Degeneracy in the Discrete Inverse Problem

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Dedicated to the memory of Ernie Esser: discoverer of the n-to-1 phenomenon.

Contents

1 Introduction

The discrete inverse problems asks to recover the interior of a graph from its boundary. More precisely, one has a response map $L_G: (\mathbb{R}^+)^G \to (\mathbb{R}^+)^{\partial G}$ (Section 3). When L_G is injective, the graph G is called **recoverable**.

Most graphs are not recoverable; instead, they exhibit degeneracy. The dimensions of the fibers of L_G provide a coarse invariant of degeneracy. In this paper, we study a finer invariant applicable when L_G has relative dimension 0. This is the **arity** of a graph, defined by

$$\kappa(G) = \sup\left\{|L_G^{-1}(\partial\Gamma)| : \partial\Gamma \in (\mathbb{R}^+)^{\partial G}\right\}.$$

We compute $\kappa(G)$ for the family of quad-graphs (Theorem 5.4). This produces a novel proof that there exist graphs of every finite arity. In addition, it yields a family of exotic recoverable graphs. Lastly, we conjecture a tight arity inequality which would imply our construction is extremal.

2 Quad-Graphs

The graphs under consideration are built in two stages: first, strips are built out of modified quadrilaterals. Then the strips are glued together along their endpoints. We call the resulting family of graphs **quad-graphs**.

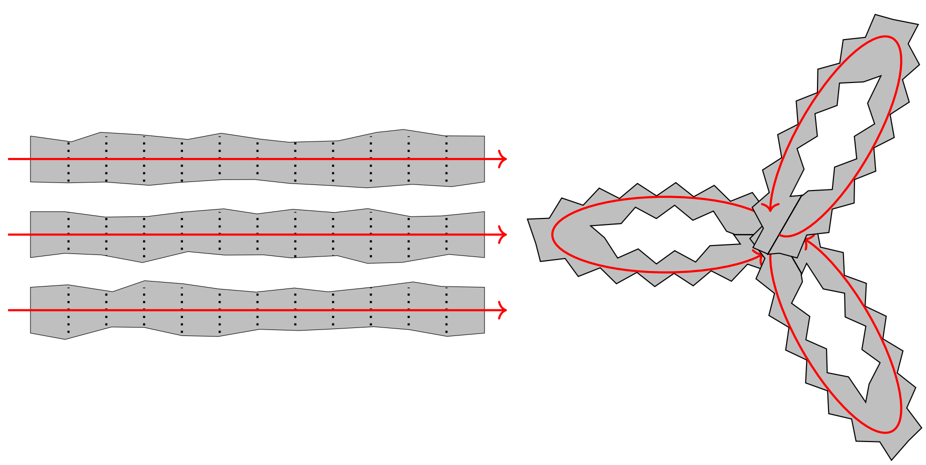


Figure 1: Gluing strips along their endpoints to obtain a quad-graph (strip lengths may differ)

2.1 Strips

For the remainder of this paper, we exclusively study graphs with boundary; the definition follows.

Definition 2.1 (Decorated Graphs)**.**

(a) A **graph with boundary** *is a tuple* $(\operatorname{Int} V, \partial V, E)$ *such that* $(\operatorname{Int} V \sqcup \partial V, E)$ *is loopless, simple, and undirected. Elements of* ∂V *(resp.* $\operatorname{Int} V$*) are the* **boundary** *(resp.* **interior***) vertices.*

(b) *Let i and j denote boundary vertices of G. A* **path through the interior of** G *consists of a path from i to j such that all internal nodes belong to* $\operatorname{Int} V$*.*

(c) *Each graph with boundary G has the* **boundary induced graph** $\partial G = (\varnothing, \partial V, \partial E)$ *where*

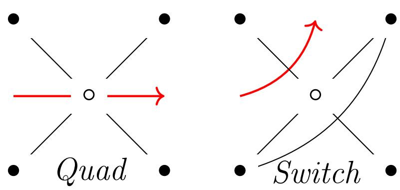
$$\partial E = \{(i,j) : i \neq j \text{ and there is a path through the interior from } i \text{ to } j\}.$$

(d) *An edge $(i,j) \in \partial E$ is* **recoverable** *if there is a* unique *path through the interior from i to j.*

(e) *A* **graph with flow** *is a tuple* $(G, \text{source}, \text{sink})$ *where G is a graph with boundary and* source *(resp.* sink*) are edges of ∂G.*

We use the symbol '$\bullet$' to denote boundary vertices, and '$\circ$' to denote interior vertices. A graph with flow is depicted by drawing a red arrow from *source* to *sink*.

Definition 2.2. *The* **quad** *and* **switch** *are the following graphs with flow:*



We refer to them collectively as **subunits***.*

Definition 2.3 (Chaining)**.** *Let $\{(G_i, source_i, sink_i)\}_{i=1}^n$ be a sequence of graphs with flow. We identify the vertices of successive sources and sinks to obtain the graph*

$$G = \bigsqcup_i G_i \Big/ \{sink_i \sim source_{i+1}\}$$

The **chaining** *of the sequence $\{(G_i, source_i, sink_i)\}_{i=1}^n$ is the graph with flow $(G, source_1, sink_n)$.*

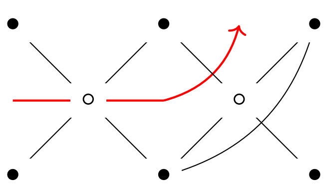


Figure 2: Chaining a quad to a switch

Given a chain of subunits, the **initial** (resp. **terminal**) **subunit** contains the source (resp. sink) edge of the chain. An **internal subunit** is any subunit that is neither initial nor terminal.

Definition 2.4. *A* **strip** *is a graph with flow obtained by*

(a) *chaining 3 or more subunits,*

(b) *such that all internal subunits are* quads*.*

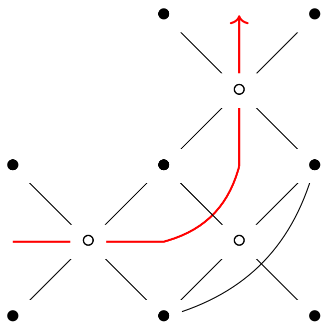


Figure 3: This is not a strip: condition (a) holds but not condition (b)

2.2 Constructing the Quad-Graph

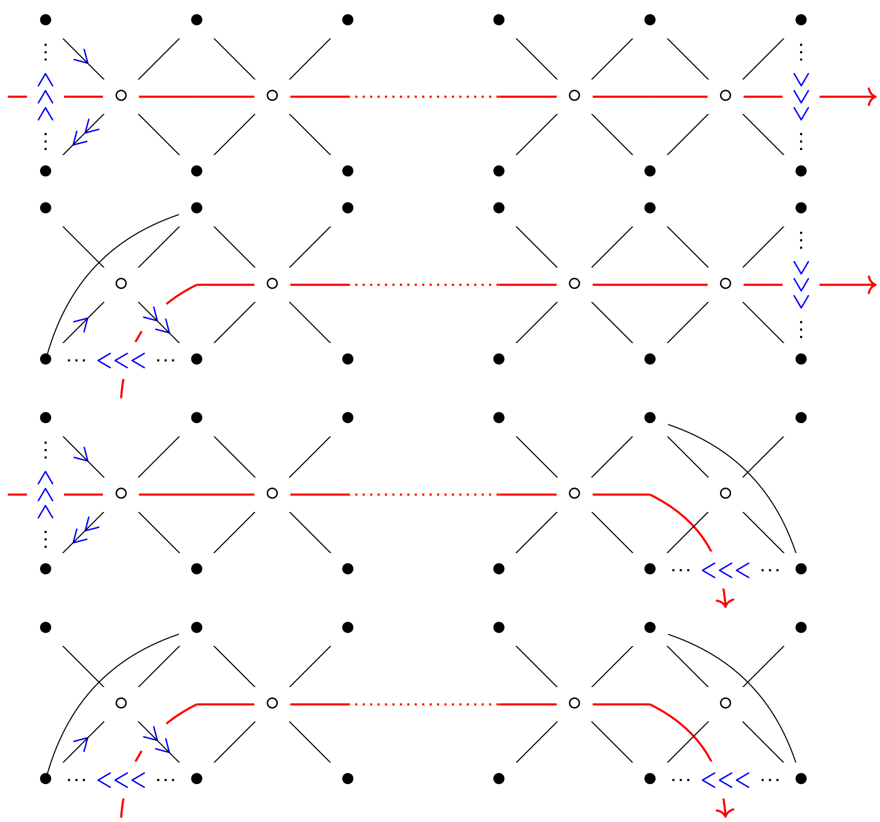
Definition 2.5. *A* **quad-graph** *built out of a collection of strips* $\{S_i\}_{i=1}^n$ *is the graph obtained by:*

(a) Gluing the strips S_i *as prescribed below.*

(b) Adding auxiliary edges as prescribed below.

