Nonlinear Discrete Laplace and Heat Equations for Electrical Networks

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Contents

1 Laplace's Equation

1.1 Existence and Uniqueness

A *graph with boundary* G is a triple, (V, E, I), where V is a set of vertices, $E \subset V \times V$ is the set of edges, and $I \subset V$ is a set of vertices designated as *interior*. The remaining vertices $B = V \setminus I$ are *boundary vertices*. We assume the vertices are indexed by integers $1, \ldots, N$ with the boundary vertices listed first. We assume G is finite and connected and that B is nonempty. An *electrical network* Γ consists of a graph with boundary and a conductivity function γ. For each directed edge $p \to q$, γ assigns a function $\gamma_{pq} : \mathbb{R} \to \mathbb{R}$ such that

- γ_{pq} is continuous and weakly increasing,
- $\gamma_{pq}(0) = 0$,
- $\gamma_{qp}(x) = -\gamma_{pq}(-x)$.

A *potential* or *voltage* is a function $u : V \to \mathbb{R}$. For a given potential, the *current function* $\iota : E \to \mathbb{R}$, defined on directed edges, is given by

$$\iota(p \to q) = \gamma_{pq}\big(u_p - u_q\big).$$

The *net current function* $J : V \to \mathbb{R}$ is given by

$$J(p) = \sum_{q \sim p} \iota(p \to q).$$

Thinking of u and J as vectors in $\mathbb{R}^V$, we define *Kirchhoff function* $K : \mathbb{R}^V \to \mathbb{R}^V$ by $u \mapsto J$. Clearly, K is continuous.

I will write $u_p = u(p)$ for $p \in V$, and let $\pi_I : \mathbb{R}^V \to \mathbb{R}^I$ and $\pi_B : \mathbb{R}^V \to \mathbb{R}^B$ be the projections onto the interior and boundary vertices respectively. A potential ϕ is called γ*-harmonic* if $J(p) = 0$ for all $p \in I$ or $\pi_I(J) = 0$. The *Dirichlet problem* is this: Given $\phi : B \to \mathbb{R}$, does there exist a γ-harmonic function u such that $\pi_B(u) = \phi$? The answer is proved in Theorem 2.4 of [2], which I restate here with slight changes of notation.

Theorem 1 (Will Johnson)**.** *Let Γ be an electrical network and suppose $\phi : B \to \mathbb{R}$.*

(i) There exists a γ-harmonic function u with $u|_B = \phi$.

(ii) The current ι is uniquely determined by ϕ.

(iii) (Maximum principle) u can be chosen so that $\max_{p\in V}|u_p| = \max_{p\in B}|\phi_p|$.

Condition (iii) below is not stated in the theorem, but the u constructed in the proof satisfies this condition. Letting $\|u\|_\infty = \max_{p\in V}|u_p|$, we can write this as $\|u\|_\infty = \|\phi\|_\infty$.

Define the *Dirichlet-to-Neumann map* $\Lambda : \mathbb{R}^V \to \mathbb{R}^V$ by $\Lambda(\phi) = \pi_B(J)$, where J is the net current function for a solution to the Dirichlet problem. This is well-defined by (ii).

1.2 Continuity and Convergence

Proposition 2. *Λ is a continuous function of ϕ.*

Proof. Continuity at 0 follows directly from condition (iii) above. Indeed, for $\phi \in \mathbb{R}^B$, let $U(\phi)$ be some solution of the Dirichlet problem satisfying the maximum principle. Then $U(0) = 0$ and $\lim_{f\to 0} U(\phi) = 0$. Since K is continuous, $\Lambda = K \circ U$ is continuous at 0.

For continuity at an arbitrary $\phi_0 \in \mathbb{R}^B$, we use a translation argument. Let u_0 be any solution of the Dirichlet problem. Define

$$\widehat{\gamma}_{pq}(x) = \gamma_{pq}(u_0(p) - u_0(q) + x) - \gamma_{pq}(u_0(p) - u_0(q));$$

it is easy to verify $\widehat{\gamma}_{pq}$ satisfies the necessary conditions to be a conductance function. Let $\widehat{U}(\widehat{\phi})$ map to $\widehat{u} \in \mathbb{R}^B$ to a solution of the Dirichlet problem satisfying the maximum principle. Then for any ϕ, $u_0 + \widehat{U}(f - f_0)$ is γ-harmonic. To see this, let $\widehat{u} = \widehat{U}(f - f_0)$ and $u = u_0 + \widehat{u}$. Then

$$\begin{aligned} J(p) &= \sum_{q\sim p} \gamma_{pq}\big(u(p) - u(q)\big) \\ &= \sum_{q\sim p} \gamma_{pq}\big(u_0(p) - u_0(q) + \widehat{u}(p) - \widehat{u}(q)\big) \\ &= \sum_{q\sim p} \widehat{\gamma}_{pq}\big(\widehat{u}(p) - \widehat{u}(q)\big) + \sum_{q\sim p} \gamma_{pq}\big(u_0(p) - u_0(q)\big) \\ &= 0. \end{aligned}$$

Also,

$$u|_B = u_0|_B + \widehat{u}|_B = \phi_0 + \phi - \phi_0 = \phi.$$

Thus, u is a solution of the Dirichlet problem for γ and ϕ. Hence,

$$\Lambda(f) = K(u)|_B = K\big(u_0 + \widehat{U}(\phi - \phi_0)\big)|_B.$$

Since $\widehat{U}$ is continuous at 0, Λ is continuous at ϕ_0. $\square$

Proposition 3. *There exists a continuous* $U : \mathbb{R}^B \to \mathbb{R}^V$ *such that* $U(\phi)$ *is a solution to the Dirichlet problem,* $\pi_B(U(\phi)) = \phi$*, and*

$$\|U(\phi_1) - U(\phi_2)\|_\infty = \|\phi_1 - \phi_2\|_\infty .$$

Proof. Suppose first that the conductances are strictly increasing. Then because the currents are uniquely determined by the boundary potentials, the solution to the Dirichlet problem must be unique. Let $U(\phi)$ be the solution to the Dirichlet problem. For any ϕ_0, we can define $\widehat{U}$ as above, and then $U(\phi_0) + \widehat{U}(\phi - \phi_0)$ is a solution to the Dirichlet problem for ϕ and hence $U(\phi) = U(\phi_0) + \widehat{U}(\phi - \phi_0)$. By construction of $\widehat{U}$,

$$\|U(\phi) - U(\phi_0)\|_\infty = \left\|\widehat{U}(\phi - \phi_0)\right\|_\infty = \|\phi - \phi_0\|_\infty ,$$

which shows that U is continuous and establishes the desired estimate.

