Number of Pieces in Percolated Hexagonal Grids

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September 15, 2015

Abstract

Percolate the faces of a hexagonal lattice with probability p. Let $E(m,n,p)$ denote the expected number of pieces. Let $\lambda(m,n,p) = \frac{E(m,n,p)}{mn}$. This paper shows that the sequence $\{\lambda(m,n,p)\}$ is decreasing and uses this fact to show that $\lambda(p) = \lim_{m,n\to\infty} \frac{E(m,n,p)}{mn}$ exists. Further, this paper gives a uniform bound on the convergence of $\{\lambda(m,n,p)\}$ and proves the relationship $\lambda(p) - \lambda(1-p) = (1-p)^3 p - p^3(1-p)$.

1 Introduction

Given a graph, select every vertex with probability p. Visually, the procedure can be thought of coloring the selected vertices black and the rest white. This process is called percolation. This paper will be concerned with the expected number of connected components resulting from percolating a hexagonal lattice. As a preliminary example, I will look at trees.

Example The expected number of connected black components in every percolated tree with n vertices is $p + (n-1)p(1-p)$

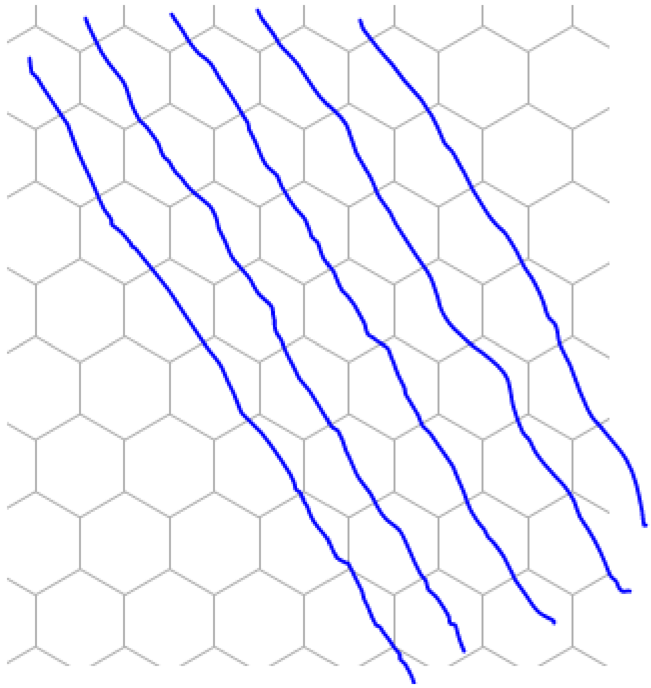
Proof. This is a proof by induction. In a one vertex tree, the expected number of pieces after percolating is p.

As the induction hypothesis, assume that the expected number of pieces in a percolated $n-1$ vertex tree is $p + (n-2)p(1-p)$. As the induction step, add another node to a tree with $n-1$ vertices. This new node will add a piece to the tree only if it is colored black and its neighbor is colored white. The probability of both events is p(1-p). Thus using the induction hypothesis, the expected number of pieces in a n vertex tree is $p + (n-1)p(1-p)$ $\square$

2 Hexagonal Grids

2.1 Terminology and Notation

Consider an m×n hexagonal board. The board is rhombus shaped. I will view the hexagons as appearing in rows and columns:



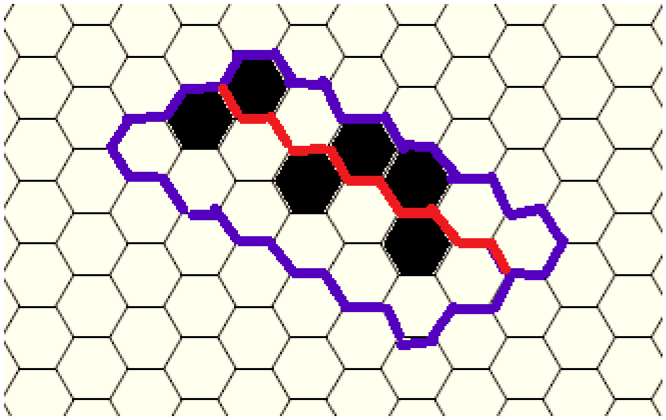
Let $E(m,n,p)$ denote the expected number of pieces in a $m \times n$ hexagonal grid percolated with probability p. I will denote $\frac{E(m,n,p)}{mn} = \lambda(m,n,p)$ and $\lim_{m,n\to\infty} \lambda(m,n,p) = \lambda(p)$. In the next section, I will show that this limit exists. Let $E(m,n)$ be a function of p with $E(m,n)(p) = E(m,n,p)$. Define $\lambda(m,n)$ and λ analogously. This notation is handy because a lot of the following arguments actually don't depend on p.