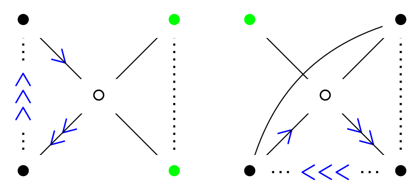
In the following diagrams, dotted edges belong to ∂S_i and not to the quad-graph.

Gluing the strips. Perform the following edges identifications to every strip:



□

Adding auxiliary edges. First color the vertices of the initial subunits as follows:



Next, introduce auxiliary edges connecting each pair of vertices that

(a) belong to the initial subunit of some strip,

(b) have the same color,

(c) are not already connected by a solid or dotted edge.

$\square$

Definition 2.6. *The* **central edge** *of the quad-graph is the edge labelled* >>> *above.*

Remark 2.7. *Each strip is a graph with flow, but the quad-graph itself is not.*

2.3 Labelling a Quad-Graph

Let G be a quad-graph. We label G using the matrix $W(G) \in M_2(\mathbb{N})$ with entries

$$W(G)_{jk} = \#\text{ of strips with initial subunit } j \text{ and terminal subunit } k,$$

where 'subunit 1' encodes 'quad' and 'subunit 2' encodes 'switch'.

3 Fibers of the Response Map

An **electrical network on** G consists of an assignment of positive conductances to the edges of G. Henceforth we will assume that G contains no pair of adjacent interior nodes (see the **Note** below). The set of networks on G is denoted by $(\mathbb{R}^+)^G$. Given a network $\Gamma \in (\mathbb{R}^+)^G$, let $\partial(\Gamma) \in (\mathbb{R}^+)^{\partial G}$ denote the network such that the conductance on the edge (v, w) is the current which exits Γ at w when vertex v is held at unit voltage relative to the rest of the boundary (this definition is intentionally left informal).

Note: Using the notation of [2], a formal definition of $\partial(\Gamma)$ is the following: Consider the Schur complement of the Kirchhoff matrix of the network Γ. This matrix is itself the Schur complement of an electrical network, namely $\partial(\Gamma)$.

Provided the graph G contains no pair of adjacent interior nodes, the boundary graph ∂G in Definition 2.1(c) is the underlying graph of $\partial(\Gamma)$. (If G contains adjacent interior nodes, the graph-theoretic description of $\partial(\Gamma)$ is more complicated.) For this reason, in the remainder of the paper we work exclusively with graphs lacking adjacent interior nodes.

Definition 3.1. *The* **response map** $L_G \colon (\mathbb{R}^+)^G \to (\mathbb{R}^+)^{\partial G}$ *is the map* $\Gamma \mapsto \partial(\Gamma)$.

Definition 3.2. *The* **arity** *of G is the quantity*

$$\kappa(G) = \sup\left\{|L_G^{-1}(\partial\Gamma)| : \partial\Gamma \in (\mathbb{R}^+)^{\partial G}\right\}.$$

The graph G is said to be **recoverable** if $\kappa(G) = 1$. Observe that when G lacks interior vertices, the response map L_G is the identity and G is recoverable. The simplest non-trivial response map occurs when G is a star. This graph is recoverable and L_G is in fact a diffeomorphism ([1], p.129).

3.1 The Star-K Transform

The graph $K_{1,n}$ with a single central interior vertex is called an n-**star**. The response map of a star is known as the **Star-K transform** [3].

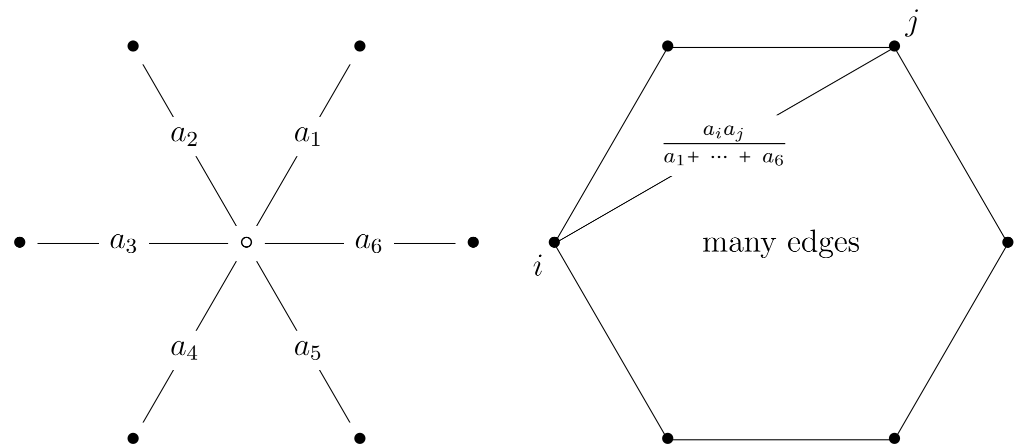


Figure 4: The Star-K transform from a 6-star to a K_6.

Consider a star network with conductance a_i between the interior node and boundary vertex i. We will always take $\Sigma = \sum_i a_i$ to be the sum of the conductances incident to an interior vertex. When vertex i is held at unit potential relative to the rest of the boundary, the potential of the interior node is $\frac{a_i}{\Sigma}$. Thus a current of $\frac{a_i a_j}{\Sigma}$ exits the network at j. This is the conductance of the edge (i, j) in the boundary network K_n, and provides a formula for the Star-K transform.

Claim 3.3 (Quadrilateral rule)**.** *Consider the Star-K transform from a 4-star to a K_4. Then the product of any pair of opposite edges of K_4 is independent of the choice of pair.*

Proof. Said product is $\frac{a_1 a_2 a_3 a_4}{\Sigma^2}$. □

The following definitions describe the method of network population used in Subsection 3.4.

Definition 3.4. *The* **source** *(resp.* **sink***)* **conductance** *of a network Γ on a graph with flow G is the conductance of the source (resp. sink) edge of $\partial\Gamma$ on ∂G.*

Definition 3.5. *Given a subunit, a coefficient c, and a source conductance x, the* **standard method of population** *comprises the assignment of conductances depicted in Figure 5.*

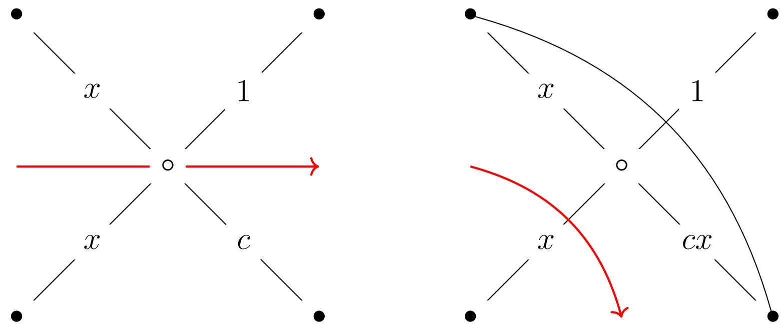


Figure 5: Populating a subunit. Edge weights will be scaled by $\frac{\Sigma}{x}$.