To remove our restrictions on γ_{pq}, note that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and weakly increasing with $f(0) = 0$, then there is a sequence $f_n \to f$ uniformly on compact sets such that f_n is C^1, $f_n(0) = 0$, and $f_n' > 0$. In particular, we can take

$$f_n(x) = \int_{x-1/n}^{x+1/n} f - \int_{-1/n}^{1/n} f + \frac{x}{n}.$$

The first term converges uniformly to f on compact sets by uniform continuity of f, the second term approaches zero and does not depend on x, and the third term approaches 0 uniformly on compact sets. Also,

$$f_n'(x) = f(x + 1/n) - f(x - 1/n) + \frac{1}{n} > 0.$$

For each γ_{pq}, let γ_{pq}^n be the sequence thus constructed. Let $U_n(\phi)$ be the solution to the Dirichlet problem for conductances γ_{pq}^n. Because $\|U_n(\phi_1) - U_n(\phi_2)\|_\infty \leq \|\phi_1 - \phi_2\|_\infty$ and $\|U_n(\phi)\|_\infty \leq \|\phi\|_\infty$, the sequence $\{U_n\}$ is equicontinuous and pointwise bounded. Therefore, by the Arzela-Ascoli theorem, there is a subsequence $\{U_{n_k}\}$ converging uniformly on compact sets to a function U. If K_{n_k} is the Kirchhoff function corresponding to the conductances γ_{pq}^n, then $K_{n_k} \to K$ uniformly on compact sets. It follows by an easy argument, which is included after the proposition, that $K_{n_k} \circ U_{n_k} \to K \circ U$ uniformly on compact sets, which implies $K(U(\phi)) = 0$, so $U(\phi)$ is a solution to the Dirichlet problem. It also satisfies $\|U(\phi_1) - U(\phi_2)\|_\infty = \|\phi_1 - \phi_2\|_\infty$ since each U_n satisfies the corresponding estimate. $\square$

Lemma 4. *Suppose $f_n : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ and $g_n : \mathbb{R}^{d_2} \to \mathbb{R}^{d_3}$ are continuous functions $\mathbb{R}^m \to \mathbb{R}^m$. If $f_n \to f$ uniformly on compact sets and $g_n \to g$ uniformly on compact sets, then $g_n \circ f_n \to g \circ f$ uniformly on compact sets.*

Proof. Let $K \subset \mathbb{R}^{d_1}$ be a compact. Then $f_n(K)$ is compact for each n and $f(K)$ is compact. In particular these sets are bounded by constants M_n and M. There is an N such that $n \geq N$ implies $|f_n(x) - f(x)| \leq 1$ for all $x \in K$. Thus, $M_n \leq M + 1$ for all $n \geq N$. Let $M' = \max\{M_1, \ldots, M_{N-1}, M + 1\}$. Let B be the closed ball of radius M'. Then $f_n(K) \subset B$ for all n and B is compact. Thus, $g_n \to g$ uniformly on B.

Choose $\epsilon > 0$. Since g is continuous, it is uniformly continuous on B, so there is a δ such that $|x - y| < \delta$ implies $|g(x) - g(y)| < \epsilon/2$ for all $x, y \in B$. There is also an N_1 such that $n \geq N_1$ implies $|f_n(x) - f(x)| < \delta$ for all $x \in K$, and an N_2 such that $n \geq N_2$ implies $|g_n(y) - g(y)| < \epsilon/2$ for all $y \in B$. Then for all $n \geq \max\{N_1, N_2\}$ and all $x \in K$, we have

$$|g_n(f_n(x)) - g(f(x))| \leq |g_n(f_n(x)) - g(f_n(x))| + |g(f_n(x)) - g(f(x))|.$$

Since $f_n(x) \in B$, we have $|g_n(f_n(x)) - g(f_n(x))| < \epsilon/2$. Since $|f_n(x) - f(x)| < \delta$ and $f(x), f_n(x) \in B$, we have $|g(f_n(x)) - g(f(x))| < \epsilon/2$, which completes the proof. □

Theorem 5. *Suppose that γ^n and γ^0 are conductances on a graph G and Λ_n and Λ_0 are the corresponding Dirichlet-to-Neumann maps. If $\gamma^n_{pq} \to \gamma^0_{pq}$, then $\Lambda_n \to \Lambda_0$ uniformly on compact sets.*

For the proof, we actually need $\gamma^n_{pq} \to \gamma^0_{pq}$ uniformly on compact sets, but this happens automatically as a result of the following lemma:

Lemma 6. *Suppose g_n and g are increasing functions $\mathbb{R} \to \mathbb{R}$ and $g_n \to g$. If g is continuous, then the convergence is uniform on compact sets.*