2.2 Existence of $\lambda(p)$

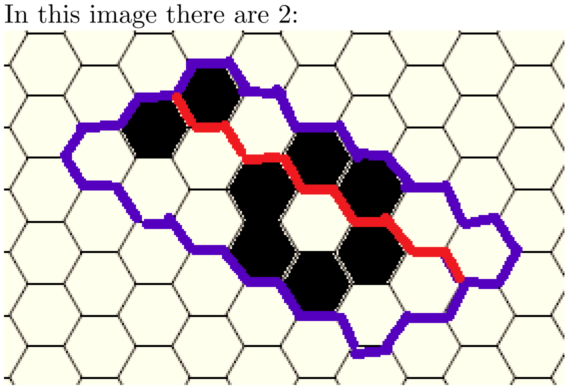
Jacob Richey [1] showed the existence of $\lambda(\frac{1}{2})$ using the subadditive ergodic theorem. This proof is a lot longer but it gives more information about the convergence of the sequence $\lambda(m,n)$, namely, that it is decreasing.
Consider an $m \times i$ grid. Let X_i be a random variable denoting the difference between the number of pieces in the full grid and the number of pieces in the $m \times (i-1)$ subgrid starting at the second column and extending to the end. Thus $E(X_i) = E(m,i) - E(m,i-1)$.

I will show that $\{E(X_i)\}$ is a increasing sequence.

Note that $E(m,1) = p+(m-1)p(1-p)$. Thus, $E(X_i)$ is the expected number of piece consolidations subtracted from $E(m,1)$.For example, in the following image there are three consolidations:



In this image there are 2:

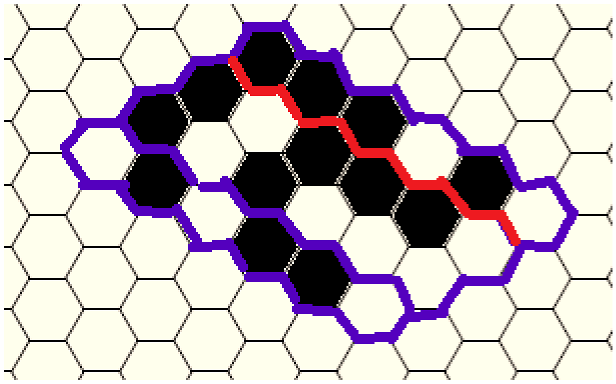


Let Z_i denote the number of consolidations in the ith row.

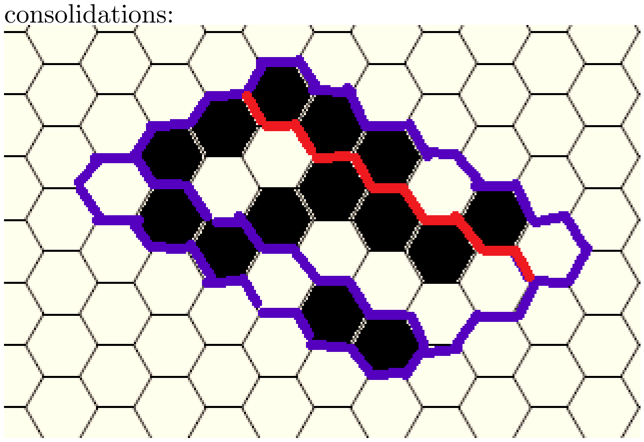
The sequence $\{Z_i\}$ is a decreasing.

For the proof, consider a $n-1 \times m$ grid. Z_{n-1} represents the number of consolidations in the $n-1$th column. Z_n is the number of consolidations in the nth column after adding another column before the first column.