After assigning conductances as depicted in Figure 5, all conductances are scaled by a factor of $\frac{\Sigma}{x}$ where Σ is the sum of all conductances incident to the interior vertex of the subunit. Therefore $\Sigma = 2x + c + 1$ for a quad and $\Sigma = (2 + c)x + 1$ for a switch.

Lemma 3.6. *The standard method of population yields a source conductance of x. The sink conductance is $\frac{c}{x}$ for a quad and cx for a switch.*

Proof. Before applying the scaling of $\frac{\Sigma}{x}$ to the edges in Figure 5, the source conductance is $\frac{x^2}{\Sigma}$, the sink conductance of a quad is $\frac{c}{\Sigma}$, and the sink conductance of a switch is $\frac{cx^2}{\Sigma}$. Observe that L_G is homogeneous of degree 1. Hence scaling everything by $\frac{\Sigma}{x}$ yields the result. □

We will apply the standard method of population to subunits of a quad-graph G. The assigned conductances are then entries of a network $\Gamma \in (\mathbb{R}^+)^G$. It must be checked that our assignment is consistent with the edge identifications performed in Subsection 2.2. Indeed, the only edges of G which are identified are $>$ and $>>$ (see Subsection 2.2). These edges are each assigned a conductance of $x \cdot \frac{\Sigma}{x}$ in Γ, regardless of the initial subunit used in a strip. Note that after identifications, the initial subunits form a $(2n+2)$-star and Σ contains $2n+2$ summands (instead of just 4, as a cursory glance at Figure 5 might suggest), provided there are n strips present in G.

Recall Definition 2.1 (d) regarding recoverable edges. In the construction of the quad-graph, the only non-recoverable edges of ∂G arise due to

(a) chaining of subunits (where source and sink edges are identified),

(b) auxiliary edges introduced in Subsection 2.2, and

(c) the boundary-boundary edge of any switch.

Lemma 3.7. *Consider a network Γ on the quad-graph G. Each recoverable edge of ∂G is assigned a conductance involving only the coefficient c in the standard population scheme.*

The effect of this Lemma is that information about the source conductance x of any subunit is not 'leaked out' to the boundary network. We have tailored both our standard method of population and our choice of auxiliary edges to ensure this Lemma holds.

Proof. First we consider non-initial subunits:

- For a quad, there are four recoverable edges with conductances $c, 1, c, 1$ respectively.
- For a switch, there are three recoverable edges with conductances $1, 1, c$ respectively.

Next consider the initial subunits, which form a $(2n+2)$-star after identifications. Our choice of scaling in Definition 3.5 implies that the boundary edges are assigned conductances $\frac{a_i a_j}{x}$. As the a_i for the black vertices in Subsection 2.2 are degree 1 monomials in x and green vertices a_j are degree 0 monomials, it follows that all black-green edges are independent of x. As every monochromatic edge is masked (by either an auxiliary edge or subunit chaining), the result follows. □

3.2 Strip Functions and Strip Words

Let S denote any graph with flow, equipped with network $\Gamma \in (\mathbb{R}^+)^S$. We are primarily concerned with the following scenario:

(a) The graph S is the induced subgraph of a strip of G.

(b) It is equipped with a network $\Gamma|_S$ induced from Γ on G.

(c) No edge identifications from the construction of G have been applied to S (Subsection 2.2).

Note that S is a graph with flow by Definition 2.4.

Applying the response map L_S produces the boundary network $\Sigma = L_S(\Gamma) \in (\mathbb{R}^+)^{\partial S}$. Let s (resp. t) in ∂S denote the source (resp. sink) edge. Denote by $x(\Sigma)$ the conductance of Σ at s.

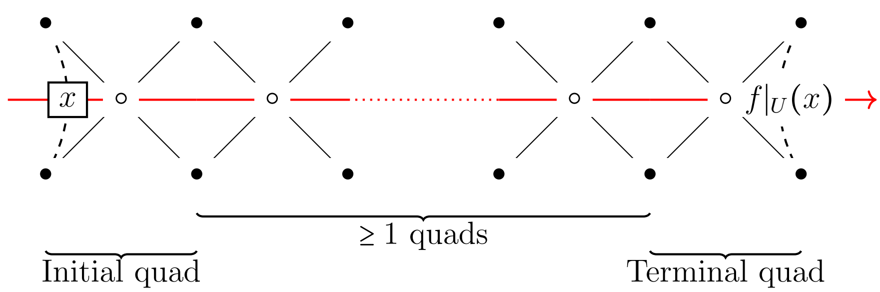


Figure 6: Strip S with function class $w(S) = [(ba)^{n-1}b]$ (see Definition 3.11)

By Definition 3.4, the conductance $x(\Sigma)$ is the source conductance of Γ. Consider the projection map $\pi: (\mathbb{R}^+)^{\partial S} \to (\mathbb{R}^+)^{\partial S \setminus \{s,t\}}$ which 'forgets' the value of the boundary network at its source and sink edges. Each fiber of π is parametrized by the value of the source conductance.

Definition 3.8. *Given $\Sigma^\circ \in (\mathbb{R}^+)^{\partial S \setminus \{s,t\}}$, the* **partial function** $f|_U: U \to \mathbb{R}^+$ *consists of*

(a) the set $U = \{x(\Sigma): \Sigma \in (\mathbb{R}^+)^{\partial S}, \pi(\Sigma) = \Sigma^\circ\}$, and

(b) the function $f|_U(x)$ is the sink conductance of Γ when the source conductance is x.

In other words, the function $f|_U$ records the flow from source to sink of S. The set U is nontrivial because certain values of x may propagate to negative values during the chaining of S, hence corresponding to a boundary network $\Sigma \notin (\mathbb{R}^+)^{\partial S}$. We emphasize this interpretation by referring to U as the **domain of positivity** of $f|_U$.

Definition 3.9 (Composition of partial functions)**.** *Given partial functions $f|_U$ and $g|_V$, we set*

$$(f|_U) \circ (g|_V) = (f \circ g)|_{V \cap g^{-1}(U)}.$$

Observe that composition of partial functions is associative.

Claim 3.10. *Suppose that S and T are graphs with flow with partial functions $f|_U$ and $g|_V$. Then the chaining of T followed by S has partial function $f|_U \circ (c-x)|_{(0,c)} \circ g|_V$ for some $c \in \mathbb{R}^+$.*

Proof. We take c to be the conductance of the identified source and sink edges in the chained boundary network. It is clear that the function values are correct. To verify the domain of positivity, it suffices to observe that a source conductance x of the chained network induces a positive chained boundary network if and only if it induces a positive boundary network on T such that $g(x) < c$ and $c - g(x)$ induces a positive boundary network on S. □

Building on the ideas of Subsection 3.1, the quad and switch subunits have partial functions

$$\left.\frac{c}{x}\right|_{\mathbb{R}^+} \text{ and } cx|_{\mathbb{R}^+} \text{ respectively.}$$

Here c is a coefficient determined by the values of $(\mathbb{R}^+)^{\partial S \setminus \{s,t\}}$. For quads, it is the product of the diagonals. For switches it is the ratio of the unmasked sides. Inducting on Claim 3.10, it follows that every partial function $f|_U$ obtained from a strip S is a composition of the partial functions

$$(c-x)|_{(0,c)}, \qquad \left.\frac{c}{x}\right|_{\mathbb{R}^+}, \qquad cx|_{\mathbb{R}^+}.$$

Definition 3.11 (Function classes)**.** *Let $w, v \in \{a,b\}^*$ be words with characters a and b. We construct the* **function class** *$[w]$ as follows:*

(a) Set $[wv] = \{f|_U \circ g|_V : g|_V \in [w], f|_U \in [v]$ and $V \cap g^{-1}(U) \neq \varnothing\}$.

(b) Set $[a] = \{(c-x)|_{(0,c)} : c \in \mathbb{R}^+\}$.

(c) Set $[b] = \{\left.\frac{c}{x}\right|_{\mathbb{R}^+} : c \in \mathbb{R}^+\}$.

