\mathfrak{a}$, we have $\Delta_\Gamma u = \Delta_{\Gamma/\mathfrak{a}} u$. □

3.2. **Exactness of $\mathcal{U}(\Gamma, -)$.** Another way to measure the torsion of Υ is to test whether Υ is projective. Recall that for an R-module N, $\mathrm{Hom}(N, -)$ is always left exact, and N is called **projective** if it is also right exact. The failure of N to be projective is measured by the functors $\mathrm{Ext}_j(N, -)$, which are the right derived functors of $\mathrm{Hom}(N, -)$, and N is projective if and only if $\mathrm{Ext}^1(N, -) = 0$. Free modules are always projective. If R is a PID and N is a finitely generated R-module, then N is torsion-free if and only if it is projective.[3]

The fundamental module Υ is projective if and only if $\mathrm{Hom}(\Upsilon(\Gamma), -) = \mathcal{U}(\Gamma, -)$ is right exact. Concretely, right exactness asks: given a surjective map $M \to N$ between R-modules, is $\mathcal{U}(\Gamma, M) \to \mathcal{U}(\Gamma, N)$ a surjection? In other words, does any N-valued harmonic function on Γ lift to an M-valued harmonic function?

Example 1. $\mathcal{U}(\Gamma, -)$ fails to be right exact for the $\mathbb{Z}$-network in Figure 2. Consider the surjection $\mathbb{Z}/4 \to \mathbb{Z}/2$. The corresponding map $\mathcal{U}(\Gamma, \mathbb{Z}/4) \to \mathcal{U}(\Gamma, \mathbb{Z}/2)$ is not surjective. If $u \in \mathcal{U}(\Gamma, \mathbb{Z}/4)$, then $2u(B) = u(A) + u(D) = 2u(C) \bmod 4$, and hence $u(B) = u(C) \bmod 2$. However, not all $\mathbb{Z}/2$-valued harmonic functions satisfy $u(B) = u(C)$; for instance, 1_B is harmonic in $\mathbb{Z}/2$.

[3]One can deduce this from the classification of finitely generated modules over a PID.

3.3. **Summary of Homological Properties.** If Γ is a non-degenerate R-network, then as noted in the proof of Proposition 1, Υ has a free resolution given by $0 \to RV^\circ \to \ker\epsilon \to \Upsilon \to 0$. In particular, this means that Υ has projective homological dimension 1 (for definition, see [33, Chapter 4]), which is consistent with our topological intuition that weighted graphs are 1-dimensional.

Our results thus far provide a lexicon giving "harmonic" or "electrical" interpretations of the homological properties of Υ for non-degenerate networks.

(1) $\mathrm{Hom}(\Upsilon(\Gamma), M) = \mathcal{U}(\Gamma, M)$ is the module of M-valued harmonic functions.
(2) $\mathrm{Tor}_1(\Upsilon(\Gamma), M) = \mathcal{U}_0(\Gamma, M)$ is the module of finitely supported harmonic functions with vanishing potential and current on the boundary.
(3) $\mathrm{Ext}^1(\Upsilon(\Gamma), -)$ is the right-derived functor of $\mathcal{U}(\Gamma, -)$. It measures the failure of N-valued harmonic functions to lift to M-valued harmonic functions when $M \to N$ is surjective.
(4) Using our free resolution of Υ, we can also compute $\mathrm{Ext}^1(\Upsilon(\Gamma), M)$ as the cokernel of $\Delta : M^V \to M^{V^\circ}$. In other words, it is the module of M-valued functions on V° modulo those that arise as the Laplacian (or net current) of M-valued potentials on V.

Remark. For non-degenerate networks, the reduced module $\widetilde{\Upsilon}$ defined in §2.2 has a free resolution $0 \to RV^\circ \to \ker\epsilon \to \widetilde{\Upsilon}$. Since $\widetilde{\Upsilon} \cong \Upsilon \oplus R$, the Tor_1 and Ext^1 functors are the same for Υ and $\widetilde{\Upsilon}$.

Consequently, we have many different ways of computing the torsion submodule of Υ when R is a PID such as $\mathbb{Z}$:

Proposition 5. *Let R be a PID, F its field of fractions, Γ a finite non-degenerate network over R. Then the following are (non-canonically) isomorphic:*

(1) The torsion submodule of Υ.
(2) The cokernel of $\Delta\colon R^V \to R^{V^\circ}$.
(3) The F/R-valued harmonic functions modulo the Laplacian image of the F-valued harmonic functions.
(4) $\mathcal{U}_0(\Gamma, F/R) = \mathrm{Tor}_1(\Gamma, F/R)$.

Proof. Note (2) and (3) are two different ways of evaluating $\mathrm{Ext}^1(\Upsilon(\Gamma), R)$; (2) uses the projective resolution of Υ and (3) uses the injective resolution of R given by $0 \to R \to F \to F/R \to 0$. To show that the torsion submodule of Υ is isomorphic to $\mathrm{Tor}_1(\Upsilon, F/R)$ and $\mathrm{Ext}^1(\Upsilon, R)$, decompose Υ as the direct sum of cyclic modules. □

Example 2 (Complete Bipartite Graphs). Consider the complete bipartite graph whose parts have m boundary vertices and n interior vertices respectively.[4] View the graph as a $\mathbb{Z}$-network with edge weights 1. The Laplacian $\Delta : R^V \to R^{V^\circ}$ is

$$\Delta = \left(\begin{array}{ccccc|ccc} 1 & \cdots & \cdots & 1 & & m & & \\ \vdots & \ddots & \ddots & \vdots & & & \ddots & \\ \vdots & \ddots & \ddots & \vdots & & & & \ddots \\ 1 & \cdots & \cdots & 1 & & & & m \\ \hline & \multicolumn{3}{c}{\underbrace{\qquad\qquad}_{\partial V}} & & \multicolumn{3}{c}{\underbrace{\qquad\qquad}_{V^\circ}} \end{array}\right)$$

[4]We are grateful to Will Dana, Collin Litterell, and Austin Stromme for allowing us to include this example which they initially studied at the University of Washington Mathematics REU, Summer 2015.

and its cokernel is isomorphic to $(\mathbb{Z}/m)^{n-1}$. Dually,

$$\mathcal{U}_0(\Gamma, \mathbb{Q}/\mathbb{Z}) = \left\{ u \in (\mathbb{Q}/\mathbb{Z})^{V^\circ} : mu = 0, \sum_{x \in V^\circ} u(x) = 0 \right\} \cong (\mathbb{Z}/m)^{n-1}.$$

3.4. **Application to the Sandpile Group.** As observed in §2.4, we can obtain sandpile group of a finite connected graph by viewing it as a $\mathbb{Z}$-network with edge weights 1 and optionally assigning one boundary vertex x (it does not matter which one). If we do not assign a boundary vertex, then the network is degenerate because constant functions are harmonic, but if we assign a boundary vertex, the network is non-degenerate by the maximum principle. It follows that Δ has rank $|V| - 1$ and hence $\widetilde{\Upsilon}(\Gamma)$ is a torsion module. In particular, the sandpile group is isomorphic to the torsion submodule of Υ, which can be computed using Proposition 5 with $\partial V = \{x\}$, $R = \mathbb{Z}$, and $F = \mathbb{Q}$.

4. Harmonic Morphisms of R-Networks

The sandpile group is known to be both a covariant and a contravariant functor of the graph, using the harmonic graph morphisms of [30], [4]. We will define two types of harmonic morphisms of R-networks. *Algebraic harmonic morphisms* (AHMs), which allow a vertex to map to a linear combination of vertices, are the natural morphisms to make Υ a functor of the network. The more concrete *geometric harmonic morphisms* (GHMs) are a direct analogue of the harmonic morphisms in the literature, and they will allow us to pull back layer-stripping operations functorially (§6).

An **algebraic harmonic morphism (AHM)** $f : \Gamma_1 \to \Gamma_2$ is an R-module homomorphism $RV_1 \to RV_2$ which maps $\ker \epsilon_1$ into $\ker \epsilon_2$ and $\Delta(RV_1^\circ)$ into $\Delta(RV_2^\circ)$. These are exactly the R-module morphisms $RV_1 \to RV_2$ which induce maps $\Upsilon(\Gamma_1) \to \Upsilon(\Gamma_2)$ and $\widetilde{\Upsilon}(\Gamma_1) \to \widetilde{\Upsilon}(\Gamma_2)$. They are the largest class of R-module morphisms $RV_1 \to RV_2$ which preserve harmonic functions and constant functions under pullbacks. We define R-**net** as the category of R-networks and AHMs. Then

Observation 6. *$\Upsilon(-)$ is a covariant functor R-**net** $\to$ R-**mod**. Thus, $\mathrm{Hom}(\Upsilon(-), -)$, $\mathrm{Ext}^1(\Upsilon(-), -)$, and $\mathrm{Tor}_1(\Upsilon(-), -)$ are functors on R-**net** $\times$ R-**mod**.*

The flexibility of AHMs to map vertices to linear combinations of vertices makes them sufficiently general that any map $\widetilde{\Upsilon}(\Gamma_1) \to \widetilde{\Upsilon}(\Gamma_2)$ can be realized by an AHM.[5]

Geometric harmonic morphisms are defined both for ∂-graphs without assigned edge weights, and for R-networks. A **GHM of ∂-graphs** $f : G_1 \to G_2$ is a map $f : V_1 \cup E_1 \to V_2 \cup E_2$ such that

(1) f maps vertices to vertices.
(2) f maps interior vertices to interior vertices.
(3) If $f(e)$ is an edge, then $f(e_\pm) = f(e)_\pm$ and $f(\overline{e}) = \overline{f(e)}$.
(4) If $f(e)$ is a vertex, then $f(\overline{e}) = f(e)$ and $f(e_\pm) = f(e)$.
(5) For any $x \in V_1^\circ$, the restricted map

$$\{e \in E_1 \colon e_+ = x \text{ and } f(e) \text{ is an edge}\} \to \{e \in E_2 : e_+ = f(x)\}$$

[5]Indeed, since $\ker \epsilon_1$ is free and $\ker \epsilon_2 \to \Upsilon(\Gamma_2)$ is surjective, the composite map $\ker \epsilon_1 \to \Upsilon(\Gamma_1) \to \Upsilon(\Gamma_2)$ lifts to a map $\ker \epsilon_1 \to \ker \epsilon_2$, which in turn can be extended to a map $RV_1 \to RV_2$.

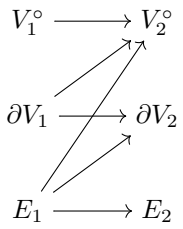

FIGURE 3. Schematic overview of a harmonic morphism.

has constant fiber size. In other words, it is n-to-1 for some $n \geq 0$ (which may depend on x).

A **GHM of R-networks** is a GHM of the underlying ∂-graphs which preserves the edge weights in the sense that $w(f(e)) = w(e)$.

Conditions (1), (3), and (4) say that f is a graph homomorphism except that it allows some edges to be "collapsed" to a vertex; in other words, if we view G_1 and G_2 as cell complexes, then f is a continuous cellular map. Condition (5) says that f restricts to an n-fold covering of the "star" $\{e \in E_1 : e_+ = x\}$ onto the "star" $\{e \in E_2 : e_+ = f(x)\}$, after ignoring collapsed edges. Condition (5) is precisely what is needed to guarantee that f locally preserves harmonicity in the sense that if $u : V_2 \to R$ is harmonic at $f(x)$ for some $x \in V_1^\circ$, then $u \circ f$ is harmonic at x (see proof of Lemma 7 below). Condition (2) is reasonable if we want functions harmonic on the interior to pull back to functions harmonic on the interior. Figure 3 shows what f does to the sets ∂V, V°, and E.