For each coloring of the $n-1 \times m$ grid, there are 2^m colorings of the $n \times m$ grid, corresponding to the 2^m different colorings of the added column. Note that adding the additional column at the other end of the grid cannot increase the number of consolidations. However, it is possible that this additional column will decrease the number of consolidations. In this picture, the leftmost column causes 3 consolidations:



In this figure, the only the leftmost column has been changed and there are 2 consolidations:



Hence, for each of these 2^m colorings, the number of consolidations for the $n-1 \times m$ grid is larger than or equal to the number of consolidations for the $n \times m$ grid. Thus $E(Z_n) \leq E(Z_{n-1})$

Lemma 1. *$\{E(X_n)\}$ is an increasing sequence, and the limit $L(m) = \lim_{n\to\infty} E(X_n)$ exists.*

Proof. As mentioned before, $E(X_n) = E(m,1) - E(Z_n)$. Because $E(Z_n) \leq E(Z_{n-1})$, $E(X_n) = E(m,1) - E(Z_n) \geq E(m,1) - E(Z_{n-1}) = E(X_{n-1})$. This argument proves that $\{E(X_n)\}$ is increasing.
Because there are m vertices in the nth column, $E(n,m) - E(n-1,m) \leq m$ for all n. The sequence $\{E(X_n)\}$ is increasing and bounded above, so $\lim_{n\to\infty} E(X_n)$ exists by the monotone sequence theorem. $\square$

Lemma 2. $\frac{E(m,n)}{n} \geq L(m)$

Proof. This proof will involve a lot of epsilons.
First choose $\epsilon > 0$, and then N so that $0 \leq L(m) - E(X_n) < \epsilon$, which is possible by lemma 1. The expected number of additional pieces from adding n columns to a $m \times N$ grid is $\sum_{i=N+1}^{N+n} E(X_i)$. This quantity can also be calculated by

adjoining a $m \times n$ grid to a $m \times N$ grid. The expected number of additional pieces is E(m,n) minus the expected number of piece consolidations that happen on the border of the 2 grids. Thus, $E(m,n) \geq \sum_{i=N+1}^{N+n} E(X_i)$, so $\frac{E(m,n)}{n} \geq \frac{\sum_{i=N+1}^{N+n} E(X_i)}{n}$. Summing n copies of the inequality $L(m) - E(X_i) < \epsilon$ with i ranging from $N+1$ to $N+n$, results in $\sum_{i=N+1}^{N+n} L(m) - E(X_i) \leq n\epsilon$, so $\frac{1}{n}\sum_{i=N+1}^{n} E(X_i) \geq L(m) - \epsilon$
Combining these two inequalities results in $\frac{E(m,n)}{n} \geq L(m) - \epsilon$ for all $\epsilon > 0$. Thus $\frac{E(m,n)}{n} \geq L(m)$. □

Theorem 1. *The sequence* $\{\frac{E(m,n)}{mn}\}$ *is decreasing in both* m *and* n*.*

Proof. By lemmas 1 and 2, $\frac{E(m,n)}{n} \geq L(m) \geq E(X_n) = E(m,n) - E(m,n-1)$. Ignoring the middle terms of the inequality and simplifying results in $nE(m,n-1) \geq (n-1)E(m,n)$. Dividing by $mn(n-1)$ results in $\frac{E(n-1,m)}{(n-1)m} \geq \frac{E(n,m)}{nm}$. $\lambda(n,m)$ will denote $\frac{E(m,n)}{mn}$. Thus $\lambda(m,n)$ is decreasing in n. It is also decreasing in m because $\lambda(m,n)$ is symmetric in m and n. □

Because $\{\lambda(m,n)\}$ is decreasing in n and bounded below by zero, the limit $l(m) = \lim_{n\to\infty} \lambda(m,n)$ exists.
For a similar reason, the sequence $\{\lambda(m,n)\}$ is decreasing implies $\{l(m)\}$ is decreasing. Showing the contrapositive, if $l(m) > l(m-1)$, then for sufficiently large n, $\lambda(m,n) > \lambda(m-1,n)$. This statement implies that the sequence $\{\lambda(m,n)\}$ is not decreasing in m.

The relationship between $L(m)$ and $l(m)$ is $\frac{L(m)}{m} = l(m)$, which will not be proved here.