(d) Set $[\varnothing] = \{\mathrm{id}|_{\mathbb{R}^+}\}$.

This is well-defined due to associativity of partial function composition. An immediate consequence of our definition is that $[b^2] = \{cx|_{\mathbb{R}^+} : c \in \mathbb{R}^+\}$.

Given a strip S, form a word $w(S)$ by writing b for each quad, b^2 for each switch, and a for each chaining operation (traversed from source to sink).

Proposition 3.12. *A strip word is a word of the form $w(S)$. The set of strip words is:*

$$\begin{matrix} (ba)^n b & b(ba)^n b \\ (ba)^n b^2 & b(ba)^n b^2 \end{matrix}, \quad n \geq 2.$$

Proof. This is a restatement of Definition 2.4. □

Proposition 3.13. *Let S be a strip. The set of partial functions $f|_U$ with $U \neq \varnothing$ induced by some $\Sigma^\circ \in (\mathbb{R}^+)^{\partial S \setminus \{s,t\}}$ is $[w(S)]$.*

Proof. Observe that quads and switches attain all partial functions in their respective function classes. Moreover, the coefficients associated to distinct subunits are independent of one another. Hence the result follows by Claim 3.10. □

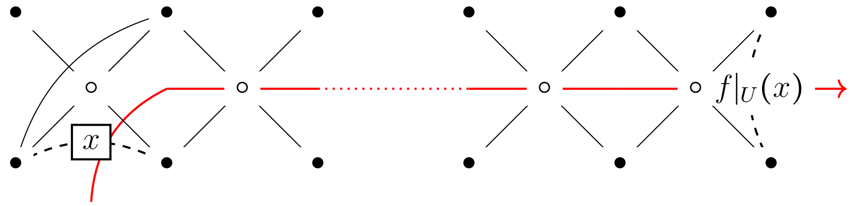


Figure 7: Strip S with $w(S) = [b^2 a \cdots ab]$

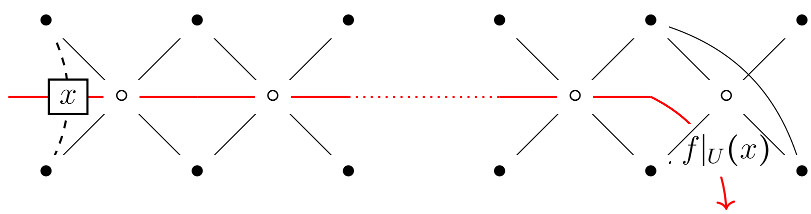


Figure 8: Strip S with $w(S) = [ba \cdots ab^2]$

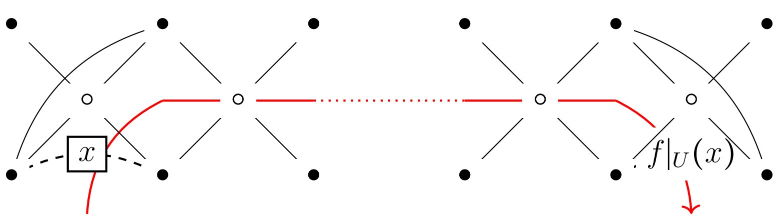


Figure 9: Strip S with $w(s) = [b^2 a \cdots ab^2]$

3.3 Characterization of Function Classes

This is a technical section which may be skipped on first reading (use Theorem 3.16 as a black box).

Consider a Möbius function $f\colon \overline{\mathbb{R}} \to \overline{\mathbb{R}}$.

Definition 3.14. *The function f is* **factorizable** *if $f(0), f(\infty), f^{-1}(0), f^{-1}(\infty) \in \mathbb{R}^+$.*

Claim 3.15 (Three points suffice)**.** *Suppose that f is a Möbius function for which*

$$|\{f(0), f(\infty), f^{-1}(0), f^{-1}(\infty)\} \cap \mathbb{R}^+| \geq 3.$$

Then f is factorizable.

Proof. This follows by the Intermediate Value Theorem (draw a picture on the torus $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$). $\square$

Theorem 3.16. *Let w_i be a strip word. The partial function $f_i|_{U_i}$ belongs to $[w_i]$ if and only if f_i is factorizable and*

$$U_i = \begin{cases} (0, f_i^{-1}(\infty)) \text{ if } w_i = (ba)^n b, & (f_i^{-1}(0), \infty) \text{ if } w_i = b(ba)^n b, \\ (0, f_i^{-1}(0)) \text{ if } w_i = (ba)^n b^2, & (f_i^{-1}(\infty), \infty) \text{ if } w_i = b(ba)^n b^2. \end{cases} \tag{1}$$

Proof. The effect of pre- (resp. post-) composing a function class by an element of b is to invert the range (resp. domain) of f. This transforms the domain U_i in a manner consistent with (1). Algebraically, this amounts to the identity

$$\left.\frac{c}{x}\right|_{\mathbb{R}^+} \circ \left.\frac{c}{x}\right|_{\mathbb{R}^+} = \mathrm{id}_{\mathbb{R}^+},$$

which allows us to 'cancel out' any elements of b which appear at initial or terminal locations of the word. Hence it suffices to verify the theorem in the case $w_i = (ba)^n b$, as the remaining cases are obtained by pre- and post- composition with b.

We prove the result for $w_i = (ba)^n b$ by induction on n, with base case $n = 2$ in which case $w_i = babab$. In light of the previous paragraph, it suffices to characterize the function class $[aba]$. We claim that

$$[aba] = \{\text{partial Möbius functions } f_U\colon f \text{ is factorizable, } U = (f^{-1}(\infty), \infty),\ U \neq \varnothing\}.$$

For the forward problem, suppose that $f|_U \in [aba]$. Then

$$f|_U = (c_1 - x)|_{(0,c_1)} \circ \left.\frac{c_2}{x}\right|_{\mathbb{R}^+} \circ (c_3 - x)|_{(0,c_3)} = c_1 - \left.\frac{c_2}{c_3 - x}\right|_{(c_3,\infty)}.$$

Factorizability is verified by substituting values into $f(x)$, and the interval U is seen to be $(f^{-1}(\infty), \infty)$ by direct computation.

For the inverse problem, consider an arbitrary partial Möbius function $f|_U$ with $U = (f^{-1}(\infty), \infty)$ and f factorizable. We have the identity

$$f|_U = (f(\infty) - x)|_{(0,f(\infty))} \circ \left.\frac{f^{-1}(\infty) \cdot [f(\infty) - f(0)]}{x}\right|_{\mathbb{R}^+} \circ (f^{-1}(\infty) - x)|_{(0,f^{-1}(\infty))}.$$

Indeed, the functions agree at the three distinct points $0, \infty, f^{-1}(\infty)$ (they are distinct because $f(\infty) \neq \infty$). Thus the Möbius functions agree everywhere.

To verify the equality of the domain on both sides, begin by observing that that the domains have the same endpoints. Since the function on each side is orientation-reversing, it follows that the same circular arc of $\overline{\mathbb{R}}$ appears on each sides. Thus the domains agree and our classification of $[aba]$ is complete. Consequently by pre- and post- composition with b, the classification of $[(ba)^2 b]$ follows.

For the inductive step, we start with the characterization of $[(ba)^n b]$ and obtain a characterization of $[a(ba)^{n+1}]$. Fix $c_1, c_2 \in \mathbb{R}^+$ and for any Möbius function f, let $\widetilde{f} = c_1 - f(c_2 - x)$. Then f is decreasing if and only if $\widetilde{f}$ is decreasing. Moreover, the partial function $f|_U$ has domain $U = (0, f^{-1}(\infty))$ if and only if the corresponding partial function $\widetilde{f}|_{\widetilde{U}}$ has domain $\widetilde{U} = (f^{-1}(\infty), \infty)$. Provided that f is decreasing, we see that f is factorizable if and only if $\widetilde{f}$ is factorizable. Lastly U is empty if and only if $\widetilde{U}$ is empty. These are all the ingredients we need.