Proof. It suffices to show that the convergence is uniform on any compact interval $[a, b]$. By multiplying g_n and g by a constant and translating, we can assume $g(a) = 0$ and $g(b) = 1$. Choose $\epsilon > 0$, and k such that $1/2^k < \epsilon/2$. By the intermediate value theorem, for $j = 1, \ldots, 2^k$, there exists a $t_j \in [a, b]$ such that $g(t_j) = j/2^k$. Then $t_j < t_{j+1}$ because g is increasing. Since there are only finitely many values of j, we know $|g_n(t_j) - g(t_j)| < \epsilon/2$ for all j for n sufficiently large. Then if $t \in [t_j, t_{j+1}]$, we have

$$g_n(t) \leq g_n(t_{j+1}) \leq g(t_{j+1}) + \frac{\epsilon}{2} = g(t_j) + \frac{1}{2^k} + \frac{\epsilon}{2} < g(t) + \epsilon,$$

and by a symmetrical argument, $g_n(t) > g(t) - \epsilon$. So $|g_n(t) - g(t)| < \epsilon$ for all $t \in [a, b]$. □

I will also use the lemma

Lemma 7. *Let $f_n : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$. If every subsequence of $\{f_n\}$ has in turn a subsequence converging uniformly on compact sets to f, then $f_n \to f$ uniformly on compact sets.*

Proof. Suppose that f_n does not converge uniformly to f on compact sets. Then there is a compact set K, an $\epsilon > 0$, and a subsequence f_{n_k} such that $\sup_{x \in K} |f_{n_k}(x) - f(x)| > \epsilon$ for each k. Then $\{f_{n_k}\}$ cannot have a subsequence converging uniformly to f on compact sets. $\square$

Now we complete the proof of the theorem:

Proof. Let K_n and K_0 be the Kirchhoff functions corresponding to the conductances γ_{pq}^n and γ_{pq}^0. Let $U_n(\phi)$ and $U_0(\phi)$ be solutions to the Dirichlet problem as in Proposition 3. Let $\{\Lambda_{n_k}\}$ be a subsequence of $\{\Lambda_n\}$. Since $\{U_{n_k}\}$ is equicontinuous and pointwise bounded, there is a subsequence $\{U_{n_{k_j}}\}$ converging uniformly on compact sets to a function U_0. By Lemma 4, $K_{n_{k_j}} \circ U_{n_{k_j}} \to K_0 \circ U_0$ on compact sets, which implies that $\pi_I(K_0 \circ U_0) = 0$, so $U_0(\phi)$ is γ-harmonic. Hence, $\Lambda_0 = \pi_B(K_0 \circ U_0)$, and $\Lambda_{n_{k_j}} = \pi_B(K_{n_{k_j}} \circ U_{n_{k_j}}) \to \Lambda_0$ uniformly on compact sets. $\square$

1.3 Differentiation of Λ

In order to differentiate (i.e. linearly approximate) Λ, we need some basic results about linear conductances. Suppose there are nonnegative constants $a_{pq} = a_{qp}$ such that

$$\gamma_{pq}(x) = a_{pq}x.$$

Assume the vertices of G are indexed by integers $1, \ldots, |V|$ with the boundary vertices listed first. Let A be the Kirchhoff matrix given by

$$(A)_{ij} = \begin{cases} \sum_{k \sim i} a_{ik} & \text{if } i = j, \\ -a_{ij} & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}.$$

In this case, $K(u) = Au$ and so the Jacobian $DK(u) = A$. Write A in block form as

$$\begin{pmatrix} A_{BB} & A_{BI} \\ A_{IB} & A_{II} \end{pmatrix}.$$

where the first row/column deals with B and the second row/column deals with I. If A_{II} is invertible, then the Dirichlet problem has a unique solution.

Indeed, suppose $\phi \in \mathbb{R}^B$ and u is a solution for the Dirichlet problem. Write u in block form as $(\phi, w)^T$. Then

$$K(u) = \begin{pmatrix} A_{BB} & A_{BI} \\ A_{IB} & A_{II} \end{pmatrix} \begin{pmatrix} \phi \\ w \end{pmatrix} = \begin{pmatrix} A_{BB}\phi + A_{BI}w \\ A_{IB}\phi + A_{II}w \end{pmatrix}.$$

For the net current on the interior vertices to be zero, we need $A_{IB}\phi + A_{II}w = 0$, and thus $w = -A_{II}^{-1}A_{IB}\phi$. Then the boundary currents are

$$\Lambda(\phi) = A_{BB}\phi + A_{BI}w = (A_{BB} - A_{BI}A_{II}^{-1}A_{IB})\phi.$$

The matrix $A_{BB} - A_{BI}A_{II}^{-1}A_{IB}$ is the Schur complement A/A_{II}.

However, Λ is well-defined even if we allow $a_{pq} = 0$ for some edges. It must also be a linear map, as the reader can verify. In fact, let $\Xi : (\mathbb{R}_{>0})^E \to M_{B\times B}$ be the map $\{a_{pq}\}_{pq\in E} \to A/A_{II}$. Then Ξ extends continuously to $(\mathbb{R}_{\geq 0})^E$. If some of the a_{pq}'s are zero, we define $\Xi(\{a_{pq}\})$ to be the matrix of the Dirichlet-to-Neumann map of the linear conductances given by a_{pq}. To see that Ξ is continuous, suppose that for each pq, we have a sequence of coefficients $(a_n)_{pq} \to (a_0)_{pq}$. Let Λ_n and Λ_0 be the corresponding Dirichlet-to-Neumann maps. Then by Theorem 5, $\Lambda_n \to \Lambda_0$ uniformly on compact sets. This implies that the matrices of Λ_n converge to the matrix of Λ_0 in each entry of the matrix.

Therefore, if $a_{pq} \geq 0$ and some of the a_{pq}'s are zero, we *define* $A/A_{II} = \Xi(\{a_{pq}\})$ even if the Schur complement does not exist in the traditional sense. I will assume this definition in the rest of the paper.