Theorem 2. *The limit* $\lambda = \lim_{m,n\to\infty} \frac{E(m,n)}{mn}$ *exists.*

Proof. First choose ϵ. Then choose M and N so that $n > N$ and $m > M$ imply $0 \leq l(m) - \lambda < \frac{\epsilon}{2}$ and $0 < \lambda(m,n) - l(m) < \frac{\epsilon}{2}$Using the adding and subtracting trick,

$$0 \leq \lambda(m,n) - \lambda = (\lambda(m,n) - l(m)) + (l(m) - \lambda) \leq$$
$$\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

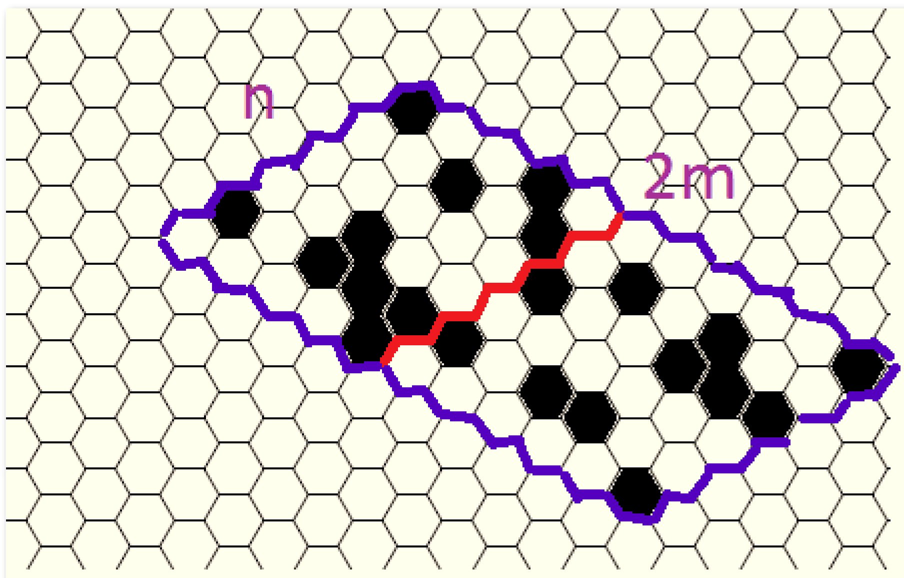
□

2.3 Uniform Bound on Convergence

Lemma 3. $n \geq 2E(m,n,p) - E(2m,n,p) \geq 0$

Proof. Look at two copies of an $m \times n$ hexagonal grid pasted together along one of the sides length n. This results in a $2m \times n$ grid. The number of pieces in the $2m \times n$ grid is less than the sum of the pieces in both grids because some pieces on the merged boundary consolidate. Taking the expectation over all possible colorings proves the right side of the inequality. Further, for any possible coloring, the difference between the number of pieces in both $m \times n$ grids and the number of pieces in the $2m \times n$ grid is the number of consolidations. The number of consolidations is at most n because there are n hexagons on the boundary. Taking the expectation over all possible colorings results in the left hand side of the inequality.

Here is a picture:



□

Lemma 4. $\frac{1}{2}(\frac{1}{n} + \frac{1}{m}) \geq \lambda(m, n, p) - \lambda(2m, 2n, p) \geq 0$

Proof. By lemma 3 and that E(m,n,p) is symmetric,

$$2m \geq 2E(2m, n, p) - E(2m, 2n, p) \geq 0$$

Further, by lemma 3

$$2n \geq 4E(m, n, p) - 2E(2m, n, p) \geq 0$$

Summing these two inequalities results in $2n+2m \geq 4E(m, n, p)-E(2m, 2n, p) \geq 0$. Dividing this inequality by 4mn results in

$$\frac{1}{2}(\frac{1}{n} + \frac{1}{m}) \geq \lambda(m, n, p) - \lambda(2m, 2n, p) \geq 0$$

□

Theorem 3. $\frac{1}{n} + \frac{1}{m} \geq \lambda(m, n, p) - \lambda(p) \geq 0$

Proof. I will look at $\lambda(m,n,p) - \lambda(2^j m, 2^j n, p)$. Adding to this expression $\lambda(2^i m, 2^i n, p) - \lambda(2^i m, 2^i n, p)$ for i between 1 and $j-1$ we get a telescoping sum

$$\sum_{i=0}^{j-1} \lambda(2^i m, 2^i n, p) - \lambda(2^{i+1} m, 2^{i+1} n, p)$$

Now applying lemma 4

$$(\frac{1}{n} + \frac{1}{m}) \sum_{i=0}^{j-1} \frac{1}{2^i} \geq \sum_{i=0}^{j-1} \lambda(2^i m, 2^i n, p) - \lambda(2^{i+1} m, 2^{i+1} n, p) \geq 0$$

Thus

$$(\frac{1}{n} + \frac{1}{m})(1 - \frac{1}{2^j}) \geq \lambda(m,n,p) - \lambda(2^j m, 2^j n, p) \geq 0$$

Taking the limit $j \to \infty$ results in the desired inequality.