Lemma 7. *If $f : \Gamma_1 \to \Gamma_2$ is a geometric harmonic morphism, then the induced map $RV_1 \to RV_2$ defines an algebraic harmonic morphism $\Gamma_1 \to \Gamma_2$.*

Proof. Since f maps V_1 to V_2, it induces an R-module map $RV_1 \to RV_2$, which we will call f by abuse of notation. Since f maps each vertex to a single vertex, it preserves $\ker \epsilon$. To show that $f\Delta RV_1^\circ \subset \Delta RV_2$, it suffices to show that $f(\Delta x) \in \Delta(RV_2^\circ)$ for each $x \in V_1^\circ$. By condition (e), there is an n for which f restricts to an n-fold covering of the star $\{e : e_+ = x\}$ onto the star $\{e : e_+ = f(x)\}$ after ignoring collapsed edges. Therefore,

$$f(\Delta x) = \sum_{\substack{e \in E_1 \\ e_+ = x}} w(e) df(e) = n \sum_{\substack{e' \in E_2 \\ e'_+ = f(x)}} w(e') \partial e' = n \Delta f(x),$$

since $df(e) = f(\partial e) = 0$ for each collapsed edge. □

There is a category R-$\mathbf{net}_{\text{geo}}$ where the objects are R-networks and the morphisms are GHMs. Lemma 7 says that we have a functor R-$\mathbf{net}_{\text{geo}} \to R$-$\mathbf{net}$. Indeed, R-$\mathbf{net}_{\text{geo}}$ is almost a subcategory of R-$\mathbf{net}$; the only thing that fails is that two different GHMs can induce the same AHM for graphs with parallel or self-looping edges, since the AHM only considers where the vertices are mapped rather than where the edges are mapped.

Adapting an analogy of [3, 30, 8], we can think of ∂-graphs and GHMs as a discrete analogue of Riemann surfaces with boundary and holomorphic maps. Just as with Riemann surfaces, the simplest class of GHMs are covering maps (see §10),

which completely preserve local structure. Like holomorphic functions, GHMs may exhibit *ramification* when a star $\{e : e_+ = x\}$ in Γ_1 is an n-fold cover of a star $\{e : e_+ = f(x)\}$ in Γ_2 for $n > 1$. This is a discrete model of the behavior of the map z^n in a neighborhood of the origin in the complex plane. The behavior of GHMs is unconstrained by condition (5) at the boundary, just as an analytic function on a Riemann surface need not be n-to-1 in the neighborhood of a boundary point. However, the ability of GHMs to collapse an edge into a vertex does not have a direct analogue in the local behavior of holomorphic functions.

An important type of GHM for the rest of this paper is the inclusion map of a harmonic subgraph. We say H is a **harmonic subgraph** of a ∂-graph G if it is a subgraph and the inclusion map is a GHM. Explicitly, H is a harmonic subgraph if $V^\circ(H) \subset V^\circ(G)$, and for each $x \in V^\circ(H)$, the star $\{e \in E(G) : e_+ = x\}$ is contained in $E(H)$. If $f : G_1 \to G_2$ is a GHM and H is a harmonic subgraph of G_2, then we can define the **pullback** $f^{-1}(H)$ as the harmonic subgraph of G_1 whose vertex and edge sets are the preimages of the vertex and edge sets of H, and whose interior vertices are $f^{-1}(V^\circ(H)) \cap V^\circ(G_1)$. **Intersections and unions of harmonic subgraphs** are defined by taking the intersections/unions of the respective sets V, V°, and E. For instance, $V^\circ(\bigcup_\alpha \Gamma_\alpha) = \bigcup_\alpha V^\circ(\Gamma_\alpha)$.

5. Completely Reducible ∂-Graphs

5.1. **Definition and Algebraic Characterization.** We prove Theorem 1 in this section. This result establishes an equivalence between several algebraic properties of Υ and the graph-theoretic property of complete reducibility, defined in terms of iterated reduction operations on a network.

Definition.

(1) A **boundary edge** is an edge e with $e_+, e_- \in \partial V$. The ∂-graph $G \setminus e$ obtained by **deleting a boundary edge** e (and leaving the sets V and V° unchanged) is a harmonic subgraph of G.
(2) A **boundary spike** is an edge e with endpoints $x \in \partial V$ and $y \in V^\circ$ such that x has degree 1. We form the ∂-graph G/e by **contracting the boundary spike**, where $E(\Gamma/e) = E(\Gamma) \setminus \{e, \overline{e}\}$, and $V(\Gamma/e) = V(\Gamma)/\sim$, where $\sim$ is the equivalence relation given by $x \sim y$. The vertex $\{x, y\}$ in Γ/e is declared to be a boundary vertex. We can (and will) identify G/e with a harmonic subgraph of G by mapping $\{x, y\}$ to y.
(3) An **isolated boundary vertex** is a boundary vertex x of degree 0, and $G \setminus x$ is the harmonic subgraph formed by deleting it.
(4) Given ∂-graphs G_i with $i \in \{1, 2\}$ and specified vertices $x_i \in \partial V_i$, the **boundary wedge sum** $G_1 \vee G_2$ is obtained by identifying x_1 with x_2 in the disjoint union. Using a similar identification as in (a), we can (and will) identify G_1 and G_2 with harmonic subgraphs of $G_1 \vee G_2$.

Intuitively, a graph G is **completely reducible** if it can be reduced to nothing by repeated applications of reduction operations (see Figure 4). Formally, **completely reducible finite ∂-graphs** are defined to be the smallest class $\mathcal{C}$ of ∂-graphs containing the empty graph such that:

- if $G' \in \mathcal{C}$ is obtained from G by deleting a boundary edge, contracting a boundary spike, or deleting an isolated boundary vertex then $G \in \mathcal{C}$, and

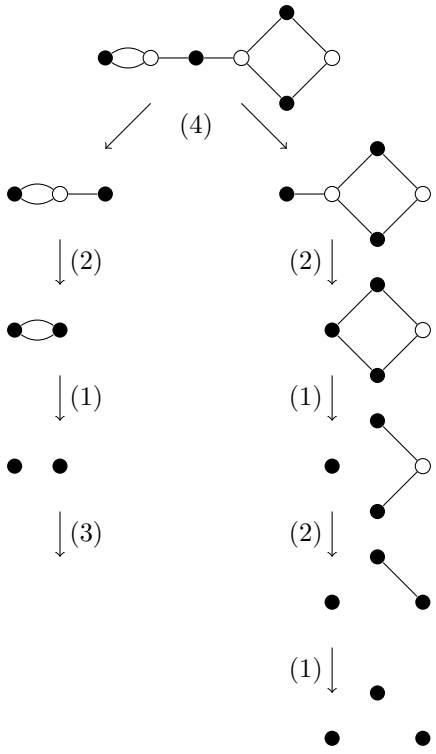

FIGURE 4. A completely reducible ∂-graph. The boundary vertices are black and interior vertices are white. The operations are (1) boundary edge deletion, (2) boundary spike contraction, (3) isolated boundary vertex deletion, (4) splitting a boundary wedge-sum.

- $\mathcal{C}$ is closed under boundary wedge-sums and disjoint unions.

A finite ∂-graph is **irreducible** if it has no boundary spikes, boundary edges, or isolated boundary vertices and is not a boundary wedge-sum or disjoint union.

The necessity of including the operation of a boundary wedge-sum in Theorem 1 rather than just the layer-stripping operations (1), (2), (3) was observed by Will Dana, Collin Litterell, and Austin Stromme.[6] We will prove the following more precise version of Theorem 1, giving several algebraic characterizations of complete reducibility:

Theorem 1. *For a finite ∂-graph G with at least one boundary vertex in each component, the following are equivalent:*

(1) G is completely reducible.
(2) Υ is free for any non-degenerate $R^\times$-network on G for all R.
(3) Υ is flat for any non-degenerate $R^\times$-network on G for all R.
(4) Any $F^\times$-network on G is non-degenerate for any field F.

[6]Discussions at the University of Washington Mathematics REU, Summer 2015.

Overview of proof. We prove (1) implies (2) in Lemma 9 below, which follows from inductive application of Lemma 8. We know (2) implies (3) since all free modules are flat. In Lemma 10, we show (3) implies (4) by equating degeneracy over F with torsion over the Laurent polynomial algebra $F[t_e^{\pm}]$ with generators indexed by the edges, using the principle of Corollary 4. In Lemma 11, we show (4) implies (1) by assuming G is not completely reducible and constructing a degenerate $F^\times$-network for any infinite field F. □

Lemma 8. *Let Γ be an $R^\times$-network. Then*

$$\Upsilon(\Gamma) \cong \begin{cases} \Upsilon(\Gamma/e), & e \text{ is a boundary spike} \\ \Upsilon(\Gamma \setminus e), & e \text{ is a boundary edge} \\ \Upsilon(\Gamma \setminus x) \oplus R, & x \text{ is an isolated boundary vertex.} \end{cases}$$

The same holds for $\widetilde{\Upsilon}$, provided that in the last case $\Gamma \setminus x$ is nonempty. For a boundary wedge-sum,

$$\widetilde{\Upsilon}(\Gamma_1 \vee \Gamma_2) \cong \widetilde{\Upsilon}(\Gamma_1) \oplus \widetilde{\Upsilon}(\Gamma_2)$$

For a disjoint union,

$$\Upsilon(\Gamma_1 \sqcup \Gamma_2) \cong \Upsilon(\Gamma_1) \oplus \Upsilon(\Gamma_2)$$
$$\widetilde{\Upsilon}(\Gamma_1 \sqcup \Gamma_2) \cong \widetilde{\Upsilon}(\Gamma_1) \oplus \widetilde{\Upsilon}(\Gamma_2) \oplus R.$$

Proof. Let e be a boundary spike with boundary endpoint x and interior endpoint y. As remarked above, the inclusion map $\phi : \Gamma/e \to \Gamma$ is a GHM. We will construct an AHM $\psi : \Gamma \to \Gamma/e$ such that $\psi_* : \Upsilon(\Gamma) \to \Upsilon(\Gamma/e)$ and $\phi_* : \Upsilon(\Gamma/e) \to \Upsilon(\Gamma)$ are inverses of each other. Set $\psi(z) = z$ for $z \neq x$, and

$$\psi(x) = y + w(e)^{-1} \sum_{\substack{e'_+ = y \\ e' \neq e}} w(e')\partial e' = x + w(e)^{-1}\Delta_\Gamma y.$$

To check that ψ is an AHM, note that it maps $\Delta RV^\circ(\Gamma)$ into $\Delta RV^\circ(\Gamma/e)$ because $\psi(\Delta y) = 0$. It is easy to see that $\psi \circ \phi = \mathrm{id}$ on $RV(\Gamma/e)$. On the other hand, $\phi \circ \psi = \mathrm{id}$ modulo $\Delta RV^\circ(\Gamma)$ since $\phi \circ \psi(x) = x + w(e)^{-1}\Delta y$. Therefore, ϕ_* and ψ_* are inverses. The same argument applies to $\widetilde{\Upsilon}$.

The case of a boundary edge is immediate since RV, $\ker \epsilon$, and ΔRV° remain unchanged by the deletion. For an isolated boundary vertex, the extra R summand arises from RV (in the case of Υ) or $\ker \epsilon$ (in the case of $\widetilde{\Upsilon}$) while ΔRV° remains unchanged.