Applying the inductive hypothesis, it follows that $[a(ba)^{n+1}]$ is characterized by $U = (f^{-1}(\infty), \infty)$. Pre- and post- composition with elements of b yields that $U = (0, f^{-1}(\infty))$ for the function class $[(ba)^{n+1}b]$. This completes the induction. □

3.4 The Concatenation Map

For any boundary network $\partial\Gamma \in (\mathbb{R}^+)^{\partial G}$, the **fiber lying over** $\partial\Gamma$ is the set $L_G^{-1}(\partial\Gamma)$. It consists of all the networks $\widetilde{\Gamma} \in (\mathbb{R}^+)^G$ which satisfy $\partial(\widetilde{\Gamma}) = \partial\Gamma$. Consequently, the fibers of L_G are parametrized by specifying a network $\partial\Gamma \in (\mathbb{R}^+)^{\partial G}$.

When G is a quad-graph, all the pertinent information needed to describe a fiber is contained in the *behavior of the strips* and *how the strips are glued together*. More precisely, we will focus on the partial functions associated to each of the strips of G as well as the conductance of the central edge of G, denoted by c in Definition 3.17.

Formally, we set $B = \mathbb{R}^+ \times \prod_{i=1}^n [w_i]$, where w_i are the words associated to the strips of G. Any element $F \in B$ contains sufficient information to study an associated fiber. We formalize this statement using the following definition.

Definition 3.17. *The* **concatenation map** $\Xi\colon (\mathbb{R}^+)^{\partial G} \to B$ *is defined by setting*

$$\Xi(\partial\Gamma) = (c, f_1|_{U_1}, \ldots, f_n|_{U_n})$$

where $f_i|_{U_i}$ is the partial function associated to strip i and c is the conductance of the boundary network $\partial\Gamma$ across the central edge of G (Definition 2.6).

For notational convenience, we also set $f_0|_{U_0} = \mathrm{id}|_{\mathbb{R}^+}$. Since each U_i is non-empty open and each partial function $f_i|_{U_i}$ is rational, we may analytically extend $f_i|_{U_i}$ to a rational function $f_i\colon \overline{\mathbb{R}} \to \overline{\mathbb{R}}$.

Note: The fibers of L_G are parametrized by boundary networks $\partial\Gamma$. Applying the concatenation map produces the element $F = \Xi(\partial\Gamma) \in B$. The tuple representing F contains sufficient information to study the fiber $L_G^{-1}(\partial\Gamma)$. For convenience, **we will henceforth refer to F simply as a fiber**. All future usage of the word 'fiber' will refer to elements of B.

Definition 3.18. *The following quantities are associated to any fiber $F \in B$:*

(a) The **function** $g(x) \coloneqq \sum_{i=0}^n f_i(x) - c$ (mapping $\overline{\mathbb{R}} \to \overline{\mathbb{R}}$).

(b) The **root set** $R \coloneqq g^{-1}(0)$, the **domain** $U \coloneqq \cap_{i=1}^n U_i$, and the **cardinality** $|F| \coloneqq |R \cap U|$.

(c) The **poles** $e_i := f_i^{-1}(\infty)$ and **signs** $\sigma_i := \operatorname{sgn} \operatorname{Res} f_i(x)|_{x=e_i}$, for $0 \le i \le n$.

Since $f_i(U_i) \subset \mathbb{R}^+$, one has $e_i \notin U_i$.

Claim 3.19. *We have the inequality* $|L_G^{-1}(\partial\Gamma)| \le |F|$*, where* $F = \Xi(\partial\Gamma)$*.*

Proof. Consider any network $\Gamma \in L_G^{-1}(\partial\Gamma)$. Let x denote the source conductance of Γ. By construction of the concatenation map, it follows that $x \in R \cap U$. To show that the mapping $\Gamma \mapsto x$ is injective, it suffices to show that Γ can be recovered given $\partial\Gamma$ and x.

Restrict attention to the identified initial subunit of G. Clearly the unmasked edges may be recovered. All that remain are the monochromatic edges (see Subsection 2.2 as well as Lemma 3.7). The central edge of conductance x has been vacuously recovered. The only remaining black monochromatic edges are the sink edges of initial switches. To recover these edges, apply the quadrilateral rule (Claim 3.3) to the four non-diagonal edges of the switch. To recover a green monochromatic edge, consider the quadrilateral containing said edge along with the central edge. All edges besides the green edge are known, allowing recoverability of the green edge. Consequently all edges of the initial subunit have been recovered.

Next, restrict attention to each of the strips in turn. Since we have recovered the initial subunit, all the internal and terminal quads may be recovered sequentially. Finally, a terminal switch may be recovered by observing that the conductance of the masked diagonal edge may be recovered from the unmasked diagonal. Hence all edges of G have been recovered.

□

Claim 3.20. *For any* $F \in B$*, there is a* $\partial\Gamma \in \Xi^{-1}(F)$ *for which* $|L_G^{-1}(\partial\Gamma)| \ge |F|$*.*

Proof. Fix any $x \in R \cap U$. We apply the standard method of population (Definition 3.5) to produce a network $\Gamma \in (\mathbb{R}^+)^G$. Assign x to the source conductance of each of the strips of G. For each of the functions $f_i|_{U_i}$ associated to the fiber F, use the fact that $f_i|_{U_i} \in [w_i]$ to obtain a factorization with coefficients $c_{i,j}$. Populate subunit j of strip i using the coefficient $c_{i,j}$. Since $x \in R \cap U$, we observe that $\Xi(\partial\Gamma) = F$. By Lemma 3.7, the network $\partial\Gamma$ is independent of x. Since the map $x \mapsto \Gamma$ is clearly injective, it follows that $|F| \le |L_G^{-1}(\partial\Gamma)|$.

□

Theorem 3.21. *We have the equality* $\kappa(G) = \sup\{|F| : F \in B\}$*.*

Proof. By Claim 3.19, we have $\kappa(G) \le \sup_F |F|$. Claim 3.20 furnishes the reverse inequality. □

4 Deformation of Fibers

In this section, we show how to remove fiber pathologies by performing *deformations*. The following claim rests on the fact that we are using strip words, as opposed to general words.

Claim 4.1. *Let* F *be a fiber with* $U \ne \varnothing$*. Then* g *is non-constant.*

Proof. Since $U \ne \varnothing$, each $U_i \ne \varnothing$. Thus by Theorem 3.16, the poles $e_i = f_i^{-1}(\infty)$ are finite for $i \ne 0$. Hence only the term f_0 in the summation yielding g has a pole at ∞. Therefore no cancellation occurs, whence g has a pole at ∞. □

4.1 Deformation I: Distinct Poles

Theorem 4.2. *For each F, there is a fiber G with distinct poles such that $|F| \le |G|$.*

Proof. We start by showing how to perturb the fiber F to produce a new fiber F_δ in which the pole of f_1 at e_1 shifts to $e_1 + \delta$.

Recall that $f_1(x)$ is a Möbius function. Therefore the auxiliary function $h: \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ given by

$$h(x) = x - f_1^{-1}\left(c - \sum_{i \neq 1} f_i(x)\right)$$

is well-defined. Moreover

$$h(x) = 0 \iff c - \sum_{i\neq 1} f_i(x) = f_1(x) \iff g(x) = 0.$$

Thus for any root $r_j \in R \cap U$, one has $h(r_j) = 0$.

Claim 4.3. *There are neighborhoods $r_j \in N_j \subset \overline{N_j} \subset U$ and $0 \in I_j$ such that*

$$|h_{N_j}^{-1}(\delta)| + |h_{N_j}^{-1}(-\delta)| = 2, \qquad \textit{for all } \delta \in I_j.$$

Proof. If $R \cap U = \varnothing$, the claim holds vacuously. Henceforth suppose $R \cap U \neq \varnothing$. In particular, h vanishes at $r_j \in R \cap U$. We deduce that h is non-constant as follows: If h were constant, then $h \equiv 0$ so $g \equiv 0$. By Claim 4.1, it follows that $U = \varnothing$, contradicting our assumption that $R \cap U \neq \varnothing$.