□

Note that this also implies that $\lambda(p)$ is continuous.

2.4 Relationship Between $\lambda(p)$ and $\lambda(1-p)$

Theorem 4.

$$\lambda(p) = (1-p)^3 p - p^3(1-p) + \lambda(1-p)$$

Proof. In a hexagonal grid, every plane animal is surrounded by a loop of the opposite color. Here are two different ways to count finite plane animals.

1. Increase the count of plane animals every time a new plane animal is started

2. Increase the count of plane animals every time a plane animal becomes fully enclosed in a loop of the opposite color.

Let $E_1(m,n,p)$ denote the expected number of pieces counting in way 1 and $E_2(m,n,p)$ the expected number of pieces counting in way 2. Note that number 2 doesn't count plane animals that border the edges of the grid. There are O(m+n) of these since there are 2m+2n-4 hexagons on the border and in a specific coloring of the grid, there can't be more pieces touching the border than there are hexagons on the border. Thus $E_1(m,n,p) = E_2(m,n,p) + O(m+n)$, so

$$\lambda(p) = \lim_{m,n\to\infty} \frac{E_1(m,n,p)}{mn} = \lim_{m,n\to\infty} \frac{E_2(m,n,p) + O(m+n)}{mn}$$
$$= \lim_{m,n\to\infty} \frac{E_2(m,n,p)}{mn}$$

Thus we can use either E_1 or E_2 when calculating $\lambda(p)$.

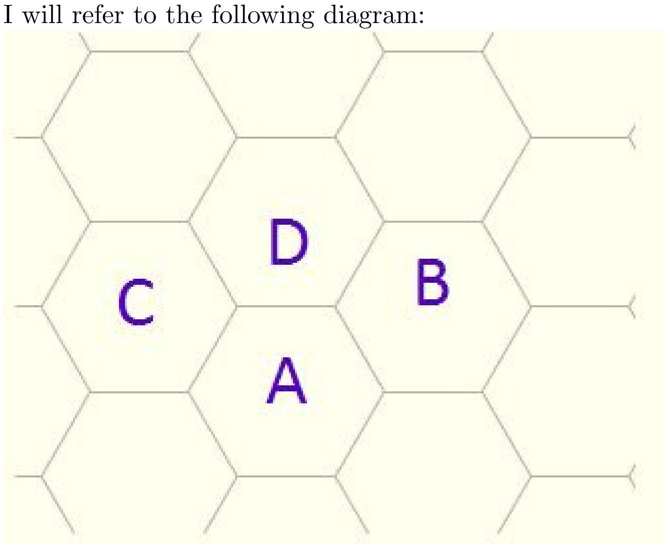
Now I will build this grid inductively one hexagon at time. First I will put down the hexagon at (1,1), then at (1,2), then (1,3) up until (1,m). Then I will move on to row 2 and continue as before. Let $(l,k)^+$ denote the hexagon that comes after (l,k) in this sequence. Formally,

$$(l,k)^+ = \begin{cases} (l,k+1) & k < m \\ (l+1,1) & k = m \end{cases}$$

Let $e(l,k)$ denote the expected number of plane animals after the (l,k)th hexagon has been put down. I will calculate how many additional plane animals we expect to get in the grid with one additional hexagon added. $Z_{(l,k)}$ will be the random variable representing the additional number of plane animals. Thus

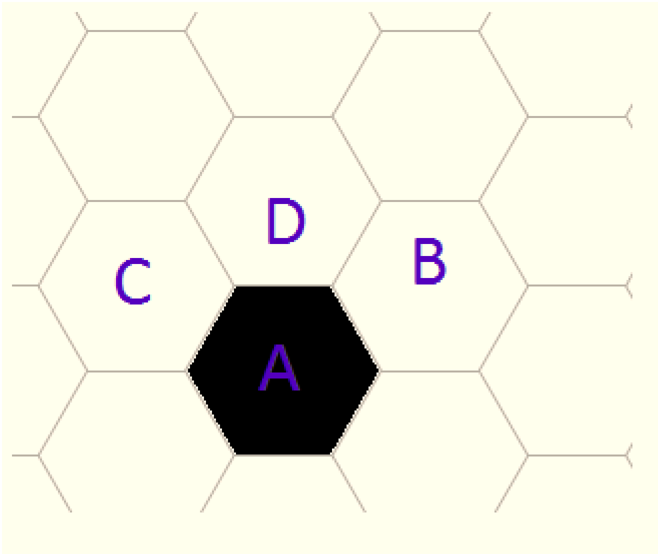
$$E(m,n) = \sum_{(l,k):1\le l,k\le n} Z_{(l,k)}$$