For a boundary wedge-sum, we have $\ker \epsilon_\Gamma = \ker \epsilon_{\Gamma_1} \oplus \ker \epsilon_{\Gamma_2}$. Since $\Delta RV^\circ \subset \ker \epsilon$ for each network, this implies

$$\Delta RV^\circ(\Gamma_1) \cap \Delta RV^\circ(\Gamma_2) = 0.$$

Together with $V^\circ(\Gamma) = V^\circ(\Gamma_1) \cup V^\circ(\Gamma_2)$, this yields

$$\Delta RV^\circ(\Gamma) = \Delta RV^\circ(\Gamma_1) \oplus \Delta RV^\circ(\Gamma_2),$$

hence $\widetilde{\Upsilon}(\Gamma) \cong \widetilde{\Upsilon}(\Gamma_1) \oplus \widetilde{\Upsilon}(\Gamma_2)$. The case of disjoint unions is similar. □

Lemma 9. *If Γ is an $R^\times$-network on a completely reducible finite ∂-graph G, then $\Upsilon(\Gamma)$ is a free module of rank $|\partial V|$.*

Proof. It suffices to show that $\widetilde{\Upsilon}$ is a free module of rank $|\partial V|-1$ for any nonempty ∂-graph G (and it is zero for the empty graph). This holds for all completely reducible ∂-graphs by inductive application of Lemmas 8. □

This completes the proof that (1) implies (2), and now we show (3) implies (4).

Lemma 10. *Let G be a finite ∂-graph, F a field. Let $R = F[t_e^{\pm 1}]$ be the Laurent polynomial algebra with generators indexed by the edges, and let $\Gamma^* = \Gamma^*(G, F)$ be the $R^\times$-network on G with $w(e) = t_e$. Then Γ^* is non-degenerate. If $\Upsilon(\Gamma^*)$ is flat, then any $F^\times$-network on G is non-degenerate.*

Proof. To prove that Γ^* is non-degenerate, we want to show that $\Delta : RV^\circ \to RV$ is injective. Choose a boundary vertex x, and then it suffices to show that $\Delta_x : R(V \setminus x) \to R(V \setminus x)$ is injective. By the weighted matrix-tree theorem (see [17]),

$$\det \Delta_x = \sum_{T \in \mathrm{Span}(G)} \prod_{e \in T} t_e \neq 0,$$

where $\mathrm{Span}(G)$ denote the set of spanning trees of G. Since R is an integral domain, it follows that Δ is injective.

Suppose $\Upsilon(\Gamma^*)$ is flat and let (G, w) be an $F^\times$-network. Let $\mathfrak{a}_w$ be the ideal in R generated by $t_e - w(e)$ for $e \in E(G)$, and note that $F_w = R/\mathfrak{a}_w$ is a field isomorphic to F as an F-algebra. If we identify F_w with F, then $\Gamma^*/\mathfrak{a}_w = (G, w)$ since the edge weights are $t_e = w(e) \bmod \mathfrak{a}_w$. Moreover, $\Gamma^*/\mathfrak{a}_w$ is non-degenerate by Corollary 4. □

The next lemma shows that (4) implies (1) in Theorem 1.

Lemma 11. *Let G be a finite ∂-graph that is not completely reducible. Then for any infinite field F, there exists a degenerate $F^\times$-network on G.*

Proof. If G is not completely reducible, then we can obtain a nonempty irreducible subgraph by repeated applications of reduction operations, and by Lemma 13, it suffices to create a degenerate network on this subgraph. Therefore, we can assume without loss that G is irreducible.

Our strategy will be to choose a potential function u first with $u|_{\partial V} = 0$, and *then* choose edge weights such that $\Delta u \equiv 0$. Let S be the set of edges in G that are contained in a cycle. Define u to be zero on any component of $G \setminus S$ that contains boundary vertices of G, and assign u a distinct, non-zero value on each of the other components. Any edge in S must have endpoints in distinct components of $G \setminus S$.

To guarantee that u does not vanish on any interior vertex, it suffices to show that any edge (x, y) with $x \in \partial V$ and $y \in V^\circ$ must be contained in a cycle. By hypothesis, (x, y) is not a boundary spike. Thus there is some $z \neq x$ for which $(x, z) \in E$. Since G is not a boundary wedge-sum, deleting x leaves G connected. Thus, there is a path from y to z avoiding x. Hence there is a cycle containing (x, y) and (z, x). Consequently, u is nonzero on all the interior vertices.

Now we choose the edge weights. Choose oriented cycles $C_1, \ldots, C_k$ such that $S = \bigcup_{j=1}^k C_j \cup \overline{C}_j$. For each j, define

$$w_j(e) = w_j(\overline{e}) = \begin{cases} 1/du(e), \text{ for } e \in C_j \\ 0, \text{ for } e \notin C_j \cup \overline{C}_j. \end{cases}$$

Then $w_j(e)du(e)$ is 1 on C_j and -1 on $\overline{C}_j$ and vanishes elsewhere. Thus, $\Delta_{w_j} u = 0$. For each $e \in S$, there is a weight function w_j with $w_j(e) \neq 0$. Since F is infinite

and the graph is finite, we may choose $\alpha_j \in F$ such that $\sum_{j=1}^k \alpha_j w_j(e) \neq 0$ for all $e \in S$ simultaneously.

Set $w = 1_S + \sum_{j=1}^k \alpha_j w_j$. Then $w(e) \neq 0$ for each e. Since $\Delta_{w_j} u = 0$ for each j, we have $\Delta_w u = 0$ by linearity. Thus, (G, w) is the desired degenerate network. □

We have now completed the proof of Theorem 1. Theorem 1, Lemma 10, and Lemma 11 yield the following corollary, which shows that we only have to use one ring and one set of edge weights to test complete reducibility:

Corollary 12. *Let G be a finite ∂-graph. Let F be an infinite field and $\Gamma^*(G, F)$ the $F[t_e^{\pm 1}]$-network with edge weights t_e. Then G is completely reducible if and only if $\Upsilon(\Gamma^*)$ is free if and only if $\Upsilon(\Gamma^*)$ is flat.*

Remark. For any F, $\Upsilon(\Gamma^*(G, F))$ is functorial on harmonic morphisms ∂-graphs in the following sense: Let $R_G = F[t_e^{\pm 1}]$, and let $f : G \to H$ be a GHM. Then f induces a map $\phi : R_G \to R_H$ given by $t_e \mapsto t_{f(e)}$ if $f(e)$ is an edge and $t_e \mapsto 0$ otherwise. Moreover, f induces a map $\Upsilon(\Gamma^*(G, F)) \to \Upsilon(\Gamma^*(H, F))$ which is an R_G-module homomorphism if we view $\Upsilon(\Gamma^*(H, F))$ as an R_G-module by pulling back the module structure through $\phi : R_G \to R_H$.

Moreover, we can modify condition (4) in Theorem 1 to include all rings rather than all fields:

Proposition 13. *G is completely reducible if and only if any $R^\times$-network on G is non-degenerate for all R.*

Proof. The "if" direction was proved in Theorem 1. To prove the "only if" direction, it suffices to show that for $R^\times$-networks, degeneracy and non-degeneracy are preserved by contraction of boundary spikes, deletion of boundary edges and isolated boundary vertices, and decomposition of boundary wedge-sums and disjoint unions. We handle the cases of boundary spikes and boundary wedge-sums, leaving the other (easier) cases to the reader.

Suppose e is a boundary spike with endpoints $x \in \partial V$ and $y \in V^\circ$. Then we claim $\mathcal{U}_0(\Gamma/e, R) \cong \mathcal{U}_0(\Gamma, R)$. A function $u \in \mathcal{U}_0(\Gamma/e, R)$ on Γ/e extends to a function on Γ by setting $u(x) = 0$. On the other hand, if $u \in \mathcal{U}_0(\Gamma, R)$, then $u(y)$ must equal zero since $u(x) = 0$ and $\Delta u(x) = w(e)(u(x) - u(y)) = 0$. Thus, $u|_{\Gamma/e}$ is in $\mathcal{U}_0(\Gamma/e, R)$.

Suppose $\Gamma = \Gamma_1 \vee \Gamma_2$ and x is the common boundary vertex. Then we claim

$$\mathcal{U}_0(\Gamma, R) \cong \mathcal{U}_0(\Gamma_1, R) \oplus \mathcal{U}_0(\Gamma_2,)R.$$

Indeed, a function $u \in \mathcal{U}_0(\Gamma_1, R)$ can be extended to a function in $\mathcal{U}(\Gamma, M)$ by setting it to zero on Γ_2, and the same holds with Γ_1 and Γ_2 switched. On the other hand, if $u \in \mathcal{U}_0(\Gamma, M)$, then $u|_{\Gamma_j}$ has net current zero at each vertex other than x; since the sum of the net currents is zero, it must also have $\Delta_{\Gamma_j} u(x) = 0$. Thus, $u|_{\Gamma_j}$ has zero potential and current on the boundary for $j = 1, 2$. □

5.2. **Application to Bipartite ∂-Graphs.** The correspondence between algebraic and ∂-graph-theoretic conditions in Theorem 1 is illustrated by the following proposition about bipartite graphs, inspired by our collaboration with Will Dana. We present both an algebraic proof and an inductive ∂-graph-theoretic proof for comparison. We say a ∂-graph is **interior-boundary bipartite** if every edge has one interior endpoint and one boundary endpoint.

Proposition 14. *Suppose that G is a finite interior-boundary bipartite ∂-graph, $|V^\circ| \geq |\partial V|$, and every interior vertex has degree ≥ 2. Then G is not completely reducible.*

Algebraic proof. Let F be any field other than the field F_2 with two elements. We will construct a degenerate $F^\times$-network on G. Since each interior vertex has at least two edges incident to it and each edge is only incident to one interior vertex, we can choose $w(e) \in F^\times$ with $\sum_{e_+=x} w(e) = 0$ for each $x \in V^\circ$. If $u \in F^{V^\circ}$, then $\Delta u|_{V^\circ} = 0$ since

$$\Delta u(x) = \sum_{e:e_+=x} w(e)(u(x) - u(e_-)) = \sum_{e:e_+=x} w(e)u(x) = 0 \text{ for all } x \in V^\circ.$$

Combining this with the fact that $\operatorname{im}\Delta \subset \ker\epsilon$ yields

$$\Delta(F^{V^\circ}) \subset \left\{ \phi \in F^{\partial V} : \sum_{x\in\partial V} \phi(x) = 0 \right\}.$$

Therefore, $\dim \Delta(F^{V^\circ}) \leq |\partial V| - 1 < |V^\circ|$, since we assumed $|\partial V| \leq |V^\circ|$. Therefore, by the rank-nullity theorem,

$$\mathcal{U}_0(\Gamma, F) = \ker(\Delta : F^{V^\circ} \to F^V) \neq 0. \qquad \square$$

∂-graph-theoretic proof. We will use the fact, proved later in §6, that any harmonic subgraph of a completely reducible ∂-graph is also completely reducible. We proceed by induction on the number of vertices. Note G must have at least one boundary vertex and one interior vertex. If it has one of each, then it must have several parallel edges between them; then G is irreducible. Suppose G has $n > 2$ vertices and divide into cases:

- If G is irreducible, we are done.
- Suppose G has a boundary spike (x, y) with $x \in \partial V$ and $y \in V^\circ$. In G/e, y is a boundary vertex and by assumption all its neighbors are boundary vertices in G. Thus, we can delete the boundary edges incident to y and then delete the now isolated boundary vertex y to obtain a harmonic subgraph G' which satisfies the original hypotheses. G' is nonempty because $|V(G)| > 2$. By inductive hypothesis, G' is not completely reducible.
- If G can be split apart as a boundary wedge sum or a disjoint union, then each piece is interior/boundary bipartite with interior vertices that have degree ≥ 2. Moreover, one of the two subgraphs must have $|\partial V| \leq |V^\circ|$, and hence is not completely reducible by inductive hypothesis.
- G has no boundary edges by assumption. Moreover, if G has an isolated boundary vertex, that can be treated as a special case of disjoint unions.

