Annular Networks: Radial Networks, Mixed Problems, and Connections

David Jekel

July 15, 2013

1 Perfect Propagation

In radial networks, voltage-covoltage information on all of one boundary circle propagates completely and consistently to the whole network. If Γ has no type 0 geodesics or self-intersecting type 1 geodesics, then perfect propagation guarantees Γ is radial.[1] This property can interfere with recovery by making easy to overdetermine the network.

Theorem 1.1. *Suppose Γ is a radial network with no simply connected lenses. Then every mixed problem specifying complete Dirichlet and Neumann data on one boundary circle has a unique solution.*

Proof. Assume without loss of generality we want to specify data on the inner boundary. Consider the medial cells in the universal cover of the annulus. Let ϕ be a function on the upper boundary cells specifying voltage-covoltage data with periodicity conditions consistent for the annulus. Choose a cell c_0 on the upper boundary and let c_n be the nth cell to the right of c_0. Let $X_0 = \{c_0\}$. Assign the (co)voltage $f(c_0)$ to be $\phi(c_0)$. Then $\overline{X_0} = X_0$ and f specifies consistent data on X_0. For $n \geq 1$, let $Y_n = \overline{X_{n-1} \cup \{c_n\}}$. Define $f(c_n) = \phi(c_n)$ and extend f consistently on Y_n; Will Johnson's theorem guarantees this is possible. Then let $X_n = \overline{Y_n \cup \{c_{-n}\}}$ and extend f on X_n.

We show that for each cell $c \in M$, f will be defined and analytic at c for n sufficiently large. Let S be the set of c and all cells sharing a side or corner with c. Let $D = \{a < \mathrm{Re}(z) < b, -\log R < \mathrm{Im}\, z < -\log r\}$ where a and b are chosen so that $S \subset D$. Let $\mathcal{F}$ be the family of all geodesics (in the universal cover) intersecting D. Choose n large enough that the

[1] This may be true when there are self-intersecting type 1 geodesics, but I have not proved it.

upper endpoints of geodesics in $\mathcal{F}$ fall within X_n. Since X_n is closed and connected, it is the intersection of half-planes, but none of the geodesics in $\mathcal{F}$ can form part of the boundary of X_n. Hence, $X_n \supset D \supset S$. Therefore, f is defined on S for n sufficiently large, and it gives consistent data on S.

Take $n \to \infty$ and f will be defined and consistent on all of M. To show that f defines a consistent function on the annulus, we only have to show that the voltages and the differences in covoltages are periodic. For any cell c, let $c + 2\pi$ denote the cell in the next period. Let K be the difference in covoltage from c to $c + 2\pi$ for c on the upper boundary and let χ be the characteristic function of the covoltage cells. Then $f(c+2\pi) = f(c)+K\chi(c)$ on the upper boundary. Since conductivities repeat periodically and the two functions are equal on the upper boundary, they are equal on the geodesic closure of the upper boundary, which is all of M. Thus, f is appropriately periodic and defines a consistent voltage-covoltage on the annulus. □

Theorem 1.2. *Suppose* Γ *is an annular planar network with no type 0 geodesics, simply connected lenses, or self-intersecting type 1 geodesics. Suppose that every mixed problem specifying complete Dirichlet and Neumann data on one boundary circle has a unique solution. Then* Γ *is radial.*

Proof. Suppose that G has a type 1 geodesic; assume without loss of generality that it is an outer-to-outer geodesic. We will show there is a mixed problem that does not have a unique solution. Let g be an outer-to-outer geodesic such that no other geodesic is fully contained in $S(g)$. Every geodesic segment in $S(g)$ has an endpoint on $\widehat{g}$ and exits $S(g)$ somewhere along g. By the same argument given in "Elimination of Type 1 Geodesics," we can remove all crossings out of $S(g)$ by motions of the medial graph (these are electrical equivalences which preserve the response matrix). In the transformed medial graph, $S(g)$ contains only a series of boundary cells.

Consider a mixed problem setting all (co)voltages to zero on the inner boundary. We will construct multiple γ-harmonic functions satisfying these boundary conditions. Consider functions with all (co)voltages set to zero outside $S(g)$. Let $c_1, \ldots, c_n$ be the cells in $S(g)$ in counterclockwise order. Assign an arbitrary (co)voltage x to c_1. That will uniquely determine the (co)voltage at c_2, c_3, and eventually c_n, and since we moved in order across the cells, we still have consistent data on the network. Thus, the mixed problem has a one-parameter family of solutions. □

2 Determinants, Mixed Problems, and Connections

Definition 2.1. Define

$$\begin{aligned}\Lambda_{ii} &= \Lambda(\partial_i V; \partial_i V),\\ \Lambda_{io} &= \Lambda(\partial_i V; \partial_o V),\\ \Lambda_{oi} &= \Lambda(\partial_o V; \partial_i V) = \Lambda_{io}^T,\\ \Lambda_{oo} &= \Lambda(\partial_o V; \partial_o V).\end{aligned}$$

Theorem 2.2. *Suppose Γ is an annular planar network. Every mixed problem specifying complete Dirichlet and Neumann data on one boundary circle has a unique solution if and only if Λ_{io} is invertible.*

Proof. Suppose every such mixed problem has a unique solution. Then for any ϕ and ψ, there is a unique $\mathbf{x}$ satisfying $\Lambda_{ii}\phi + \Lambda_{io}\mathbf{x} = \psi$. In particular, this is true for $\phi = \mathbf{0}$, which means that for any ψ, $\Lambda_{io}\mathbf{x} = \psi$ has a unique solution. Thus, Λ_{io} is invertible.