Then for all linear conductances, $D\Lambda(\phi) = A/A_{II} = DK(u)/DK_{II}(u)$. Actually, the formula holds in great generality. If the conductances are differentiable, then the Jacobian DK is given by

$$(DK)_{ij}(u) = \frac{\partial J(i)}{\partial u(j)} = \begin{cases} \sum_{k\sim i} \gamma'_{ik}\big(u(i) - u(k)\big) & \text{if } i = j, \\ -\gamma'_{ij}\big(u(i) - u(j)\big) & \text{if } i \sim j, \\ 0 & \text{otherwise,} \end{cases}$$

and we have

Theorem 8. *If each γ_{pq} is differentiable, then Λ is differentiable and*

$$D\Lambda(\phi) = DK(u)/DK_{II}(u),$$

where u is any γ-harmonic function with $\pi_B(u) = \phi$.

Proof. Here, I assume $\phi = 0$ and $u = 0$. The general case can be handled by the same translation argument used in Proposition 2. We want to show that $D\Lambda(0) = DK(0)/DK_{II}(0)$.

For each edge pq, γ_{pq} is differentiable at 0 and hence $\gamma_{pq}(x)/x$ extends to a continuous nonnegative function on $\mathbb{R}$. For any γ-harmonic u, define linear conductances γ^u by

$$a^u_{pq} = \frac{\gamma_{pq}(u_p - u_q)}{u_p - u_q}, \qquad \gamma^u_{pq}(x) = a^u_{pq} x.$$

Let A^u be the corresponding Kirchhoff matrix, K_u the Kirchhoff function and Λ_ϕ the Dirichlet to Neumann map. By construction,

$$\gamma_{pq}(u_p - u_q) = \gamma^u_{pq}(u_p - u_q).$$

Thus, u is γ^u-harmonic, and $K(u) = K_u(u)$. Also, $A_0 = \lim_{u \to 0} A_u = DK(0)$.

For $\phi \in \mathbb{R}^B$, let $u = U(\phi)$ be a solution of the Dirichlet problem such that U is continuous. Because u is γ^u-harmonic and $\pi_B(u) = \phi$, we can apply the formula proven above for linear conductances:

$$\Lambda(\phi) = \Lambda_u(\phi) = \big(A^u/A^u_{II}\big)\phi.$$

As noted above, A^u/A^u_{II} depends continuously on the coefficients a^u_{pq}, which depend continuously on u, which depends continuously on ϕ. Therefore, Λ is differentiable at 0 and

$$D\Lambda(0) = \big(A^0/A^0_{II}\big) = DK(0)/DK_{II}(0). \qquad \square$$

Corollary 9. *If each γ_{pq} is C^1, then Λ is C^1.*

Proof. Let $U(\phi)$ be a solution to the Dirichlet problem continuous in ϕ. Then $D\Lambda(\phi) = DK(U(\phi))/DK_{II}(U(\phi))$ is the composition of continuous functions. $\square$

Combining this with results of [1] and [2], we have

Corollary 10. *Suppose γ_{pq} is differentiable and $\gamma'_{pq}(x) > 0$. Then*

(i) $DK_{II}(u)$ is invertible for all $u \in \mathbb{R}^V$.

(ii) For all $\phi \in \mathbb{R}^B$, the Dirichlet problem has a unique solution $U(\phi)$.

(iii) $DU = \begin{pmatrix} I \\ -DK_{II}^{-1} DK_{IB} \end{pmatrix}$, where DK is evaluated at U.

(iv) If γ_{pq} is C^n, then U and Λ are C^n.

Proof. (i) follows from noticing that $DK(\phi)$ is a Kirchhoff matrix for a set of positive linear conductances on G, and then applying Lemma 3.8 of [1]. For (ii), note that by Theorem 1, the current across each edge is uniquely determined by ϕ. Since the conductance functions are strictly increasing, the voltage drop across each edge is uniquely determined, and since the graph is connected, the voltages are uniquely determined. To prove (iii), consider the case $\phi = 0$ and $u = 0$, and let γ^u be as in the previous theorem. Since u is γ^u-harmonic,

$$u = \begin{pmatrix} I \\ -(A^u_{II})^{-1} A^u_{IB} \end{pmatrix} \phi$$

by the results for linear conductances, and (iii) follows by the same reasoning as above. Alternatively, (iii) can be deduced from the implicit function theorem. (iv) follows from the previous corollary, (iii), and repeated application of the chain rule. $\square$

Remark. In fact, [1] shows that DK is symmetric and positive semi-definite and every principal proper submatrix is positive definite. Actually, DK is the Hessian matrix of a convex function. In the process of proving his Theorem 2.4, [2] defines the pseudopower

$$Q(u) = \sum_{p \sim q} q_{pq}(u_p - u_q), \text{ where } q_{pq}(x) = \int_0^x \gamma_{pq}(t)\, dt,$$

and shows it is convex. The Kirchhoff function is the gradient of $\frac{1}{2}Q$, and DK is its Hessian.

Remark. (iii) can be viewed as a nonlinear PDE which is satisfied by U. Given a PDE with a similar form, we can show it has a unique solution using Theorem 1 and Corollary 10.

2 The Inverse Problem

The *inverse conductivity problem* is to find the conductivity function of a network Γ given the graph and the Dirichlet-to-Neumann map. If this is possible, then Γ (or γ) is said to be *recoverable*. To say that all linear conductances are recoverable means that any linear conductance is recoverable *on the assumption that it is linear*. In other words, no two *linear* conductances produce the same Dirichlet-to-Neumann map. The same definition holds with "linear" replaced by "differentiable with positive derivative."