$\square$

In light of the relationship between torsion over a ring and degeneracy over its quotient rings, our algebraic proof has the following corollary:

Corollary 15 (Will Dana)**.** *If G is as in the previous Proposition, then there is a non-degenerate $\mathbb{Z}[e^{2\pi i/3}]^\times$-network Γ on G such that $\operatorname{Tor}_1(\Upsilon(\Gamma), \mathbb{Z}[e^{2\pi i/3}]/2) \neq 0$.*

Proof. Let $R = \mathbb{Z}[e^{2\pi i/3}]$. We can choose $w(e) \in R^\times$ such that $\sum_{e_+=x} w(e) = 2$, which implies $\Delta u = 2u$ on V° for all $u \in R^{V^\circ}$, hence the network is non-degenerate. Since $2 = 0$ in $R/2$, the reduced network $\Gamma/2$ over the field $R/2$ is degenerate by

the same argument as before, and hence $\operatorname{Tor}_1(\Upsilon(\Gamma), R/2) = \mathcal{U}_0(\Gamma/2, R/2) \neq 0$ as in Corollary 4. □

6. Functoriality of Layer-Stripping

Our next goal will be to show that layer-stripping is functorial on unramified harmonic morphisms (UHMs) of ∂-graphs. An important consequence will be that if $f : G \to H$ is a UHM, then layerability of H implies layerability of G (Proposition 17). Functoriality is also a key step in the proof of Theorem 2, which shows that every ∂-graph can be layer-stripped to a unique flower, and that this construction is functorial on UHMs.

6.1. **Definitions.** Using the terminology of §4, we say a GHM $f : G \to H$ is an **unramified harmonic morphism** (**UHM**) if f restricted to $\{e \in E(G) : e_+ = x, f(e) \text{ is an edge}\}$ is a bijection onto $\{e \in E(H) : e_+ = f(x)\}$ for each $x \in V^\circ(G)$ and an injection for $x \in \partial V(G)$. We remark that UHMs include covering maps and inclusions of harmonic subgraphs. Moreover, UHMs are closed under pullbacks of subgraphs: If $f : G \to H$ is a UHM and $H' \subset H$, then f restricts to a UHM $f^{-1}(H') \to H'$.

The correct choice of definitions that will guarantee functoriality of layer-stripping is subtle. The individual layer-stripping operations defined in §5.1 do not always pull back to individual layer-stripping operations of the same type, the main culprit being boundary spike contraction. Suppose $f : G \to H$ is a UHM, and that H' is obtained from H by contracting the boundary spike $e = (x, y)$ with $x \in \partial V(H)$ and $y \in V^\circ(H)$. Then $G' = f^{-1}(H')$ is not necessarily obtained from G by boundary spike contraction, since the following are all possible:

- $f^{-1}(e)$ might contain multiple boundary spikes.
- There might be some $e' \in f^{-1}(e)$ whose endpoint $y' \in f^{-1}(y)$ is a boundary vertex instead of being interior like y Thus, in a neighborhood of e', the transformation from G to G' is not precisely a boundary spike contraction, but could be expressed as deleting a boundary edge e', then deleting the endpoint $x' \in f^{-1}(e)$ of e' which is now an isolated boundary vertex.
- There might be some $x' \in f^{-1}(x)$ with no edge attached to it, and in this case, the boundary spike contraction has pulled back to an isolated boundary vertex deletion.
- There may be some collapsed edges of G in $f^{-1}(x)$. Thus, in order to remove the vertices in $f^{-1}(x)$, one must first delete the edges in $f^{-1}(x)$, which must be boundary edges.

However, as some careful case-checking will show, these are essentially the only things that can go wrong. The transformation $G \mapsto G'$ can always be expressed as a sequence of three steps: deleting boundary edges, contracting boundary spikes, and deleting isolated boundary vertices. This motivates the following definition:

Definition. We take the terms **boundary spike contraction**, **boundary edge deletion**, and **isolated boundary vertex deletion** to allow multiple (even infinitely many) contractions or deletions and to include the trivial identity transformation (removing zero boundary spikes, boundary edges, or isolated boundary vertices). For a contraction of multiple boundary spikes, we require that the spikes do not share any endpoints.

We define a **layer-stripping operation** as a boundary edge deletion followed by a boundary spike contraction followed by an isolated boundary vertex deletion.

We say a ∂-graph is **layerable** if there exists a finite sequence of layer-stripping operations that will transform it to the empty graph. Note this is a stronger condition than complete reducibility, which also allows us to split apart boundary wedge-sums.

This definition makes layer-stripping functorial on UHMs in the following sense:

Lemma 16. *If $f : G \to H$ is unramified, $H_1 \subset H_2 \subset H$ and H_1 is obtained from H_2 by a layer-stripping operation, then $f^{-1}(H_1)$ is obtained from $f^{-1}(H_2)$ by a layer-stripping operation.*

Proof. The main ideas were given above and the details are left to the reader. □

Proposition 17. *Let G and H be finite ∂-graphs, $f : G \to H$ an unramified harmonic morphism. If H is layerable, then G is layerable.*

Proof. If H is layerable, then there is a filtration of subgraphs

$$H = H_0 \supset H_1 \supset \cdots \supset H_n = \varnothing$$

where H_{j+1} is obtained from H_j by a layer-stripping operation. By the Lemma,

$$G = f^{-1}(H_0) \supset \cdots \supset f^{-1}(H_n) = \varnothing$$

is a layer-stripping filtration of G, hence G is layerable. □

Remark. As in §5.1, if G' is obtained from G by a sequence of layer-stripping operations, then G' is a harmonic subgraph of G. Moreover, Lemmas 8 and 13 generalize to layer-stripping operations, even with infinitely many edges.

Remark. The process of splitting apart boundary wedge-sums is not functorial on UHMs because if $\Gamma_1 \cap \Gamma_2$ consists of a single vertex x, then $f^{-1}(x)$ might contain multiple vertices. However, splitting apart boundary wedge-sums and disjoint unions is functorial on inclusion maps, and thus, any subgraph of a completely reducible ∂-graph is also completely reducible. In fact, G is completely reducible if and only if it has no nonempty irreducible harmonic subgraphs.

6.2. **Layerable Equivalence, Reduction to Flowers.** Layer-stripping can be viewed as a loose discrete analogue of a deformation retraction; it is a reduction to a smaller space that leaves our "topological" invariant unchanged. This inspires the following analogue of homotopy equivalence. We say two finite ∂-graphs G and G' are **layerably equivalent** if there is a finite sequence of ∂-graphs $G = G_0, G_1, \ldots, G_n = G'$ such that for each j, either 1) G_j is obtained from G_{j+1} by a layer-stripping operation or 2) G_{j+1} is obtained from G_j by a layer-stripping operation.

As usual, we apply the same terminology to networks. If two $R^\times$-networks are layerably equivalent, then by Lemma 8 their Υ modules are isomorphic up to adding and removing free direct summands.

Our next goal is to find a canonical representative for each equivalence class. A natural candidate is a ∂-graph with no boundary spikes, boundary edges, or disconnected boundary vertices, which we will call a *flower* (because many planar examples look like flowers). Any finite ∂-graph is layerably equivalent to a flower: We can keep removing boundary spikes, boundary edges, and isolated boundary vertices until there are no more left. The result is a sequence of layer-stripping

operations that transforms G into a flower $❀(G)$. We claim that in fact this flower is *unique* and the map $G \mapsto ❀(G)$ is *functorial* on UHMs.

Theorem 2.

(1) Any finite ∂-graph G can be layer-stripped to a unique flower $❀(G)$.
(2) There is exactly one flower in each layerable equivalence class.
(3) If $f : G \to H$ is a UHM, then $❀(G) \subset f^{-1}(❀(H))$.
(4) $❀$ is a functor on the category of finite ∂-graphs and UHMs, and the inclusion $❀(G) \to G$ is a natural transformation $❀ \to \mathrm{id}$.

Proof. To prove the uniqueness claim of (1), suppose that we have two flowers H and $H' \subset G$, and let

$$G = H_0 \supset H_1 \supset \cdots \supset H_n = H, \qquad G = H'_0 \supset \cdots \supset H'_m = H'$$

be the corresponding filtrations of subgraphs obtained through layer-stripping. Applying Lemma 16 to the inclusion map $H \to G$, we see that

$$H = H \cap H'_0 \supset \cdots \supset H \cap H'_0 = H \cap H'$$

is another layer-stripping filtration. Since H is a flower, the filtration must be trivial, so that $H = H \cap H'$. By symmetry $H' = H \cap H' = H$.

To prove (2), suppose G' is obtained from G by a layer-stripping operation. We just showed $❀(G)$ is independent of the sequence of layer-stripping operations, so that $❀(G)$ is obtained by performing the layer-stripping operation $G \mapsto G'$, then reducing G' to a flower. Hence, $❀(G) = ❀(G')$. In general, if we have a layerable equivalence sequence $G = G_0, G_1, \ldots, G_n = G'$, then $❀(G_j)$ and $❀(G_{j+1})$ are equal (based on the identification of G_j as a subgraph of G_{j+1} or vice versa) and hence $❀(G) = ❀(G')$.

(3) follows from Lemma 16, and (4) follows from (3). □

As a consequence, the study of all "non-free" properties of finite $R^\times$-networks can be reduced in a functorial manner to the study of flowers:

Corollary 18.

(1) There is a corresponding flower functor on finite $R^\times$-networks and UHMs.
(2) $\Upsilon(\Gamma) = \Upsilon(❀\Gamma) \oplus R\,[\partial V(\Gamma) \setminus \partial V(❀\Gamma)]$
(3) If F is any functor on R-modules that commutes with direct sums and vanishes on free modules, then the inclusion natural transformation $❀ \to \mathrm{id}$ on $R^\times$-networks induces a natural isomorphism $F \circ \Upsilon \circ ❀ \to F \circ \Upsilon$.

7. Layer-Stripping and Harmonic Continuation

In this section, we interpret layer-stripping as a geometric model for harmonic continuation, and give two applications.

7.1. **Finding a Basis for Υ.** Our proof that Υ is free for any layerable $R^\times$-network can be (1) generalized to infinite networks, (2) sharpened to give an explicit basis, and (3) interpreted "dually" as a statement about step-by-step continuation of harmonic functions.

Lemma 8 has the following interpretation in terms of harmonic functions: For an $R^\times$-network Γ with boundary spike e with endpoints $x \in \partial V$ and $y \in V^\circ$, any harmonic function on Γ/e extends uniquely to a harmonic function on Γ

by setting $u(x) = u(y) + w(e)^{-1}\Delta_{\Gamma/e}u(y)$. For a boundary edge e, a harmonic function on $\Gamma \setminus e$ is also harmonic on Γ. For an isolated boundary vertex x, $\Upsilon(\Gamma) = RV(\Gamma)/\Delta(RV^\circ(\Gamma))$ is the internal direct sum $\Upsilon(\Gamma/e) \oplus Rx$. Dually, any harmonic function u on $\Gamma \setminus x$ extends to a harmonic function on Γ which is unique up to the choice of the one parameter $u(x)$.