Consequently h is non-constant, differentiable, and finite near r_j. Dependent on whether $h'(r_j)$ vanishes, h attains either a local extremum at r_j or else is injective near r_j. Thus for some open $N_j \ni r_j$ and $I_j \ni 0$, we have

$$|h_{N_j}^{-1}(\delta)| + |h_{N_j}^{-1}(-\delta)| = \begin{cases} 1+1, & \text{(injective or } \delta = 0) \\ 2+0, & \text{(extremum, } \delta \neq 0) \end{cases} \qquad \text{for all } \delta \in I_j.$$

$\square$

Shrink the neighborhoods N_j to ensure they are pairwise disjoint for $1 \le j \le |R \cap U| = |F|$. Set $N = \bigcup_j N_j$ and $I = \bigcap_j I_j$. Note that $\overline{N} = \bigcup_j \overline{N_j} \subset U$.

Define the perturbation $f_{1,\delta}(x) = f_1(x-\delta)$ for $\delta \in I$ (δ may be negative). Since $f_{1,\delta}^{-1}(x) = f_1^{-1}(x) + \delta$ and f_1 is factorizable, so is $f_{1,\delta}$ (after shrinking I). Define

$$U_{1,\delta} = \begin{cases} (0, f_{1,\delta}^{-1}(\infty)) & (f_{1,\delta}^{-1}(0), \infty) \\ (0, f_{1,\delta}^{-1}(0)) & (f_{1,\delta}^{-1}(\infty), \infty) \end{cases} \qquad \text{chosen according to } U_1 \text{ in (1, Subsection 3.3)}$$

Since the endpoints of $U_{1,\delta}$ vary linearly in δ, it follows that for small δ we have $U_{1,\delta} \subset \mathbb{R}^+$. Thus $f_{1,\delta}|_{U_{1,\delta}} \in [w_1]$. Replacing $f_1|_{U_1}$ with $f_{1,\delta}|_{U_{1,\delta}}$ in the fiber F produces a new fiber $F_\delta \in B$. Let g_δ, U_δ, and R_δ denote the function, domain, and root set of the fiber F_δ. Since

$$U_\delta = U_{1,\delta} \cap \bigcap_{i=2}^{n} U_i,$$

it follows that the endpoints of U_δ also vary linearly in δ. Therefore the condition $\overline{N} \subset U$ yields that $N \subset U_\delta$ for $\delta \in I$ (after shrinking I). Thus $R_\delta \cap N \subset R_\delta \cap U_\delta$. In addition, we remark that $R_\delta = g_\delta^{-1}(0) = h^{-1}(\delta)$. Therefore

$$|F_\delta| = |R_\delta \cap U_\delta| \geq |R_\delta \cap N| = \sum_{j=1}^{|F|} |R_\delta \cap N_j| = \sum_{j=1}^{|F|} |h_{N_j}^{-1}(\delta)|. \tag{2}$$

Shrink I such that it is symmetric about 0. Then for all $\delta \in I$,

$$\begin{aligned} |F_\delta| + |F_{-\delta}| &\overset{(2)}{\geq} \sum_{j=1}^{|F|} |h_{N_j}^{-1}(\delta)| + |h_{N_j}^{-1}(-\delta)| \\ \text{(Claim 4.3)} \qquad &= 2|F|. \end{aligned}$$

In particular, it follows that $|F| \leq |F_\delta|$ for some $\delta \in I \setminus \{0\}$.

Iterating this procedure, we may perturb the poles of F one at a time. This yields a fiber G with distinct poles for which $|F| \leq |G|$. □

4.2 Deformation II: Distinguished Fibers

Definition 4.4. *Suppose all poles of the fiber F are distinct, and $U \neq \varnothing$.*

(a) We order the poles in increasing order:

$$\sup U \leq \underbrace{e_{-n_-} < e_{-n_-+1} < \cdots < e_{-1}}_{n_-} < \underbrace{e_0}_{\infty} < \underbrace{e_1 < \cdots < e_{n_+-1} < e_{n_+}}_{n_+} \leq \inf U.$$

Note that $n_- + n_+ = n$. We index modulo $n+1$.

(b) The **tuple of signs** $\sigma \in \{\pm 1\}^{n+1}$ is defined to be $\sigma = (\sigma_{-n_-}, \cdots, \underbrace{\sigma_0}_{1}, \cdots, \sigma_{n_+})$.

(c) A **trapped interval** is an open interval (e_i, e_{i+1}) with $\sigma_i = \sigma_{i+1}$. *Observe that each trapped interval contains a root of g, by the Intermediate Value Theorem.*

(d) The fiber is called **distinguished** if g has a unique root in each trapped interval, all remaining roots in U, and $n+1$ global roots. (It turns out that in this case g is an $(n+1)$-sheeted covering of $\overline{\mathbb{R}}$, however we will not need this fact.)

We write $\widetilde{F}$ for a distinguished fiber (along with $\widetilde{g}, \widetilde{R}$, and $\widetilde{U}$ for its function, root set, and domain).

Lemma 4.5. *If F has at least as many trapped intervals as $\widetilde{F}$, then $|F| \leq |\widetilde{F}|$.*

The reason why 'typical' fibers have *fewer* roots in U than distinguished fibers is that they have *more* roots outside of U. We prove this by partitioning U^c according to the poles $\{e_i\}$, and observing the behavior of g between the poles.

Proof. Suppose for contradiction that $|F| > |\widetilde{F}|$. Since each trapped interval contains a root of g, we have $|R \setminus U| \geq \#\{\text{trapped intervals of } F\}$. Applying the hypothesis yields

$$\begin{aligned} |R| &= |F| + |R \setminus U| \\ &> |\widetilde{F}| + \#\{\text{trapped intervals of } \widetilde{F}\} \\ &= n+1, \end{aligned}$$

since $\widetilde{F}$ is distinguished. Hence g has more roots than its degree, so it is constant. Applying Claim 4.1, we conclude $U = \varnothing$. Thus $|F| = 0$, contradicting our assumption that $|F| > |\widetilde{F}|$. ☐

Theorem 4.6. *For each F, there is a distinguished fiber $\widetilde{F}$ with $|F| \leq |\widetilde{F}|$.*

Proof. Suppose that $U \neq \varnothing$. By Theorem 4.2, we may suppose that all poles e_i of F are distinct. Construct a set $\widetilde{R}$ with $n+1$ elements as follows. Append an element of each trapped interval to $\widetilde{R}$. As there are at most n trapped intervals, we may select the remaining elements from U.

We are ready to perform the main construction. For $1 \leq i \leq n$, let

$$\widetilde{g}(x) = \frac{\prod_{r\in\widetilde{R}}(x-r)}{\prod_{i=1}^{n}(x-e_i)}, \qquad \widetilde{f}_i(x) = \frac{\operatorname{Res}\widetilde{g}(x)|_{x=e_i}}{x-e_i} + d_i \tag{3}$$

with d_i left unspecified for the moment.

Claim 4.7. *The function $\widetilde{f}_i$ is factorizable for d_i sufficiently large.*

Proof. Observe that $\widetilde{f}_i^{-1}(\infty) = f_i^{-1}(\infty) \in \mathbb{R}^+$. Choosing d_i sufficiently large ensures that $\widetilde{f}_i(0), \widetilde{f}_i(\infty) \in \mathbb{R}^+$. As factorizability need only be checked at 3 points, the claim follows. ☐

Claim 4.8 (Sign Coherence)**.** *We have the identity* $\operatorname{sgn}\operatorname{Res}\widetilde{g}(x)|_{x=e_i} = \sigma_i$.

Proof. Using the indexing of Definition 4.4, we consider indices $i \in [-n_-, n_+]$. We compute that

$$\begin{aligned} \operatorname{Res}\widetilde{g}(x)|_{x=\infty} &= \lim_{x\to\infty}\frac{\widetilde{g}(x)}{x} \\ &= \lim_{x\to\infty}\frac{\prod_{r\in\widetilde{R}}\left(1-\frac{r}{x}\right)}{\prod_{i\neq 0}\left(1-\frac{e_i}{x}\right)} \\ &= 1. \end{aligned}$$

Consequently the claim holds for $i=0$.