I will refer to the following diagram:

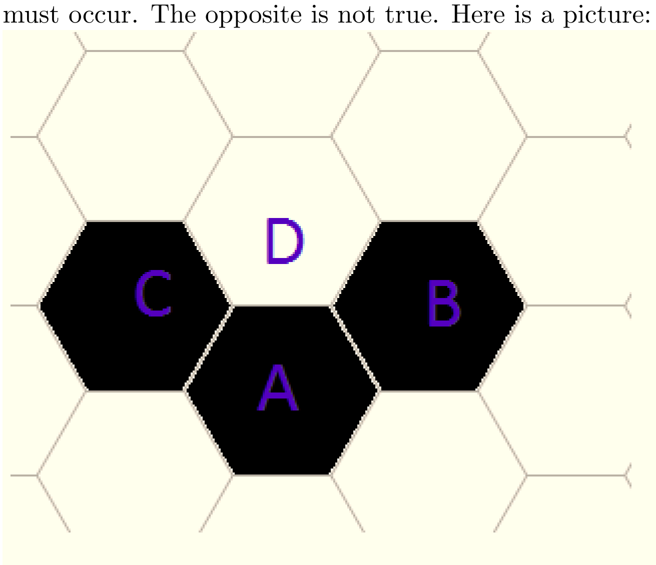


Note that this diagram only makes sense if A is not on the top or leftmost border. In other words, $l \neq 1$ and $k \neq 1$ A is the last hexagon added, and B, C, D are its neighbors. There are 2^4 different ways to color the hexagons A,B,C,D. Let X be a random variable that denotes the colorings of A,B,C,D. I plan to use the conditional expectation formula on these 16 scenarios. x_i will denote specific scenarios.

If a certain coloring of A,B,C, and D increases the number of pieces, A must be black because adding a white hexagon will not change the number of pieces. If B,C, or D is black,the number of pieces did not increase because the new hexagon became part of another piece. Thus, Coloring B,C,D white and A black will add a piece to the board. I will call this configuration x_1. This argument shows that if the number of pieces on the board increased by coloring A, then configuration x_1 occurs. Clearly, the opposite is also true, if configuration x_1 occurs, then coloring A increased the number of pieces in the board. Here is a picture:



It is also possible that coloring A will subtract pieces. In this case A must still be black because a white hexagon does not change the number of pieces. Further, B and C must belong to distinct black pieces so B and C must be black and D must be white. We will refer to this configuration by x_2. This argument shows that if coloring A decreased the number of components, configuration x_2 must occur. The opposite is not true. Here is a picture:



Thus, all other colorings will not change the number of pieces, so these are the only two cases we need to analyze.

I will use the conditional expectation formula on these two scenarios. Let x_1 be the first scenario described and x_2 the second.x_3 through x_{16} will be the values representing the other 12 colorings.

The configuration x_2 (coloring D white and A,B,C black) will subtract a piece only if they are not in the same component. If they are in the same piece, then hexagon A just closed a black loop and enclosed a white plane animal.

Thus for $l, k \neq 1$, by conditioning over x_i we get

$$E_1(Z_{(l,k)}) = \sum_{i=1}^{16} E(\text{additional peices}|X = x_i) \times P(X = x_i)$$

Using $E(\text{additional peices}|X = x_i) = 0$ for $i > 2$,

$$= \sum_{i=1}^{2} e(\text{additional peices}|X = x_i) \times P(X = x_i)$$

$$= p(1-p)^3 * 1 - p^3(1-p) * P(\text{no path from B to C excluding A}|x_2)$$

The probability of configuration x_1 is $p(1-p)^3$ and 1 piece is always added. The probability of configuration x_2 is $p^3(1-p)$, and a piece is subtracted only if there is no black path connecting B and C.