Lemma 8 and the foregoing observations lead to the following:

Proposition 19. *Let Γ be an $R^\times$ network. Consider a filtration of subnetworks $\varnothing = \Gamma_0 \subset \Gamma_1 \subset \dots$ indexed by $\mathbb{N}$, in which Γ_j is obtained from Γ_{j+1} by a layer-stripping operation, and which exhausts Γ in the sense that $\bigcup_j \Gamma_j = \Gamma$. Let S be the set of vertices that are isolated boundary vertices at some step of the filtration.*

1. *Γ is non-degenerate.*
2. *Υ is free and S is a basis for it.*
3. *For any function $\phi : S \to M$, there is a unique harmonic function $u : V \to M$ with $u|_S = \phi$.*
4. *The function u can be found by computing $u|_{\Gamma_n}$ through inductive extension from Γ_j to Γ_{j+1}.*

Proof. The claims hold for each Γ_j, with $V_j \cap S$ instead of S, by inductive application of the foregoing observations, so it only remains to pass to the limit. Γ is non-degenerate because any finitely supported harmonic function must be supported in some Γ_j. Moreover, RV is the direct limit of RV_j and $\Delta(RV^\circ)$ is the direct limit of $\Delta(RV_j^\circ)$ with respect to the obvious inclusion maps. Since direct limits commute with quotients, this implies $\Upsilon(\Gamma)$ is the direct limit of $\Upsilon(\Gamma_j)$. The natural map $R(S \cap V_j) \to \Upsilon(\Gamma_j)$ is an isomorphism, and passing to the limit shows that the map $RS \to \Upsilon(\Gamma)$ is also an isomorphism. The statements about harmonic functions follow from the proof or can be verified directly. □

Example 3. Consider an $R^\times$-network on the lattice graph $\mathbb{Z}^2$ where all the vertices are interior. A filtration is given by the subnetworks Γ_n induced by the vertices $\mathbb{Z} \times \{-n+1, \dots, n\}$ (for $n \geq 1$) with boundary $\mathbb{Z} \times \{-n+1, n\}$. Isolated boundary vertices are only added at the first step from $\Gamma_0 = \varnothing$ to Γ_1. Thus, the vertices $\mathbb{Z} \times \{0, 1\}$ form a basis for Υ.

7.2. **Finding** $\mathrm{Tor}_1(\Upsilon(\Gamma), M)$**.** Harmonic continuation via layer-stripping has applications to non-layerable networks as well. For instance,

Proposition 20. *Suppose Γ is a finite non-degenerate $R^\times$-network over a PID. Suppose that a layerable network Γ' is obtained from Γ by changing n vertices from interior to boundary. Then the torsion submodule of $\Upsilon(\Gamma)$ has at most n invariant factors.*

Proof. Suppose that Γ' is obtained from Γ by assigning $S \subset V^\circ(\Gamma)$ to be boundary vertices in Γ'. We know that $\mathrm{Tor}_1(\Upsilon(\Gamma'), M)$ is trivial, or equivalently any harmonic u with vanishing potential and current on $\partial V'$ is identically zero. This implies that if $u \in \mathcal{U}(\Gamma, M)$ with vanishing potential and current on the boundary, and if $u|_S = 0$, then $u \equiv 0$. In other words, there is an injection $\mathrm{Tor}_1(\Upsilon(\Gamma), M) \to M^S$. Thus, $\mathrm{Tor}_1(\Upsilon(\Gamma), M)$ has at most $|S|$ invariant factors. □

We claim that, in fact, we can use harmonic continuation to solve for u in terms of $u|_S$. We describe how to use harmonic continuation to express the submodule of M^S corresponding to $\mathrm{Tor}_1(\Upsilon(\Gamma), M)$ explicitly in the form $\{\phi \in M^S : A\phi = 0\}$

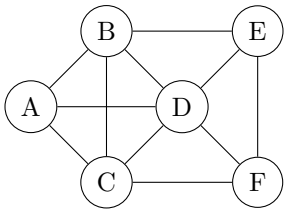

FIGURE 5. The graph appearing in Example 4.

where A is some $(|\partial V| + |S|) \times |S|$ matrix. Thus, we may determine the invariant factor decomposition of $\mathrm{Tor}_1(\Upsilon(\Gamma), M)$ from the Smith normal form of A rather than Δ.

Our harmonic continuation process will start with the values of u on $\partial V'$ and move inward from the boundary, rather than outward as in Proposition 19. Given a layer-stripping filtration

$$\Gamma' = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_n \supset \Gamma_{n+1} = \varnothing,$$

we define a complementary filtration of subnetworks Σ_j by

$$V(\Sigma_j) = V(\Gamma') \setminus V^\circ(\Gamma_j), \quad V^\circ(\Sigma_j) = V(\Gamma') \setminus V(\Gamma_j), \quad E(\Sigma_j) = E(\Gamma') \setminus E(\Gamma_j).$$

We will find u in terms of $u|_S$ by harmonic continuation through the filtration $\{\Sigma_j\}$.

We assume for simplicity that $\Gamma_j \mapsto \Gamma_{j+1}$ is a boundary edge deletion or boundary spike contraction for $j = 0, \ldots, n-1$, and that $\Gamma_n \mapsto \Gamma_{n+1}$ is an isolated boundary vertex deletion (this can always be arranged). Σ_0 consists of the isolated boundary vertices $\partial V' = \partial V \cup S$. If Γ_{j+1} is obtained from Γ_j by removing a boundary spike or boundary edge, then Σ_{j+1} is obtained from Σ_j by adding a boundary spike or boundary edge. At the last step, $\Gamma_{n+1} = \varnothing$ is obtained from Γ_n by deleting isolated boundary vertices, while Σ_{n+1} is obtained from Σ_n by changing these vertices from boundary to interior.

Σ_{n+1} has the same vertex and edge sets as Γ, but all the vertices are interior. Hence,

$$\begin{aligned}\mathcal{U}_0(\Gamma, M) &= \{u \in \mathcal{U}(\Sigma_{n+1}, M) : u|_{\partial V} = 0\} \\ &= \{u \in \mathcal{U}(\Sigma_n, M) : u|_{\partial V} = 0, \Delta u|_{\partial V_n} = 0\}.\end{aligned}$$

Any function on Σ_0 extends uniquely to a harmonic function Σ_n. In particular, for any $\phi \in M^S$, there is unique harmonic function u on Σ_n such that $u|_S = \phi$ and $u|_{\partial V} = 0$. Through the harmonic continuation process, we can inductively solve for $u|_{V_j}$ as a linear function of $\phi = u|_S$, and hence obtain a formula for $\Delta u|_{\partial V_n}$ as a linear function of ϕ. If A is the matrix representing the transformation $\phi \to \Delta u|_{\partial V_n}$, then

$$\mathrm{Tor}_1(\Upsilon(\Gamma), M) = \{u \in \mathcal{U}(\Sigma_n, M) : u|_{\partial V} = 0, \Delta u|_{\partial V_n} = 0\} \cong \{\phi \in M^S : A\phi = 0\}.$$

When solving for $\Delta u|_{\partial V_j}$ in terms of $u|_S$, it is not necessary to keep track of all the values of $u|_{V_j}$ at each stage. Instead, we can simply keep track of the boundary data $(u|_{\partial V_j}, \Delta_{\Sigma_j} u|_{\partial V_j})$. If we write the boundary data of Σ_j as a column vector a_j with $2|\partial V_j|$ entries, and if Σ_{j-1} is obtained from Σ_j by adding a boundary spike, then a_{j+1} is obtained from a_j by a single elementary row operation; for adding a boundary edge, there are four row operations.

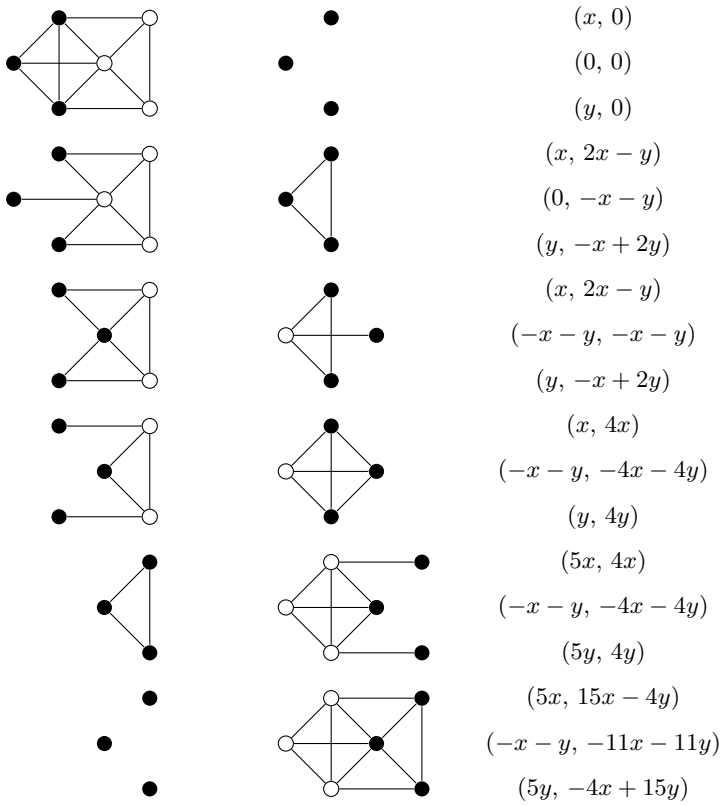

FIGURE 6. Complementary filtrations of the network Γ'. Left column: Γ_j. Middle column: Σ_j. Right column: (potential, current) pairs representing the boundary data of a harmonic function extended along the filtration $\{\Sigma_j\}$.

Example 4. We compute the sandpile group of the graph shown in Figure 5 by choosing one boundary vertex, setting the edge weights to 1, and finding $\mathcal{U}_0(\Gamma, \mathbb{Q}/\mathbb{Z})$ (see 3.4). We declare A to be a boundary vertex of Γ, then define Γ' by changing B and C to boundary vertices as well. A layerable filtration $\{\Gamma_j\}$ of Γ' is shown in Figure 6, along with the complementary filtration $\{\Sigma_j\}$. We let x and y be the values of u at B and C and solve for $(u|_{\partial V_j}, \Delta u|_{\partial V_j})$ inductively, and we obtain

$$\Delta u|_{\partial V_n} = (15x - 4y, -11x - 11y, 4x - 15y),$$

and hence $\mathrm{Tor}_1(\Upsilon, \mathbb{Q}/\mathbb{Z})$ is found from the SNF of

$$\begin{pmatrix} 15 & -4 \\ -11 & -11 \\ -4 & 15 \end{pmatrix},$$

yielding $\Upsilon \cong \mathbb{Z} \oplus \mathbb{Z}/11 \oplus \mathbb{Z}/19$. Generators for the $\mathbb{Z}/11$ and $\mathbb{Z}/19$-valued harmonic functions are given by taking $(x, y) = (1, 1)$ and $(x, y) = (1, -1)$ respectively.

Harmonic continuation reduces the computation of $\mathrm{Tor}_1(\Upsilon(\Gamma), M)$ to $|\partial V_n| = |\partial V| + |S|$ equations in $|S|$ unknowns. If $|S|$ is much smaller than $|V|$, then A is

a much smaller matrix than Δ. Thus, harmonic continuation is a useful approach when Γ can be made layerable by changing only a few vertices from interior to boundary. On the other hand, for dense graphs (for example, a complete graph with no boundary vertices) one needs to change almost all the vertices to boundary to achieve layerability. In this case, harmonic continuation does not simplify the SNF computation, and we expect the torsion submodule of Υ to have a large number of invariant factors.