For convenience, let $\widetilde{\sigma}_i = \operatorname{sgn}\operatorname{Res}\widetilde{g}(x)|_{x=e_i}$. We consider the ratio

$$\begin{aligned} \frac{\operatorname{Res}\widetilde{g}(x)|_{x=e_i}}{\operatorname{Res}\widetilde{g}(x)|_{x=e_{i+1}}} &= \frac{\prod_{r\in\widetilde{R}}(e_i-r)}{\prod_{j\neq i,0}(e_i-e_j)} \Bigg/ \frac{\prod_{r\in\widetilde{R}}(e_{i+1}-r)}{\prod_{j\neq i+1,0}(e_{i+1}-e_j)} \\ \implies \frac{\widetilde{\sigma}_i}{\widetilde{\sigma}_{i+1}} &= -\prod_{r\in\widetilde{R}}\operatorname{sgn}\left(\frac{e_i-r}{e_{i+1}-r}\right) \\ &= (-1)^{|\widetilde{R}\cap(e_i,e_{i+1})|+1}. \end{aligned}$$

By construction of $\widetilde{R}$, we have

$$|\widetilde{R}\cap(e_i,e_{i+1})| = \begin{cases} 1, & (e_i,e_{i+1}) \text{ is trapped} \\ 0, & \text{otherwise.} \end{cases}$$

Consequently $\frac{\widetilde{\sigma}_i}{\widetilde{\sigma}_{i+1}} = \frac{\sigma_i}{\sigma_{i+1}}$. Thus the claim follows by induction on $|i|$. ☐

By Claims 4.7 and 4.8, it follows that $\widetilde{f}_i|_{\widetilde{U}_i} \in [w_i]$ for some $\widetilde{U}_i$. In fact, we have

$$\widetilde{U}_i = \begin{cases} (0, \widetilde{f}_i^{-1}(\infty)) & (\widetilde{f}_i^{-1}(0), \infty) \\ (0, \widetilde{f}_i^{-1}(0)) & (\widetilde{f}_i^{-1}(\infty), \infty) \end{cases} \qquad \text{chosen according to } U_i \text{ in (1, Subsection 3.3)}$$

Set $\widetilde{U} = \bigcap_i \widetilde{U}_i$.

Claim 4.9. *For all sufficiently large d_i in (3), we have $U \subset \widetilde{U}$.*

Proof. It suffices to show that $U_i \subset \widetilde{U}_i$ for d_i sufficiently large.

First observe that if $\widetilde{f}_i^{-1}(\infty)$ is used as an endpoint of $\widetilde{U}_i$, the claim is immediate. Indeed,

$$e_i = \widetilde{f}_i^{-1}(\infty) = f_i^{-1}(\infty)$$

so in this case $U_i = \widetilde{U}_i$.

Next, suppose $\widetilde{f}_i^{-1}(0)$ is used an endpoint; say $\widetilde{U}_i = (0, \widetilde{f}_i^{-1}(0))$ for specificity. For such a function class, f_i and $\widetilde{f}_i$ must be increasing functions on U_i and $\widetilde{U}_i$ respectively. Consequently $f_i^{-1}(0) < e_i$. On the other hand,

$$\widetilde{f}_i^{-1}(0) = e_i - \frac{\operatorname{Res} \widetilde{g}(x)|_{x=e_i}}{d_i}.$$

As $\operatorname{Res} \widetilde{g}(x)|_{x=e_i} < 0$ (since $\widetilde{f}_i$ is increasing), it follows that $\widetilde{f}_i^{-1}(0)_{d_i} > f_i^{-1}(0)$ for all sufficiently large d_i. Thus $\widetilde{U}_i \supset U_i$. The remaining case $\widetilde{U}_i = (\widetilde{f}_i^{-1}(0), \infty)$ follows a symmetrical argument. □

Choose $\widetilde{c} = \sum_{i=1}^n (d_i - e_i) + \sum_{r \in \widetilde{R}} r$, and ensure that $\widetilde{c} > 0$ by increasing d_i if necessary. Thus we have produced the data of a fiber $\widetilde{F} = (\widetilde{c}, \widetilde{f}_1|_{\widetilde{U}_1}, \dots, \widetilde{f}_n|_{\widetilde{U}_n}) \in B$.

Claim 4.10. *The function associated to $\widetilde{F}$ is $\widetilde{g}$.*

Proof. We have rigged our construction to produce the identity

$$\widetilde{g}(x) = \sum_{i=0}^{n} \widetilde{f}_i(x) - \widetilde{c} \tag{4}$$

a posteriori. Both sides have simple poles at e_i for $0 \le i \le n$, so we may establish equality by comparing first the residues then the constant terms. Note that the residue at $e_0 = \infty$ is defined to be the coefficient of x.

We claim that for any $0 \le i, j \le n$,

$$\operatorname{Res} \widetilde{f}_j(x)\Big|_{x=e_i} = \begin{cases} 0, & i \neq j \\ \operatorname{Res} \widetilde{g}(x)|_{x=e_i}, & i = j. \end{cases}$$

The $i \neq j$ case follows since the poles are distinct. If $i = j = 0$ the claim is immediate. Lastly if $i = j \ge 1$, apply (3) to conclude. Therefore

$$\operatorname{Res} \widetilde{g}(x)|_{x=e_i} = \operatorname{Res} \left(\sum_{i=0}^{n} \widetilde{f}_i(x) - c \right) \Bigg|_{x=e_i}, \qquad 0 \le i \le n.$$

It remains to show that the two sides have the same constant term. To this end, we compute

$$\begin{aligned}\widetilde{g}(x) &= \frac{\prod_{r\in\widetilde{R}}(x-r)}{\prod_{i=1}^{n}(x-e_i)} \\ &= x + \sum_{i=1}^{n} e_i - \sum_{r\in\widetilde{R}} r + o(1).\end{aligned}$$

Since $\widetilde{f}_i(x) = d_i + o(1)$ for $i \geq 1$ and $\widetilde{c} = \sum_{i=1}^{n}(d_i - e_i) + \sum_{r\in\widetilde{R}} r$, we have

$$\begin{aligned}\sum_{i=0}^{n} \widetilde{f}_i(x) &= x + \sum_{i=1}^{n} d_i + o(1) \\ &= \widetilde{g}(x) + \widetilde{c} + o(1).\end{aligned}$$

Thus the two sides of (4) have the same constant term and residues, so the functions agree. □

Since $\widetilde{g}$ has been constructed with root set $\widetilde{R}$, Claim 4.10 implies $\widetilde{R}$ is the root set of $\widetilde{F}$. Moreover, it is immediate from our construction of $\widetilde{U}$ that it is the domain of $\widetilde{F}$. Lastly, the fibers F and $\widetilde{F}$ have the same poles and signs (provided the poles of F are distinct; see the first line of this proof). Thus we have located all relevant quantities associated to $\widetilde{F}$.

By construction of $\widetilde{R}$, it follows that $\widetilde{F}$ has $n+1$ global roots, a unique root in every trapped interval, and all remaining roots in U. Since $U \subset \widetilde{U}$ (Claim 4.9), it follows that $\widetilde{F}$ is a distinguished fiber. As the fibers F and $\widetilde{F}$ have the same trapped intervals (Claim 4.8 and Equation 3), we may apply Lemma 4.5 to conclude that $|F| \leq |\widetilde{F}|$.