If Λ_{io} is invertible, then $\Lambda_{io}\mathbf{x} = \psi - \Lambda_{ii}\phi$ has a unique solution for any ϕ and ψ. Similarly, Λ_{oi} is invertible and $\Lambda_{oi}\mathbf{x} = \psi - \Lambda_{oo}\phi$ has a unique solution. Once $\mathbf{x}$ is known, the voltage on the whole network is determined because the Dirichlet problem has a unique solution. □

Corollary 2.3. *If Γ is radial, there is a connection between $\partial_i V$ and $\partial_o V$.*

Proof. It follows directly from the determinant connection formula. □

Remark. *The above results hold even for degenerate networks. If a medial cell with a primal vertex in it touches both boundary circles, then it must be counted on both boundaries. It is considered to be connected to itself.*

Theorem 2.4. *Suppose Γ is an annular planar network. Suppose $P \subset \partial_i V$ and $Q \subset \partial_o V$ and $|P| = |Q| = k$, where k is odd. Let P be written in clockwise order and Q in counterclockwise order. Then $\det \Lambda(P;Q) \leq 0$ and $\det \Lambda(P;Q) < 0$ if and only if there is a connection between P and Q.*

Proof. The only possible permutations for a connection are of the form $\tau(i) \equiv i + n \mod k$, where n is some integer. Since k is odd, all these permutations are even. Let T be the set of all such permutations. By the determinant connection formula,

$$\det \Lambda(P;Q) \cdot \det K(\operatorname{int} V, \operatorname{int} V) = -\sum_{\tau \in T} \sum_{\alpha : \tau_\alpha = \tau} D_\alpha \prod_{e \in E_\alpha} \gamma(e).$$

All the terms inside the sum are positive and there is at least one term if and only if there is a connection between P and Q. $\square$

Corollary 2.5. *Suppose Γ is a radial network. Suppose $P \subset \partial_i V$ and $Q \subset \partial_o V$ and $|P| = |Q| = k$, where k is odd. Let P be written in clockwise order and Q in counterclockwise order. Then $\det \Lambda(P; Q) < 0$.*

Proof. There is a connection between $\partial_i V$ and $\partial_o V$. Thus, there must be a connection between P and Q. $\square$

3 Geodesics and Connections

Lemma 3.1. *Suppose Γ' is obtained from Γ by geodesic elimination. Then the maximum size k-connection between the inner and outer boundary is the same for Γ and Γ'.*

Proof. Geodesic elimination corresponds on the primal graph to Y-Δ transformations, removing boundary-to-boundary edges on the same boundary circle, removing boundary spikes, and deleting disconnected vertices. We only have to show that these operations do not change the maximum size k-connection between the two boundaries.

This is obvious for disconnected vertex deletion. Boundary-to-boundary edges play no role in connections from one boundary circle to the other. If a boundary spike is used in a k-connection, then 1-connection must pass through the interior vertex of the spike, and that vertex cannot be used in any other 1-connections. Thus, contracting the spike will not affect the k-connections.

Now consider Y-Δ transformations. Let a, b, and c be the corners of the Y, and let d be the central vertex. Suppose a k-connection uses the edge ab. If a k-connection uses another edge, say bc, it cannot use the third; so we can change the connection to use the ac rather than ab and bc. Thus, the case where two edges are used reduces to the case where one edge is used. If the edge ab is used and ac and bc are not used, then we can change the Y to a Δ and the new connection can use ad and db. It does not matter if the connection used vertex c. To show that changing a Δ to a Y does not affect connections, we simply reverse the argument. $\square$

Theorem 3.2. *Suppose Γ is an annular planar network with no simply connected lenses, type 0 geodesics, or lenses involving type 1 geodesics. The maximum size k-connection between the inner and outer boundaries is equal to half the number of type 2 geodesics.*

Proof. Γ can be reduced by geodesic elimination to a radial network Γ' (see "Elimination of Type 1 Geodesics"), which has a connection between $\partial_i V$ and $\partial_o V$. The size of this connection is exactly half the number of type 2 geodesics. The geodesic elimination does not change the number of type 2 geodesics or the size of the maximum connection between the two boundaries. $\square$

We can think of this theorem as a version of the cut-point lemma in which the cut divides the inner boundary circle from the outer boundary circle. For each boundary circle, the number of black cells is equal to the number of reentrant geodesics plus the maximum size k-connection between the two boundaries.