8. Network Duality

8.1. **Dual Circular Planar Networks, Harmonic Conjugates.** As shown in [9], dual planar graphs have isomorphic sandpile groups. In this section, we generalize this result to circular planar $R^\times$-networks.

The following definitions are adapted from [18, 12, 25]. A **circular planar ∂-graph** G is a (finite) ∂-graph embedded in the closed unit disk $\overline{D}$ in the complex plane such that $V \cap \partial D = \partial V$. The **faces** of G are the components of $D \setminus G$. A connected circular planar ∂-graph has a **circular planar dual** $G^\dagger$ whose vertices are points in the faces of G and whose edges connect faces which share an edge. For each oriented edge $e \in E(G)$, $e^\dagger$ denotes the oriented edge from the face on the right of e to the face on the left of e. A vertex of $G^\dagger$ is considered a boundary vertex if the closure of its face in G intersects ∂D.

If $\Gamma = (G, w)$ circular planar $R^\times$-network, then the *dual network* $\Gamma^\dagger$ is the network on $G^\dagger$ with $w(e^\dagger) = w(e)^{-1}$.

Remark. The planar dual is constructed in a similar fashion for a connected planar network without boundary. To incorporate planar networks without boundary into the circular planar framework, we may designate an arbitrary vertex to be a boundary vertex.

Theorem 3. *If Γ is a connected circular planar $R^\times$-network, then*

$$\widetilde{\Upsilon}(\Gamma^\dagger) \cong \widetilde{\Upsilon}(\Gamma).$$

Theorem 3 generalizes the main result of [9] to $R^\times$-networks. Our proof uses ideas from [5, §26 - 29] and[12, Section 7].

Proof. The result follows from reformulating $\widetilde{\Upsilon}$ in terms of terms of oriented edges rather than vertices. Recall that $C_1(\Gamma)$ is the free R-module on the oriented edges E modulo the relations $\overline{e} = -e$. Then $\ker \epsilon$ can be identified the quotient of $C_1(\Gamma)$ by the submodule generated by cycles. The cycle submodule is in fact generated by the oriented boundaries of interior faces. $\Delta(RV^\circ)$ corresponds to the submodule of $C_1(\Gamma)$ generated by $\sum_{e_+=x} w(e)e$. The edges bounding an interior face of G correspond to the edges incident to an interior vertex in $G^\dagger$. Therefore,

$$\widetilde{\Upsilon}(\Gamma) \cong \frac{C_0(\Gamma)}{(\sum_{e^\dagger_+=x} e : x \in V^\circ(G^\dagger)) + (\sum_{e_+=x} w(e)e : x \in V^\circ(G))}$$

$$\widetilde{\Upsilon}(\Gamma^\dagger) \cong \frac{C_0(\Gamma^\dagger)}{(\sum_{e^\dagger_+=x} w(e^\dagger)e^\dagger : x \in V^\circ(G^\dagger)) + (\sum_{e_+=x} e^\dagger : x \in V^\circ(G))}.$$

Since $w(e^\dagger) = w(e)^{-1}$, we can define an isomorphism $\Upsilon(\Gamma) \to \Upsilon(\Gamma^\dagger)$ by $e \mapsto w(e)^{-1}e^\dagger$. □

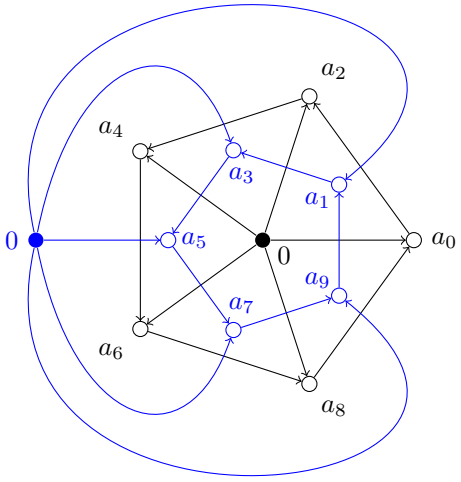

FIGURE 7. W_5 and its (isomorphic) dual. Arrows indicate the paired dual oriented edges.

Application of $\mathrm{Hom}(-, M)$ yields the following "complex-analytic" interpretation of network duality, as in [8, 14, 25].

Corollary 21. *Let Γ be a circular planar $R^\times$-network. Modulo constant functions, for any M-valued harmonic function u on Γ, there is a unique harmonic conjugate v on $\Gamma^\dagger$ satisfying the discrete Cauchy-Riemann equation $w(e)du(e) = dv(e^\dagger)$.*

Proof. Given our interpretation of $\widetilde{\Upsilon}(\Gamma)$ in the previous proof, a harmonic function modulo constants is equivalent to a map $\phi : E(\Gamma) \to M$ such that $\phi(e)$ sums to zero around any oriented cycle and $\sum_{e_+=x} w(e)\phi(e) = 0$ for each interior vertex. For any such ϕ, we can define a similar function ψ on the dual network by $\psi(e^\dagger) = w(e)\phi(e)$. □

Remark. If G is circular-planar, then contracting a boundary spike on G corresponds to deleting a boundary edge in $G^\dagger$. Layer-stripping may disconnect the graph, and in this case, the definition of dual is trickier to state, and we leave the details to the reader. Boundary wedge-sums or disjoint unions of circular planar networks will correspond to boundary wedge-sums or disjoint unions in the dual. Hence, G is completely reducible if and only if $G^\dagger$ is completely reducible. One can also prove this by appealing to Theorem 1 and Theorem 3.

8.2. **Wheel Graphs.** Consider the wheel graph W_n embedded in the complex plane with vertices at $\left\{e^{2\pi ik/n}\right\}_{k\in\mathbb{Z}}$ and at 0. Edges connect 0 to $e^{2\pi ik/n}$ and $e^{2\pi ik/n}$ to $e^{2\pi i(k+1)/n}$ for all $k \in \mathbb{Z}$. Figure 7 depicts W_5 and its planar dual.

The sandpile group of W_n is computed in [6] using chip-firing, induction, and the symmetry of the graph, and a connection with Lucas sequences is uncovered. We present an alternate approach to compute the sandpile group using harmonic continuation and planar duality.

Proposition 22 ([6] Theorem 9.2)**.** *Let W_n be the wheel graph, considered as a boundary-less $\mathbb{Z}$-network with edge weights 1. Let $F_0 = 0$, $F_1 = 1$, $F_2 = 2$, ... be the Fibonacci numbers. The sandpile group of W_n is*

$$\widetilde{\Upsilon}(W_n) \cong \begin{cases} \mathbb{Z}/(F_{n-1}+F_{n+1}) \times \mathbb{Z}/(F_{n-1}+F_{n+1}), & n \text{ odd,} \\ \mathbb{Z}/F_n \times \mathbb{Z}/5F_n, & n \text{ even.} \end{cases}$$

Proof. By Proposition 5 and §3.4, it suffices to compute the $\mathbb{Q}/\mathbb{Z}$-valued harmonic functions that vanish at the central "hub" vertex. This is equivalent to finding the $\mathbb{Z}$-module of pairs (u, v) on W_n and $W_n^\dagger$ that satisfy Cauchy-Riemann and vanish at two "hub" vertices (colored solid in Figure 7). Let $a_0, a_1, a_2, \dots$ be the potentials on the "rim" vertices of W_n and $W_n^\dagger$ in counterclockwise order as shown in the Figure 7, with indices taken modulo $2n$. The Cauchy-Riemann equations tell us that

$$a_{j+1} - a_{j-1} = a_j - 0.$$

In other words, the numbers a_j satisfy the Fibonacci recurrence, so that

$$\begin{pmatrix} a_{j+1} \\ a_j \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_j \\ a_{j-1} \end{pmatrix}.$$

Note that a harmonic pair (u, v) is uniquely determined by (a_1, a_0). More precisely, if A is the 2×2 matrix of the recursion, then $(a_1, a_0) \in (\mathbb{Q}/\mathbb{Z})^2$ will produce a harmonic pair (u, v) through the iteration process if and only if it is a fixed point of A^{2n}. The module of harmonic pairs (u, v) is thus isomorphic to the kernel of $A^{2n} - I$ acting on $(\mathbb{Q}/\mathbb{Z})^2$. So the invariant factors of the sandpile group are given by the Smith normal form of $A^{2n} - I$, which is the same as the SNF of $A^n - A^{-n}$ because $\det A = -1$. For $n \geq 1$,

$$A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \qquad A^{-n} = (-1)^n \begin{pmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{pmatrix},$$

If n is odd, then

$$A^n - A^{-n} = (F_{n+1} + F_{n-1})I,$$

and if n is even, then

$$A^n - A^{-n} = \begin{pmatrix} F_{n+1} - F_{n-1} & 2F_n \\ 2F_n & F_{n-1} - F_{n+1} \end{pmatrix} = F_n \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}.$$

From here, the computation of the invariant factors is straightforward. □

9. Analysis of Chain-Link Fence Networks

9.1. **Motivation and Set-Up.** We compute Υ for a family of ∂-graphs resembling a chain-link fence on the cylinder, which play a crucial role in the cylindrical electrical inverse problem [26]. We proceed by harmonic continuation, also exploiting symmetry and subgraphs.

To simplify the main computation, we initially consider a slightly different family of ∂-graphs from [26]. Consider a ∂-graph $\textsc{clf}(m, n)$ with $V = \mathbb{Z}/m \times \{0, \dots, n\}$ and $\partial V = \mathbb{Z}/m \times \{0\}$ and edges defined by

$$(j, k) \sim (j + 1, n - k + 1) \text{ for } k \geq 1$$
$$(j, k) \sim (j + 1, n - k) \text{ for } k \geq 0,$$

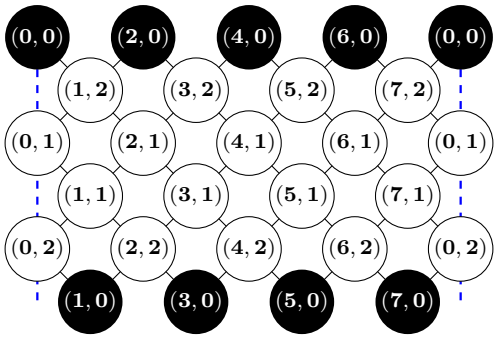

FIGURE 8. The graph $\mathrm{CLF}(8,2)$. Boundary vertices are black, interior vertices are white, and the vertices lying on blue dashed lines are identified.

as shown in Figure 8. If m is even then the network is one of Lam and Pylyavksyy's "purely cylindrical" graphs [26]. If m is odd, then it resembles a chain-link fence shaped like a Möbius band.