The equations for $D\Lambda$ and DU in Corollary 10 allow us to reduce the inverse problem to the linear case in the following sense:

Proposition 11. *Suppose that all positive linear conductances are recoverable on a graph G. Let Γ be a network on G with differentiable conductances γ_{pq} such that $\gamma'_{pq}(x) > 0$ for all x. Then for each ϕ, $u = U(\phi)$ and the associated current ι are uniquely determined by Λ.*

Proof. The fact that linear conductances are recoverable means that any Kirchhoff matrix A for positive linear conductances on G is uniquely determined by A/A_{II}. In particular, for any f, we can determine $DK(U(\phi))$ from $D\Lambda = DK/DK_{II}$. Knowing DK, we can compute DU using (iii) of Corollary 10. U is uniquely determined by DU because $U(0) = 0$. Similarly, for any edge pq,

$$\iota(p \to q) = \gamma_{pq}(u_p - u_q), \text{ where } u = U(f),$$

so differentiating with respect to ϕ yields

$$\frac{\partial \iota(p \to q)}{\partial \phi_r} = \gamma'_{pq}(u_p - u_q)\left(\frac{\partial u_p}{\partial \phi_r} - \frac{\partial u_q}{\partial \phi_r}\right)$$

for each $i \in B$. The quantity on the right can be computed from DK and DU. Therefore, $\iota(p \to q)$ is uniquely determined for each f. $\square$

Unfortunately, this is not enough to guarantee recoverability. Suppose we want to find $\gamma_{pq}(x_0)$. *If* there is a γ-harmonic function u such that $u_p - u_q = x_0$, then the previous proposition guarantees that $\phi(p)$, $\phi(q)$, and $\iota(p \to q)$ are uniquely determined by Λ. In that case, $\gamma_{pq}(x_0)$ is also uniquely determined. However, in some cases, there is no γ-harmonic function with $u_p - u_q = x_0$, as shown in [2] section 4, and this implies Γ is not recoverable.

To guarantee recoverability, we need an additional hypothesis on the graph.

2.1 Layerable Graphs

The inverse problem has been studied much more thoroughly in the linear case. The typical approach is to determine conductance of a boundary edges (edges between two boundary vertices) and boundary spikes (edges between an interior vertex and a boundary vertex of degree one). Knowing the conductances near the boundary, we can determine some interior voltages and currents for γ-harmonic functions, which in turn gives us information about conductances deeper in the network.

Alternatively, after recovering the conductance of each edge, we can remove it from the graph and update the Dirichlet-to-Neumann map (although this may not be feasible numerically for nonlinear conductances). A boundary edge is *deleted* (simply removed from the graph). A boundary spike is *contracted*. That is, if p is the boundary vertex and q is the interior vertex, the edge is removed and p and q are replaced by a single boundary vertex q', which occupies the position of q in the graph.

We call a graph G *layerable* if at has at least two boundary vertices and there exists a sequence of graphs $G_0 = G$, G_1, $\ldots$, G_N such that

- G_{n+1} is obtained from G_n by deleting a boundary edge or contracting a spike.
- In each G_n, there are two disjoint paths from each interior vertex to the boundary.
- G_N has no interior vertices and no edges.

We will prove that for layerable graphs, the inverse problem for nonlinear conductivities reduces to the linear case. It is reasonable to assume the graph is layerable because most known recoverable graphs are layerable. We need the following lemma:

Proposition 12. *Let $\Gamma = (G, \gamma)$, where G is layerable and each γ_{pq} is strictly increasing. Choose an edge ij and a constant $C \in \mathbb{R}$. There exists a γ-harmonic potential u such that $u_i - u_j = x_0$.*

Proof. G_L be the smallest graph in the sequence described above which includes the edge ij. We show there is a γ-harmonic potential u^L on G_L with $u^L(i) - u^L(j) = x_0$, then we show that it can be extended to G_{L-1}, G_{L-2}, $\ldots$.

Notice that ij is either a boundary edge or a spike of G_L. If it is a boundary edge, then let u^L be the solution of the Dirichlet problem with $u^L(i) = x_0$ and $u^L(k) = 0$ for each other boundary vertex k.

If ij is a spike, assume without loss of generality that i is the interior vertex. By Theorem 9.4 of [2], the Neumann problem has a solution for any boundary currents which add up to zero on each connected component of the graph. By assumption, there exist two disjoint paths from i to the boundary of G_L, which implies that G_{L+1} has at least two boundary vertices in the component which includes i'. Let $r \neq i'$ be a boundary vertex in the same component as i' of G_{L+1}. Let u^{L+1} be a solution to the Neumann problem on G_{L+1} with $J_{L+1}(i') = \gamma_{ij}(x_0)$, $J^{L+1}(r) = -\gamma_{ij}(x_0)$, all other boundary currents zero, and $u^{L+1}(i') = x_0$. Then let $u^L(k) = u^{L+1}(k)$ for all $k \neq j$ and $u^L(j) = 0$. This ensures $J^L(i) = 0$.

This completes the base case. Now suppose that there is a γ-harmonic function u_n on G_n, and I will show it can be extended to a harmonic function on G_{n-1}. If G_{n-1} is obtained from G_n by adjoining a boundary edge pq, let $u^{n-1}(k) = u^n(k)$ for all k. $J^{n-1}(k)$ will be the same as $J^n(k)$ except at p and q, so u_n is γ-harmonic.

If G_{n-1} is obtained from G_n by adjoining a spike pq where p is the boundary vertex and q the interior vertex, then let $u^{n-1}(k) = u^n(k)$ for all $k \neq p, q$, let $u^{n-1}(q) = u^n(q')$, and let $u^{n-1}(p) = u^n(q') - \gamma_{pq}^{-1}(J^n(q'))$. This ensures that $J^{n-1}(q) = 0$.