Further, $p^3(1-p)P(\text{no path from B to C excluding A}|x_2) = p^3(1-p)(1 - P(\text{path from B to C excluding A}|x_2))$ If there is a black path from B to C not involving A and D is white, then A is the last piece in a black loop enclosing a white plane animal. Thus $p^3(1-p)P(\text{path from B to C excluding A}|x_2) = P(\text{A encloses a white plane animal})$. However, this probability depends on (l, k). Let $Y_{(l,k)}$ be the event that the (l,k)th hexagon enclosed a white plane animal Thus

$$E(Z_{(l,k)}) = (1-p)^3 - p^3(1-p) + 1 * P(Y_{(l,k)})$$

Which implies

$$E_1(m, n, p) = \sum_{l=1 or k=1} E(Z_{l,k}) + \sum_{l=2}^{n} \sum_{k=2}^{m} E(Z_{l,k})$$

Note that $\sum_{l=1 \text{ or } k=1} Z_{l,k}$ is the number of pieces on the boundary. As mentioned before, this quantity is O(m+n). Thus

$$E_1(m, n, p) = O(m+n) + \sum_{l=2}^{n} \sum_{k=2}^{m} p(1-p)^3 - p^3(1-p) + P(Y_{(l,k)})$$

$$= O(n+m) + (m-1)(n-1)(p(1-p)^3 - (1-p)p^3) + \sum_{l=2}^{n} \sum_{k=2}^{m} P(Y_{(l,k)})$$

However,

$$E_2(m,n) = \sum_{l=1}^{m}\sum_{k=1}^{n} P(\text{hexagon(l,k) closes a loop}) = \sum_{l=2}^{n}\sum_{k=2}^{m} P(Y_{(l,k)})$$

because any hexagon on the left or top edge does not close a loop. Thus,

$$E_1(m,n,p) = O(n+m) + (m-1)(n-1)((1-p)^3 p - p^3(1-p)) + E_2(m,n,1-p)$$

Dividing by mn and taking limits,

$$\lambda(p) = \lim_{m,n\to\infty} \frac{E_1(m,n,p)}{mn}$$

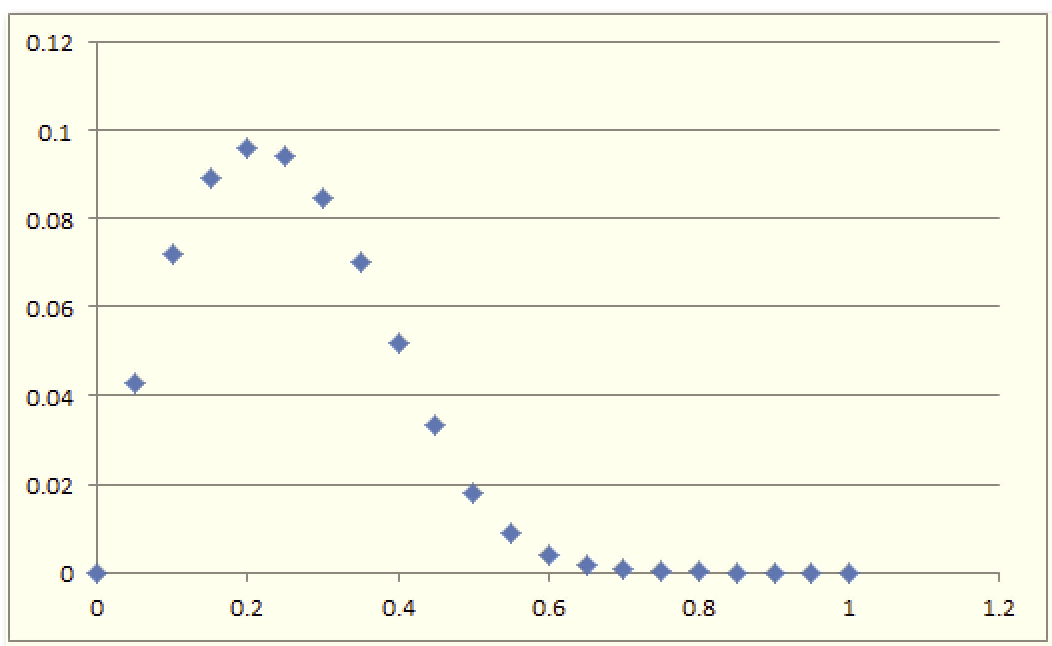
$$= \lim_{m,n\to\infty} \frac{O(n+m) + (m-1)(n-1)((1-p)^3 p - p^3(1-p)) + E_2(m,n,1-p)}{mn}$$