Consider $\mathrm{CLF}(m,n)$ as a $\mathbb{Z}$-network with edge weights 1. Since the network is non-degenerate, Υ will have a free part of rank $|\partial V|$, and thus we will restrict our attention to the torsion. By Corollary 5, the torsion submodule is isomorphic $\mathrm{Tor}_1(\Upsilon, \mathbb{Q}/\mathbb{Z}) = \mathcal{U}_0(\mathrm{CLF}(m,n), \mathbb{Q}/\mathbb{Z})$. We will show

Theorem 4. *For* $\mathrm{CLF}(m,n)$ *viewed as a* $\mathbb{Z}$*-network with edge weights* 1*,*

$$\mathcal{U}_0(\mathrm{CLF}(m,n), \mathbb{Q}/\mathbb{Z}) \cong \begin{cases} (\mathbb{Z}/2)^n, & m \text{ odd} \\ (\mathbb{Z}/2)^{2n}, & m \equiv 2 \mod 4 \\ \displaystyle\bigoplus_{j=1}^{n} \left(\frac{\mathbb{Z}}{\gcd(4^j, 2m)} \right)^2, & m \equiv 0 \mod 4. \end{cases}$$

As in §7, we observe that changing the vertices $S = \{0,1\} \times \{1, \ldots, n\}$ to boundary will make the graph layerable, and therefore, we desire to write a harmonic function $u \in \mathcal{U}_0(\mathrm{CLF}(m,n), \mathbb{Q}/\mathbb{Z})$ in terms of its values on these vertices. However, given the symmetry of the graph, there is a more convenient way to formulate harmonic continuation than using the layerable filtration. Starting with the values of u on S (the first two columns of vertices), we will proceed column by column to harmonically continue around the annulus. Thus, as in §8.2, the harmonic functions will be periodic points of a certain propagation matrix.

The harmonic continuation process does not use the condition that the net current on the boundary is zero, so we will have to test that condition separately. Note that the annulus is covered by the infinite strip, and $\mathrm{CLF}(m,n)$ is covered by a graph $\mathrm{CLF}(\infty, n)$ with vertex set $\mathbb{Z} \times \{0, \ldots, n\}$. For $\phi \in (\mathbb{Q}/\mathbb{Z})^S$ to extend to an element of $\mathcal{U}_0(\mathrm{CLF}(m,n), \mathbb{Q}/\mathbb{Z})$, two conditions must be satisfied: First, it must satisfy the periodicity condition to yield a well-defined function on $\mathrm{CLF}(m,n)$, and second, it must define a function on $\mathrm{CLF}(\infty, n)$ with net current zero on the boundary. The advantage in separating these two conditions is that the second condition is easier to analyze on the infinite graph.

In §9.2, we use harmonic continuation to write our module in a simple form (Lemma 23). Then in §9.3, we work algebraically to find the invariant factor decomposition.

9.2. **Harmonic Continuation.** Our goal is to compute the $\mathbb{Q}/\mathbb{Z}$-valued harmonic functions with $u = \Delta u = 0$ on ∂V. We start by understanding the harmonic functions with $u = 0$ on the boundary using harmonic continuation around the loop. Assume $u(j,0) = 0$ and let

$$a_j = \begin{pmatrix} u(j,1) \\ \vdots \\ u(j,n) \end{pmatrix} \in (\mathbb{Q}/\mathbb{Z})^n.$$

Then harmonicity amounts to

$$4a_j = Ea_{j-1} + Ea_{j+1},$$

where E is the $n \times n$ matrix with 1's on and above the skew-diagonal, e.g.,

$$E = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad n = 5.$$

Thus, the vectors a_j satisfy the recurrence relation

$$\begin{pmatrix} a_{j+1} \\ a_j \end{pmatrix} = \begin{pmatrix} 4E^{-1} & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a_j \\ a_{j-1} \end{pmatrix}.$$

Therefore, if we let T be the "propagation matrix,"

$$T = \begin{pmatrix} 4E^{-1} & 1 \\ -1 & 0 \end{pmatrix},$$

then harmonic functions with $u = 0$ on ∂V are equivalent to fixed points of T^m in $(\mathbb{Q}/\mathbb{Z})^{2n}$.

Next, we must determine when a fixed point of T^m will have net current zero on the boundary, which amounts to writing all the net current conditions in terms of the first two columns of vertices. We sidestep the direct computation of currents by adding an extra row of vertices at the top/bottom of $\textsc{clf}(m,n)$, viewing it as a subgraph of $\textsc{clf}(m,n+1)$ by the map $(j,k) \mapsto (j, n+1-k)$ (see Figure 9). Harmonic functions on $\textsc{clf}(m,n)$ with $u = \Delta u = 0$ on ∂V correspond to harmonic functions on $\textsc{clf}(m,n+1)$ which are zero on the boundary vertices and on the top and bottom rows of interior vertices, that is, $u(j,0) = u(k,n+1) = 0$. If $w \in (\mathbb{Q}/\mathbb{Z})^{2(n+1)}$ represents the initial data of a harmonic function on $\textsc{clf}(m,n+1)$, then $u(j,n+1)$ is the $(n+1)$-th coordinate of $T^{j-1}w$. Therefore,

Lemma 23. *$\mathcal{U}_0(\textsc{clf}(m,n), \mathbb{Q}/\mathbb{Z})$ is isomorphic to*

$$\{w \in (\mathbb{Q}/\mathbb{Z})^{2(n+1)} : T^m w = w \text{ and } (e,0) \cdot T^j w = 0 \text{ for all } j \in \mathbb{Z}\},$$

where $e = (0,\dots,1) \in \mathbb{Z}^{n+1}$ is the $(n+1)$-th standard basis row vector and T is the propagation matrix for $\textsc{clf}(m,n+1)$.

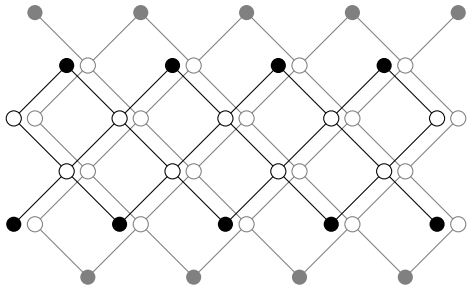

FIGURE 9. An embedding of CLF(8, 1) into CLF(8, 2). The vertices on the left and right sides are identified.

9.3. **An Explicit Basis for $\mathcal{U}_0$.** To compute the module in Lemma 23, we first consider the condition $(e,0)\cdot T^j w = 0$, disregarding the periodicity condition. We obtain

Lemma 24.

$$\{w \in (\mathbb{Q}/\mathbb{Z})^{2(n+1)} : (e,0)\cdot T^j w = 0\} \cong \bigoplus_{j=1}^{n} \left(\frac{\mathbb{Z}}{4^j}\right)^2.$$

Proof. This amounts to finding Smith normal form for the submodule of $\mathbb{Z}^{2(n+1)}$ generated by $(e,0)\cdot T^j$, $j \in \mathbb{Z}$. Note

$$T = \begin{pmatrix} 4E^{-1} & 1 \\ -1 & 0 \end{pmatrix}, \qquad T^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 4E^{-1} \end{pmatrix}.$$

Since $(e,0)$ only interacts with the top row of T^j, we first need to find $(1_{n\times n}, 0_{n\times n})\cdot \mathbb{Z}[T,T^{-1}]$. A straightforward argument shows that

$$(1,0)\cdot\mathbb{Z}[T,T^{-1}] = \mathbb{Z}[4E^{-1}](1,0) + \mathbb{Z}[4E^{-1}](0,1).$$

Therefore,

$$\frac{\mathbb{Z}^{2(n+1)}}{(e,0)\cdot\mathbb{Z}[T,T^{-1}]} \cong \left(\frac{\mathbb{Z}^{n+1}}{e\cdot\mathbb{Z}[4E^{-1}]}\right)^2.$$

Another simple computation shows that e, $e\cdot E$, …, $e\cdot E^n$ is a basis for $\mathbb{Z}^{n+1}$ and therefore, since E is invertible over $\mathbb{Z}$,

$$e, E^{-1}e, \ldots, E^{-n}e \text{ is a basis for } \mathbb{Z}^{n+1}.$$

On the other hand,

$$e, 4E^{-1}e, \ldots, 4^n E^{-n}e \text{ is a basis for } e\cdot\mathbb{Z}[4E^{-1}],$$

so that

$$\frac{\mathbb{Z}^{n+1}}{e\cdot\mathbb{Z}[4E^{-1}]} \cong \bigoplus_{j=1}^{n} \frac{\mathbb{Z}}{4^j},$$

which completes the proof. □

Now we turn to the periodicity condition $T^m w = w$. In light of the last lemma, we can restrict our attention to w in the 2-torsion submodule $T_2(\mathbb{Q}/\mathbb{Z})^{2(n+1)}$. Since we are only considering functions that vanish on the top and bottom rows of interior

vertices of $\mathrm{CLF}(m, n+1)$, we can also revert back to working $\mathrm{CLF}(m, n)$ for purposes of the periodicity computation. We claim

Lemma 25. *Suppose $m = r2^s$ with r odd. Then*

$$\{w \in T_2(\mathbb{Q}/\mathbb{Z})^{2n} : T^m w = w\} \cong \begin{cases} (\mathbb{Z}/2)^n, & m \text{ odd} \\ (\mathbb{Z}/2)^{2n}, & m \equiv 2 \bmod 4 \\ (\mathbb{Z}/2^{s+1})^{2n}, & m \equiv 0 \bmod 4. \end{cases}$$

Proof. We need to find an accurate enough 2-adic expansion of $T^m - 1$. Note that

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bmod 4.$$

Hence, when $m = 1 \bmod 4$,

$$T^m - 1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \bmod 4,$$

and thus $\{w \in T_2(\mathbb{Q}/\mathbb{Z})^{2n} : T^m w = w\} \cong (\mathbb{Z}/2)^n$, and the case where $m = 3 \bmod 4$ is similar. When $m = 2 \bmod 4$, then $T^m - I = -2 \bmod 4$, so we get $(\mathbb{Z}/2)^{2n}$. To handle the case where $m = 0 \bmod 4$, one can show by induction that

$$T^{2^s} = 1 + 2^{s+1} \begin{pmatrix} 0 & E^{-1} \\ -E^{-1} & 0 \end{pmatrix} \bmod 2^{s+2} \text{ for } s \geq 2.$$

and hence if r is odd

$$T^{r2^s} - 1 = 2^{s+1} \begin{pmatrix} 0 & E^{-1} \\ -E^{-1} & 0 \end{pmatrix} \bmod 2^{s+2} \text{ for } s \geq 2,$$

which completes the proof. □

Proof of Theorem 4. We work on $\mathrm{CLF}(m, n)$ rather than $\mathrm{CLF}(m, n+1)$. The module we want to compute is the intersection of two submodules of $(\mathbb{Q}/\mathbb{Z})^{2n}$, the first given by the boundary current zero condition without periodicity, and the second given by the periodicity condition. The theorem now follows from Lemmas 24 and 25 and a small amount of casework. □

9.4. **Chain-Link Fence Variants.** The graphs $\mathrm{CLF}(2m, n)$ are "purely cylindrical" graphs on the cylinder with $2n$ rows of vertices. As in [26], one can replace $2n$ rows of vertices with an odd number of vertices. We will call the cylindrical graph with n rows and $2m$ columns of vertices $\mathrm{CLF}'(m, n)$, so that $\mathrm{CLF}(2m, n) = \mathrm{CLF}'(m, 2n)$.