By induction, we can extend u to all of G. $\square$

Theorem 13. *Suppose G is a layerable graph on which all positive linear conductances are recoverable. If γ is a conductivity with each γ_{pq} is differentiable with $\gamma'_{pq} > 0$, then (G, γ) is recoverable.*

3 The Heat Equation

If Kirchhoff's current law is analoguous to Laplace's equation, and if K is analogous to the Laplacian, we obtain a natural analogue to the heat equation by letting u vary with respect to a real variable t. For $u : I \subset \mathbb{R} \to \mathbb{R}^V$, the (discrete nonlinear) heat equation is

$$u' = -K(u), \text{ where } u' = \frac{du}{dt}.$$

In this version, there is no distinction between interior and boundary vertices, so it models insulated boundary conditions. For a more general version, let $\phi : [0, \infty) \to \mathbb{R}^B$ and fix $u_0 \in \mathbb{R}^V$ with $\pi_B(u_0) = \phi(0)$. Then consider the initial value problem

$$\pi_I(u') = -\pi_I(K(u)), \quad \pi_B(u) = \phi, \quad u(0) = u_0$$

(homogeneous heat equation with boundary potentials). Let $w = \pi_I(u)$, $w_0 = \pi_I(u_0)$ and write $u = (\phi, w)$. Then we can rewrite this equation in the form

$$w' = -\pi_I(K(\phi, w)), \quad w(0) = w_0,$$

which is simpler and has the advantage of not assuming in its notation that $\phi = \pi_B(u)$ is differentiable. Finally, the inhomogenous problem is given by

$$w' = \theta - \pi_I(K(\phi, w)), \quad w(0) = w_0,$$

where $\theta : [0, \infty) \to \mathbb{R}^I$ is a "forcing function" depending on t.

3.1 Maximum Estimates and Existence

I will show this equation has a unique solution whenever θ and ϕ are continuous. But first, here is an estimate on the solutions, which is needed for the proof of existence. This an analogue of the maximum principle in the continuous heat equation.

Proposition 14 (Maximum estimate)**.** *Suppose ϕ is C^1, and let w be a solution to the above equation on some interval $[0,t]$ and $u = (\phi, w)$. Then*

$$\max_{(p,\tau)\in V\times[0,t]} u_p(\tau) \le \max_{(p,\tau)\in(V\times\{0\})\cup(B\times[0,t])} u_p(\tau) + \int_0^{t_0} \|\theta(\tau)\|_\infty \, d\tau.$$

Proof. Let

$$f(t) = \max_{(p,\tau)\in V\times[0,t]} u_p(\tau), \quad g(t) = \max_{(p,\tau)\in(V\times\{0\})\cup(B\times[0,t])} u_p(\tau).$$

Clearly, f and g are increasing. Now we prove they are Lipschitz for t in a compact interval $[0,t_0]$. Since u is C^1, it is Lipschitz on $[0,t_0]$, so there is an L such that $|u_p(s) - u_p(t)| \le L|s-t|$ for all $p \in V$ and $s,t \in [0,t_0]$. Suppose $0 \le s < t \le t_0$. Suppose that $f(s) < f(t)$. There must be a $(p,\tau_0) \in V\times(s,t]$ with $u_p(\tau_0) = f(t)$. Note $u(s) \le f(s)$, so by the intermediate vallue theorem, there is a $\tau_1 \in [s,\tau_0]$ with $u_p(\tau_1) = f(s)$. Thus,

$$|f(t) - f(s)| = |u_p(\tau_0) - u_p(\tau_1)| \le L|\tau_0 - \tau_1| \le L|t-s|.$$

If $f(t) = f(s)$, then trivially $|f(t) - f(s)| \le |t-s|$. Thus, f is Lipschitz with constant L. By similar reasoning, we see that g is Lipschitz with constant L as well.

This implies f and g are absolutely continuous; hence, they are differentiable almost everywhere and the fundamental theorem of calculus holds. Upon examining the definition of f and g, we see

$$f(0) = g(0) = g(0) + \int_0^0 \|\theta(\tau)\|_\infty \, d\tau.$$

Thus, if we prove $f' \le g' + \|\theta\|_\infty$ wherever the derivatives exist, the lemma will follow because

$$f(t) = f(0) + \int_0^t f' \le f(0) + \int_0^t g' + \int_0^t \|\theta\|_\infty = g(t) + \int_0^t \|\theta\|_\infty .$$

Fix $\tau_0 \in [0,t]$ and suppose $f'(\tau_0)$ and $g'(\tau_0)$ exist. Let $M = f(\tau_0)$. Let $P = \{p \in V : u_p(\tau_0) < M\}$ (which may be empty). If $p \in V \setminus P$, then there is an interval $[\tau_0, \tau_0 + \delta_p)$ in which $u_p(\tau) < M$. If $p \in P \cap I$, then

$$(K(u(\tau_0)))_p = \sum_{q \sim p} \gamma_{pq}(u_p(\tau_0) - u_q(\tau_0)) \geq 0,$$

so

$$u_p'(\tau_0) = \theta_p(\tau_0) - (K(u(\tau_0)))_p \leq \|\theta(\tau_0)\|_\infty - 0.$$

Thus, given $\epsilon > 0$, there is δ_p such that for $0 < h < \delta_p$,

$$u_p(\tau_0 + h) \leq u_p(\tau_0) + (u_p(\tau_0)' + \epsilon)h \leq M + (\|\theta(\tau_0)\|_\infty + \epsilon)h.$$

If $p \in P \cap B$, then for $h > 0$,

$$u_p(\tau_0 + h) - u_p(\tau_0) = u_p(\tau_0 + h) - M \leq g(\tau_0 + h) - g(\tau_0)$$

because $g(\tau_0) \leq f(\tau_0) = M$. There is a δ_p such that for $0 < h < \delta_p$,

$$g(\tau_0 + h) - g(0) \leq (g'(\tau_0) + \epsilon)h,$$

which implies

$$u_p(\tau_0 + h) \leq M + (g'(\tau_0) + \epsilon)h.$$

Let δ be the minimum of the δ_p's over $p \in V = (V \setminus P) \cup (P \cap I) \cup (P \cap B)$. Then using each of the three cases, we have for $0 < h < \delta$,

$$\begin{aligned}\max_{p \in V} u_p(\tau_0 + h) &\leq \max(M, M + (\|\theta(\tau_0)\|_\infty + \epsilon)h, M + (g'(\tau_0) + \epsilon)h) \\ &= M + [\max(\|\theta(\tau_0)\|_\infty, g'(\tau_0)) + \epsilon]h.\end{aligned}$$