$$= (1-p)^3 p - p^3(1-p) + \lambda(1-p) = p(1-p)(1-2p) + \lambda(1-p)$$

□

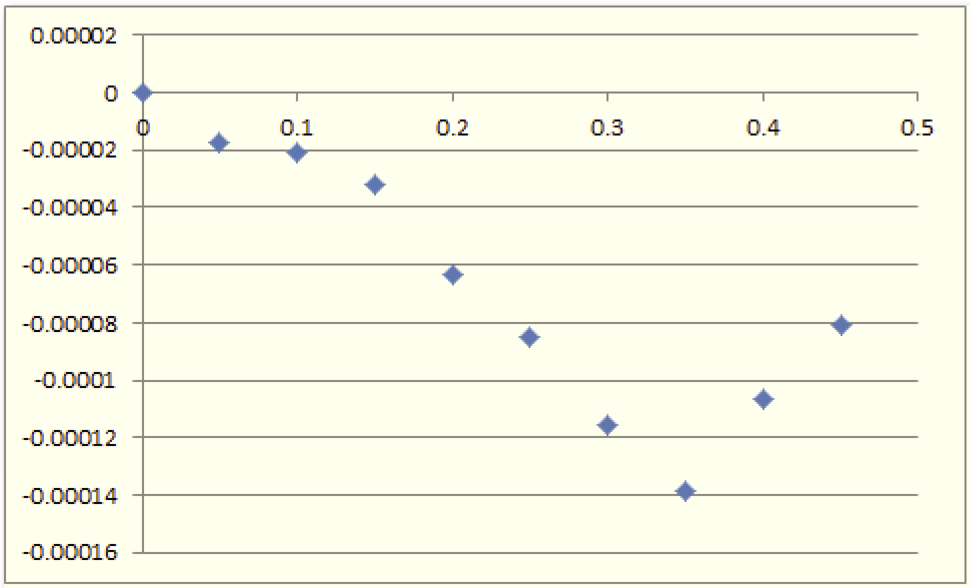
2.5 Simulations

I wrote a simulation to help visualize the function $\lambda(p)$. In the simulation a 2000 x 2000 hexagonal grid is percolated and the number of black components are counted. This is done 250 times and all the trials are averaged. By the central limit theorem this random variable approaches a normal random variable with mean $\lambda(2000, 2000, p)$. Here is a graph of the simulation's output:



Note that by section 2.3, the actual value of $\lambda(2000, 2000, p)$ is at most 0.001 larger than the value of $\lambda(p)$

To verify the result relating $\lambda(p)$ and $\lambda(1-p)$ I graphed $p(1-p)(1-2p)$ subtracted from the experimental value of $\lambda(p) - \lambda(1-p)$.



3 Extending these Results to the Square Lattice

I will focus on extending the previous result relating $\lambda(p)$ and $\lambda(1-p)$. However, it is important to first make sure that the relevant limits exist.

I will associate two different infinite graphs to the infinite square lattice.

1. there is one vertex for each face. Two vertices are connected if the corresponding faces touch along an edge. This graph will be called S

2. there is one vertex for each face. Two vertices are connected if the corresponding faces touch along an edge or share a corner. This graph will be called S'

Let $E(m, n, p, G)$ denote the expected number of connected components in an $m \times n$ section of G. Similarly, let $\lambda(m, n, p, G) = \frac{E(m,n,p,G)}{mn}$. The proof in section 2.2 does not use the fact that the lattice is hexagonal so the same proof can be used to establish the existence of these limits. Let $\lambda(p, G) = \lim_{m,n\to\infty} \lambda(m, n, p, G)$.

These two infinite graphs are duals in a certain sense. The squares bordering a plane animal in S form cycles in S' and the squares bordering a connected component in S' form cycles in S.

With this observation, using reasoning analogous to section 2.4, we get the equation $\lambda(p, S) - \lambda(1-p, S') = p(1-p)(1-p-p^2)$.

Note that using this technique, a similar equation can be found for a lot of different infinite lattice-like graphs. Further, note the uniform bound in section 2.3 also holds for both S and S', the proof did not use that the lattice was a hexagonal lattice.

References

[1] Richey, Jacob. *Counting Clusters on a Finite Grid.*