Knowing the $\mathbb{Q}/\mathbb{Z}$-valued harmonic functions with zero boundary data for $\mathrm{CLF}'(m, 2n)$, we can easily deduce what they are for $\mathrm{CLF}'(m, 2n-1)$

Proposition 26.

$$\mathcal{U}_0(\Upsilon(\mathrm{CLF}'(m, 2n-1)), \mathbb{Q}/\mathbb{Z}) \cong \begin{cases} (\mathbb{Z}/2)^{2n-1}, & m \text{ odd} \\ \bigoplus_{j=1}^{n} \frac{\mathbb{Z}}{(4^j, m)} \oplus \bigoplus_{j=1}^{n-1} \frac{\mathbb{Z}}{(4^j, m)}, & m \text{ even.} \end{cases}$$

Proof. Consider the inclusion maps $f_k : \mathrm{CLF}'(m, k) \to \mathrm{CLF}'(m, k+1)$ which include a smaller cylindrical network into the bottom n rows of the larger one. Now f_{2n-1} induces an inclusion map $\mathcal{U}_0(\mathrm{CLF}'(m, 2n-1), \mathbb{Q}/\mathbb{Z}) \to \mathcal{U}_0(\mathrm{CLF}'(m, 2n), \mathbb{Q}/\mathbb{Z})$. Let a_0 and a_1 be the data of a harmonic function on the first two columns of vertices

on $\textsc{clf}'(m,2n)$. The proof of Lemma 24 shows that the harmonic functions on $\textsc{clf}'(m,2n)$ split into a direct sum of those with $a_0 = 0$ and those with $a_1 = 0$, and one can deduce that the same direct sum decomposition holds for $\textsc{clf}'(m,2n-1)$. Functions on $\textsc{clf}'(m,2n)$ with $a_1 = 0$ must have potential zero on the top row of interior vertices in $\textsc{clf}'(m,2n)$ (see Figure 8), and therefore restrict to harmonic functions on $\textsc{clf}'(m,2n-1)$. Thus, the functions with $a_1 = 0$ are the same for $\textsc{clf}'(m,2n-1)$ and $\textsc{clf}'(m,2n)$. By similar reasoning, the functions with $a_0 = 0$ are the same for $\textsc{clf}'(m,2n-1)$ and $\textsc{clf}'(m,2n-2)$. The asserted result follows: Intuitively, $\textsc{clf}'(m,2n-1)$ gets half of the module for $\textsc{clf}'(m,2n-2)$ and half of the module for $\textsc{clf}'(m,2n)$. □

Our method of proof also allows us to determine how the various inclusion maps of CLF networks interact with the invariant factor decomposition. As in the proof of Lemma 24, everything boils down to working out the right action of the matrix E on the standard basis vectors. For instance, if we consider the inclusion of $\textsc{clf}(m,n)$ into the "middle" of $\textsc{clf}(m,n+1)$ as in Figure 9, one can show this includes each of the summands $\mathbb{Z}/4^n$ into $\mathbb{Z}/4^{n+1}$ for the networks on the infinite strip, which determines its behavior on the finite networks as well.

The CLF networks were discussed at the University of Washington Mathematics REU during the summer of 2015. Will Dana, Collin Litterell, and Austin Stromme assisted in the computation of the characteristic polynomials of matrices in §5, during the course of which we gained a better understanding for the structure of these matrices. Unfortunately we do not have an occasion to present this calculation, but we remark that there is a beautiful connection with Chebyshev polynomials of the second kind.

10. Covering Maps and Symmetry

In §5, we were primarily concerned with the existence or non-existence of torsion. Here we explain how symmetry imposes constraints on the group structure and torsion primes of Υ. We omit the straightforward proofs of the following observations, which we record for the benefit of future research.

Recall from §4 that a covering map is a geometric harmonic morphism $f : \widetilde{\Gamma} \to \Gamma$ such that $f(V_1^\circ) = V_2^\circ$, $f(\partial V_1) = \partial V_2$, and for each $x \in V_1$, f maps $\{e \in E_1 \colon e_+ = x\}$ bijectively onto $\{e \in E_2 \colon e_+ = f(x)\}$. The **fibers** of f are the sets $f^{-1}(x)$ for $x \in V(\Gamma)$. The covering map is **r-sheeted** if $|f^{-1}(x)| = r$ for all x, and it is **finite-sheeted** if $|f^{-1}(x)| < \infty$ for all x.

Observation 27. *A covering map $f : \widetilde{\Gamma} \to \Gamma$ induces a surjection $\Upsilon(\widetilde{\Gamma}) \to \Upsilon(\Gamma)$ providing the following isomorphism:*

$$\Upsilon(\Gamma) \cong \Upsilon(\widetilde{\Gamma}) \Bigg/ \sum_{\substack{x,y \in V(\Gamma) \\ f(x)=f(y)}} R(x-y)$$

Dually, there is an injection $\mathcal{U}(\Gamma, M) \to \mathcal{U}(\widetilde{\Gamma}, M)$ which identifies harmonic functions on Γ with harmonic functions on $\widetilde{\Gamma}$ that are constant on each fiber of f.

Observation 28. *Suppose that f is an r-sheeted covering map. Regard r as an element of R by applying the ring morphism $\mathbb{Z} \to R$. Suppose further that multiplication by $r \in R$ is an isomorphism of the R-module M, and let r^{-1} denote the*

inverse morphism. Then the map $\mathcal{U}(\Gamma, M) \to \mathcal{U}(\widetilde{\Gamma}, M)$ *is a split injection with left inverse given by the averaging operator*

$$Au(x) = r^{-1} \left(\sum_{y \in f^{-1}(x)} u(y) \right),$$

as in the proof of Maschke's Theorem, and hence $\mathcal{U}(\widetilde{\Gamma}, M) \cong \mathcal{U}(\Gamma, M) \oplus \ker A$.

Observation 29. *If a group* K *acts freely by* R*-network automorphisms on* Γ*, then there exists a quotient network* Γ/K *and a covering map* $\Gamma \to \Gamma/K$*. The induced* K*-action on* $\mathcal{U}(\Gamma, M)$ *has fixed-point submodule* $\mathcal{U}(\Gamma, M)^K = \mathcal{U}(\Gamma/K, M)$*. If* K *is a* p*-group, then*

$$|\mathcal{U}(\Gamma, M)| \equiv |\mathcal{U}(\Gamma/K, M)| \text{ mod } p,$$

provided both sides are finite. For example, this holds when Γ *and* M *are finite.*

For a finite-sheeted covering map $f : \widetilde{\Gamma} \to \Gamma$, there is an injection $\mathcal{U}_0(\Gamma, M) \to \mathcal{U}_0(\widetilde{\Gamma}, M)$. Indeed, if $f \colon \widetilde{\Gamma} \to \Gamma$ is a covering map and u is harmonic on Γ with vanishing boundary potential and net current, then the pullback f^*u also has vanishing boundary potential and current. Furthermore when f is finite-sheeted, the pullback of a finitely supported function is finitely supported. Observations 28 and 29 also apply with $\mathcal{U}_0(\Gamma, M)$ in place of $\mathcal{U}(\Gamma, M)$.

While these statements hold in general, the mod p counting formula is especially useful when $R = \mathbb{Z}$. For instance, suppose P is a p-group acting freely on a finite non-degenerate $\mathbb{Z}$-network Γ. Because $\operatorname{Tor}_1(\Upsilon(\Gamma), M) = \mathcal{U}_0(\Gamma, M)$, and because $|\mathcal{U}_0(\Gamma, M)| = |\mathcal{U}_0(\Gamma/S, M)| \bmod p$, we see that $\Upsilon(\Gamma)$ has p-torsion if and only if $\Upsilon(\Gamma/P)$ has p-torsion.

On the other hand, if q is relatively prime to p, then by Observation 28,

$$\mathcal{U}_0(\Gamma, \mathbb{Z}/q^k) \cong \mathcal{U}_0(\Gamma/P, \mathbb{Z}/q^k) \oplus \ker A, .$$

where A is the averaging operator on $\mathcal{U}_0$. Together with Observation 29, this yields $|\ker A| \equiv 1 \bmod p$. By Observation 29, P acts faithfully on $\ker A$. This is a significant amount of information about the q-torsion, especially if we have either bounds on the order of $\operatorname{Tor}_1(\Upsilon(\Gamma), \mathbb{Z}/q^k)$ through determinantal computations or bounds on the number of generators obtained from harmonic continuation as in §9.2.

These symmetry principles can be tweaked to yield statements for branched covering maps. We leave the general case to the reader and restrict our attention to the wheel graphs of §8.2. We declare the central vertex to be a boundary vertex. There is a natural branched covering map $W_{kn} \to W_n$ that preserves the central vertex, and W_n can be viewed as the quotient of W_{kn} by the rotation action of the cyclic group $K = \mathbb{Z}/k$.

Though this is not a covering map, Observation 27 still holds, and Observation 28 holds with $r = k$. To verify Observation 29 in this case, note $\mathcal{U}_0(W_n, M) \subset \mathcal{U}_0(W_{nk}, M)^K$ trivially. On the other hand, if $\widetilde{u}$ is harmonic on W_{kn} and constant on each fiber, then the corresponding function u on W_n is harmonic at all vertices except possibly the central vertex. But since $\operatorname{im} \Delta \subset \ker \epsilon$, it must be harmonic at the central vertex as well. Our remarks about the torsion primes of Υ thus hold for the wheel graphs.

Let q be any prime. We know from harmonic continuation that $\mathcal{U}_0(W_n, \mathbb{Z}/q)$ is generated by at most two elements. Conversely, any element of $(\mathbb{Q}/\mathbb{Z})^2$ must

be a fixed point of the iteration process in §8.2 for some n, and thus, there will eventually be q-torsion with two generators (and given the formula in §8.2, we know that for $q \neq 5$, the q-torsion will already have two generators when it first appears). At some point, we will have q-torsion in W_{pn} but not W_n, for some prime p; and applying Observation 29 with $M = \mathbb{Z}/q$ shows that $q \equiv \pm 1 \bmod p$. After that, the only way for W_{pn} to have more q-torsion than W_n is if $p = q$, since otherwise, $\mathcal{U}_0(W_{pn}, \mathbb{Z}/q^k) = \mathcal{U}_0(W_n, \mathbb{Z}/q^k) \oplus \ker A$, and $\ker A$ must be trivial since the q-torsion component from W_n already has the maximum possible number of generators.

The chain-link fence example manifests similar behavior. There are obvious covering maps $\textsc{clf}(km, n) \to \textsc{clf}(m, n)$ which induce inclusions $\mathcal{U}_0(\textsc{clf}(m, n), \mathbb{Q}/\mathbb{Z}) \to \mathcal{U}_0(\textsc{clf}(km, n), \mathbb{Q}/\mathbb{Z})$. If m is divisible by large enough powers of 2, this is an isomorphism. It is also an isomorphism when k is odd — the bigger graph can only have more 2-torsion when k is divisible by 2.

11. Open Problems

Much like the sandpile group, the module Υ blends ideas from network theory, combinatorics, algebraic topology, homological algebra, and complex analysis. We have correlated the algebraic properties of Υ with the combinatorial properties of ∂-graphs, including harmonic morphisms, layer-stripping, wedge-sums, duality, and symmetry. We applied this general theory to study a non-trivial family of ∂-graphs related to those in [26]. The following questions remain:

Question 1. *Do our algebraic invariants extend to higher dimensions, along the lines of* [16]*? What is the analogue of the electrical inverse problem in this case? What is the analogue of the layer-stripping technique?*

Question 2. *Determine the group of $\mathbb{Q}/\mathbb{Z}$-harmonic functions supported in a given subset of the $\mathbb{Z}^2$ lattice. Our analysis of* CLF *graphs resolves the case of a diagonal strip with sides parallel to the lines $y = \pm x$. A straightforward argument using harmonic continuation shows that these are the only strips with an interesting answer.*

Question 3. *Can the techniques developed herein be used to aid the computation of previously intractable sandpile groups?*

Question 4. *We have studied algebraic invariants which test complete reducibility of graphs. This is a first step to testing solvability of the inverse problem. Are there other interesting invariants of ∂-graphs which test whether or not the electrical inverse problem can be solved by layer stripping?*

Question 5. *Does Theorem 1 extend to infinite graphs? If Υ is flat for all unit edge weights, must it be free?*

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