This is also true if we replace h with any $\eta \in [0,h]$, which implies

$$\begin{aligned}f(\tau_0 + h) &\leq M + [\max(\|\theta(\tau_0)\|_\infty, g'(\tau_0)) + \epsilon]h \\ &= f(\tau_0) + [\max(\|\theta(\tau_0)\|_\infty, g'(\tau_0)) + \epsilon]h.\end{aligned}$$

Therefore,

$$f'(\tau_0) = \lim_{h \to 0^+} \frac{f(\tau_0 + h) - f(\tau_0)}{h} \leq \max(\|\theta(\tau_0)\|_\infty, g'(\tau_0)) + \epsilon.$$

Taking $\epsilon \to 0$ yields

$$f'(\tau_0) \leq \max(\|\theta(\tau_0)\|_\infty, g'(\tau_0)) \leq g'(\tau_0) + \|\theta(\tau_0)\|_\infty. \qquad \square$$

Corollary 15.

$$\min_{(p,\tau)\in V\times[0,t]} u_p(\tau) \geq \min_{(p,\tau)\in(V\times\{0\})\cup(B\times[0,t])} u_p(\tau) - \int_0^{t_0} \|\theta(\tau)\|_\infty \, d\tau.$$

Proof. Symmetrical to the proof of the proposition. □

Corollary 16.

$$\|w(t)\|_\infty \leq \max\left(\|w_0\|_\infty, \max_{0\leq\tau\leq t} \|\phi(\tau)\|_\infty\right) + \int_0^t \|\theta(\tau)\|_\infty \, d\tau.$$

Here $\|\cdot\|_\infty$ is the infinity norm on $\mathbb{R}^V$ (or $\mathbb{R}^B$ or $\mathbb{R}^I$) given by $\|u\|_\infty = \max_{p\in V} |u_p|$. This follows from combining the proposition and corollary, unwinding the definition of $\|\cdot\|_\infty$, and noting $\|w(t)\|_\infty \leq \max_{0\leq\tau\leq t} \|u(\tau)\|_\infty$. □

Theorem 17 (Existence)**.** *Let Γ be an electrical network. Let $\theta : [0,\infty) \to \mathbb{R}^I$ and $\phi : [0,\infty) \to \mathbb{R}^B$ be continuous and $w_0 \in \mathbb{R}^I$. There exists $w : [0,\infty) \to \mathbb{R}^I$ satisfying*

$$w' = \theta - \pi_I(K(\phi, w)), \quad w(0) = w_0.$$

Proof. First consider the case where each γ_{pq} is C^1 with $\gamma'_{pq} > 0$, and ϕ is C^1. Let S be the set of $t_0 \geq 0$ such that there exists a unique solution to the IVP on $[0, t_0 + \epsilon)$ for some $\epsilon > 0$. [When I say "a unique solution on $[0, t_0 + \epsilon)$," I mean that if we consider any other solution on $[0, t_1)$, the two solutions must agree on the overlap $[0, t_0 + \epsilon) \cap [0, t_1)$.] I will show S is nonempty and it is both open and closed in $[0,\infty)$. It is open by construction.

To show $0 \in S$, note that K is C^1 and hence Lipschitz on compact sets. In particular, $\theta - \pi_I(K(\phi, w))$ is Lipschitz in w and continuous in (t, w) in a neighborhood of $(0, w_0)$. Thus, by the Picard-Lindelof theorem, there is a unique solution to the IVP for t in some interval $(-\epsilon, \epsilon)$. So S is nonempty.

Since S must be an interval, to prove it is closed, it suffices to show that $[0, t_0) \subset S$ implies $t_0 \in S$. If $[0, t_0) \subset S$, then for any $t < t_0$, there is a solution w_t defined on $[0, t)$; by pasting these solutions together, we obtain a unique solution w on $[0, t_0)$. Let

$$h(t) = \max\left(\|w_0\|_\infty, \max_{0\leq\tau\leq t} \|\phi(\tau)\|_\infty\right) + \int_0^t \|\theta(\tau)\|_\infty \, d\tau.$$

This is continuous on $[0,\infty)$ and independent of w, and $\|w(t)\|_\infty \leq h(t)$ for $t \in [0, t_0)$. This implies that $w(t)$ is bounded on $[0, t_0)$. Since K is bounded on bounded sets and θ and ϕ are continuous on $[0,\infty)$, $w' = \theta - \pi_I(K(\phi, w))$ is also bounded. Therefore, w' is integrable on $[0, t_0)$, and

$$\lim_{t\to t_0^-} w(t) = w_0 + \lim_{t\to t_0^-} \int_0^t w'(\tau)\, d\tau$$

exists. Thus, w extends to a continuous function on $[0, t_0]$.

By the Picard-Lindelof theorem, there is $\tilde{w} : (t_0-\epsilon, t_0+\epsilon) \to \mathbb{R}^I$ satisfying

$$\tilde{w}' = \theta - \pi_I(K(\phi, \tilde{w})), \quad \tilde{w}(t_0) = w(t_0).$$

Let $\overline{w} = w$ on $[0, t_0]$ and $\overline{w} = \tilde{w}$ on $[t_0 + \epsilon)$. Then $\overline{w}$ is continuous and satisfies the differential equation for $t \neq t_0$, but by L'Hopital's rule,

$$\begin{aligned}
\lim_{t\to t_0} \frac{\overline{w}(t) - \overline{w}(t_0)}{t - t_0} &= \lim_{t\to t_0} \overline{w}'(t) \\
&= \lim_{t\to t_0} (\theta(t) - \pi_I(K(\phi, \overline{w}(t)))) \\
&= \theta(t_0) - \pi_I(K(\phi, \overline{w}(t_0))),
\end{aligned}$$

so $\overline{w}$ is a solution on $[0, t_0 + \epsilon)$, and it is easy to verify it is unique. Thus, $t_0 \in S$ as desired, and S must be $[0, \infty)$.