In the remaining case $U = \varnothing$, we have $|F| = 0$. Choose any fiber $\mathcal{F}$ from the image of the response map; that is, take $\mathcal{F} \in \Xi(\mathrm{Im}(L_G))$. Note this set is non-empty, since it is the image of a non-empty set. It follows that $\mathcal{F}$ has non-empty domain $\mathcal{U}$, whereupon we apply the first part of the proof to obtain a distinguished fiber $\widetilde{\mathcal{F}}$. Since $|F| \leq |\widetilde{\mathcal{F}}|$ holds vacuously, we are done. □

Corollary 4.11. *We have the equality*

$$\sup\{|F| : F \in B\} = \sup\{|\widetilde{F}| : \widetilde{F} \in B \text{ such that } \widetilde{F} \text{ is distinguished}\}$$

Proof. By containment, $\sup|\widetilde{F}| \leq \sup|F|$. The reverse follows directly from Theorem 4.6. □

5 Counting the Arity

The size of a distinguished fiber $\widetilde{F}$ turns out to depend only on its tuple of signs.

Definition 5.1. *The* **discrepancy** $d(\sigma)$ *of a tuple* σ *is* $|\{i : \sigma_i \neq \sigma_{i+1}\}|$.

Lemma 5.2. *We have* $|\widetilde{F}| = d(\sigma) + 1$*, where* σ *is the tuple of signs associated to* $\widetilde{F}$.

Proof. If $d(\sigma) = 0$, then each of the intervals (e_i, e_{i+1}) for $-n_- \leq i < i+1 \leq n_+$ are trapped. Consequently $\widetilde{g}$ has n roots in trapped intervals, leaving a single root of $\widetilde{g}$ in $\widetilde{U}$. In the general case, there are only $n - d(\sigma)$ trapped intervals, yielding $d(\sigma)+1$ roots in $\widetilde{U}$. □

Let $\Omega \subset \{\pm 1\}^{n+1}$ denote the set of tuples $\sigma = (s, 1, t)$, where $s \in \{\pm 1\}^{n_-}$ and $t \in \{\pm 1\}^{n_+}$ satisfy

$$W(G) = \begin{pmatrix} \#_{-1}(t) & \#_1(t) \\ \#_1(s) & \#_{-1}(s) \end{pmatrix} \tag{5}$$

For the remainder of this section, we write $W = W(G)$ (the definition is in Subsection 2.3).

Lemma 5.3. *The map from {distinguished fibers}* $\to \{\pm 1\}^{n+1}$ *given by* $\widetilde{F} \mapsto \sigma_{\widetilde{F}}$ *has image* Ω.

Proof. Index the strips of G using i as in Definition 4.4. Suppose strip i has initial subunit j and terminal subunit k. Then the function class $[w_i]$ contains functions of sign $(-1)^{j+k+1}$. Moreover, $\operatorname{sgn} i = (-1)^j$. Consequently

$$W_{jk} = \#\{\text{strips with } \sigma_i = (-1)^{j+k+1} \text{ and } \operatorname{sgn} i = (-1)^j\}.$$

This translates to the conditions of (5); hence $\sigma_{\widetilde{F}} \in \Omega$.

Conversely, take $\sigma \in \Omega$. Build a fiber F with arbitrary distinct poles e_i and signs σ_i, using the indexing of Definition 4.4. Then Theorem 4.6 produces a fiber $\widetilde{F}$ for which $\sigma_{\widetilde{F}} = \sigma_F = \sigma$. ☐

Theorem 5.4. $\kappa(G) = \min\{2W_{11}, 2W_{12} + 1\} + \min\{2W_{21} + 1, 2W_{22}\} + 1$

Proof. By Theorem 3.21,

$$\begin{aligned} \kappa(G) &= \sup_F |F| \\ \text{(Corollary 4.11)} \qquad &= \sup_{\widetilde{F}} |\widetilde{F}| \\ \text{(Lemma 5.2)} \qquad &= \sup_{\widetilde{F}} d(\sigma_{\widetilde{F}}) + 1 \\ \text{(Lemma 5.3)} \qquad &= \sup_{\Omega} d(\sigma) + 1. \end{aligned}$$

When $\sigma = (s, 1, t)$, we have $d(\sigma) = d((s,1)) + d((1,t))$. Therefore

$$\begin{aligned} \kappa(G) &= \sup\{d((1,t)) \mid \#_{-1}(t) = W_{11}, \#_1(t) = W_{12}\} \\ &\quad + \sup\{d((s,1)) \mid \#_1(s) = W_{21}, \#_{-1}(s) = W_{22}\} + 1 \\ &= \min\{2W_{11}, 2W_{12} + 1\} + \min\{2W_{21} + 1, 2W_{22}\} + 1. \end{aligned}$$

We elaborate on the last line. Configurations of σ which maximize discrepancy are of the form

$$\sup U \cdots \overset{+}{e_{-2}} \overset{-}{e_{-1}} \overset{+}{e_0} \overset{-}{e_1} \overset{+}{e_2} \cdots \inf U.$$

The formula for $\sup d((1,t))$ may be derived as follows. The maximum discrepancy coincides with the length of the longest alternating sign chain beginning at e_0 with the specified quantity of ± 1 terms in t. For W_{12} sufficiently large, the chain has length $2W_{11}$: indeed, each of the W_{11} negative components of t belong to the chain, and there are an equal number of 1 and -1 terms appearing in the chain. When W_{11} is sufficiently large, the chain has length $2W_{12} + 1$ as there are $W_{12} + 1$ appearances of 1 in the chain and W_{12} appearances of -1. ☐

6 Applications

Corollary 6.1. *Quad-graphs attain every finite arity.*

Proof. Choose any quad-graph G with

$$W(G) = \begin{pmatrix} \lfloor \frac{n-1}{2} \rfloor & \lceil \frac{n-1}{2} \rceil \\ 0 & 0 \end{pmatrix}.$$

Applying the formula from Theorem 5.4, it follows that $\kappa(G) = n$. □

We say that a graph G is **recoverable** if $\kappa(G) = 1$.

Corollary 6.2. *A quad-graph is recoverable if and only if*

$$W(G) = \begin{pmatrix} 0 & \star \\ \star & 0 \end{pmatrix}.$$

In other words, each strip must have exactly one switch.

Corollary 6.3 (Duality)**.** *Arity is preserved under row interchange (but not column interchange) of $W(G)$.*

6.1 Extremal Inequalities

We conjecture the following bound on κ.

Conjecture 6.4. *Suppose $G = (\operatorname{Int} V, \partial V, E)$ satisfies $1 < \kappa(G) < \infty$. Then*

$$2\kappa(G) - 1 \leq \min\{|\operatorname{Int} V|, \frac{|\partial V|}{2}\}$$

This conjecture implies that our construction produces the minimal graphs of given arity.

An **auxiliary edge** of a graph with boundary G consists of a boundary-boundary edge. Let G_{top} denote the graph G with auxiliary edges removed.

Conjecture 6.5. *Suppose equality holds in Conjecture 6.4 for some graph H. Then there is a quad-graph G with*

$$W(G) = \begin{pmatrix} \lfloor \frac{\kappa-1}{2} \rfloor & \lceil \frac{\kappa-1}{2} \rceil \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ \lfloor \frac{\kappa-1}{2} \rfloor & \lceil \frac{\kappa-1}{2} \rceil \end{pmatrix}$$

such that each strip of G contains precisely 3 subunits, for which $H_{top} = G_{top}$ after possible twisting of the orientations of the central edge identifications.

6.2 Surfaces

Given any quad-graph G, define a surface with boundary Σ_G by building a CW-complex out of G_{top} as follows. Take the 0-cells to be the vertices of G_{top}, and the 1-cells to be the edges of G_{top}. For each interior vertex of G_{top}, attach a 2-cell to the boundary of the quadrilateral surrounding the vertex.

Conjecture 6.6. *There is an embedding of G_{top} onto Σ_G and the genus of G_{top} is equal to the genus of Σ_G. Moreover, there is a simple relationship between the arity of G and the genus of Σ_G.*

References

[1] Y. Colin de Verdière. *Spectres de Graphes*, 1998.

[2] E. B. Curtis and J. A. Morrow. *Inverse Problems for Electrical Networks*, 2000.

[3] E. B. Curtis and J. A. Morrow. *Circular Planar Graphs and Resistor Networks*, 1998.