For each pq, let $\{\gamma^n_{pq}\}$ be the sequence of C^1 conductances with positive derivative converging to γ_{pq} uniformly on compact sets, and let K_n be the corresponding Kirchhoff function. Let $\{\phi_n\}$ be a sequence of C^1 functions converging uniformly on compact sets to ϕ. Then $K_n \to K$ uniformly on compact sets. For each n, there is a w_n satisfying

$$w_n' = \theta - \pi_I(K_n(\phi, w_n)), \quad w_n(0) = w_0.$$

Let $h_n(t)$ be the function corresponding to $\gamma_{n,pq}$ and w_n given by the same formula as $h(t)$. Since $\phi_n \to \phi$ uniformly on compact sets, we know that

$$\sup_{\substack{1\le n<\infty \\ 1\le\tau\le t}} \|\phi_n(\tau)\|_\infty$$

is finite for any t. It is increasing and in particular bounded on compact sets. Thus, $g(t) = \sup_{1\le n<\infty} h_n(t)$ is finite, and it is bounded for t in a compact set. Since $\|u_n(\tau)\|_\infty \le g(t)$, we know the ϕ_n's and w_n's are uniformly bounded on compact sets. It follows that $K_n(\phi_n, w_n)$ is uniformly bounded on compact sets, and so is $w_n' = \theta - K(\phi_n, w_n)$. So for any t, there is an $M(t)$ with $|w_n'(\tau)| \le M(t)$ for all $\tau \in [0, t]$ and $n \ge 1$. In particular,

$\{w_n\}$ is equicontinuous and pointwise bounded, so by the Arzela-Ascoli theorem, there is a subsequence $\{w_{n_k}\}$ converging uniformly on compact sets to a function w. By Lemma 4, the composition $K_{n_k}(\phi_{n_k}, w_{n_k}) \to K(\phi, w)$ uniformly on compact sets, and hence

$$w(t) = w_0 + \lim_{k\to\infty} \int_0^t (\theta - \pi_I(K_{n_k}(\phi, w_{n_k}))) = w_0 + \int_0^t (\theta - \pi_I(K(\phi, w))),$$

and by the fundamental theorem of calculus, $w' = \theta - \pi_I(K(\phi, w))$ as desired. $\square$

Corollary 18. *The solution constructed satisfies the estimates of Lemma 14, Corollary 15, and Corollary 16.*

Proof. Let w_{n_k} be as above and $u_{n_k} = (\phi, w_{n_k})$, $u = (\phi, w)$. Since $u_{n_k} \to u$ uniformly on compact sets, and each u_{n_k} satisfies the estimates, so does u. $\square$

3.2 Difference Estimates and Uniqueness

Let $\|u\|_\alpha = \left(\sum_{p\in V} |u_p|^\alpha\right)^{1/\alpha}$ for $\alpha \in [1, \infty)$ and $\|u\|_\infty = \max_{p\in V} |u_p|$. Observe that $\|u\|_\alpha$ is continuous in α and $\lim_{\alpha\to\infty} \|u\|_\alpha = \|u\|_\infty$.

Lemma 19 (First difference estimate)**.** *Suppose that w and $\tilde{w} : [0, \infty) \to \mathbb{R}^V$ satisfy*

$$w' = \theta - K(\phi, w), \quad \tilde{w}' = \theta - K(\phi, \tilde{w}).$$

If $\alpha \in [1, \infty]$, then $\|w(t) - \tilde{w}(t)\|_\alpha$ is a decreasing function of t.

Proof. Suppose $1 < \alpha < \infty$. Let $f(x) = |x|^\alpha$. Note that $f'(x) = \alpha|x|^{\alpha-1} \operatorname{sgn} x$ is increasing, and $f(0) = 0 = f'(0)$. Let

$$g(t) = \|w(t) - \tilde{w}(t)\|_\alpha^\alpha = \sum_{p\in I} f(w_p(t) - \tilde{w}_p(t)).$$

Differentiate and apply $w' - \tilde{w}' = \theta - \pi_I(K(\phi, w)) - \theta + \pi_I(K(\phi, \tilde{w})) = -\pi_I(K(u) - K(\tilde{u}))$:

$$\begin{aligned} g' &= \frac{d}{dt} \sum_{p\in I} f(w_p - \tilde{w}_p) = \sum_{p\in I} f'(w_p - \tilde{w}_p)(w_p' - \tilde{w}_p') \\ &= -\sum_{p\in I} f'(u_p - \tilde{u}_p)\big((K(u))_p - (K(\tilde{u}))_p\big). \end{aligned}$$

We can take the sum over V rather than I because for $p \in B$, $f'(u_p - \tilde{u}_p) = f'(\phi_p - \phi_p) = 0$. Then by definition of K,

$$\begin{aligned} g' &= -\sum_{p \in V} f'(u_p - \tilde{u}_p) \sum_{q \sim p} \big(\gamma_{pq}(u_p - u_q) - \gamma_{pq}(\tilde{u}_p - \tilde{u}_q)\big) \\ &= -\frac{1}{2} \sum_{pq \in E} \big(f'(u_p - \tilde{u}_p) - f'(u_q - \tilde{u}_q)\big)\big(\gamma_{pq}(u_p - u_q) - \gamma_{pq}(\tilde{u}_p - \tilde{u}_q)\big). \end{aligned}$$

Note that both f' and γ_{pq} are increasing, so $f'(u_p - \tilde{u}_p) - f'(u_q - \tilde{u}_q) \geq 0$ if and only if $u_p - \tilde{u}_p \geq u_q - \tilde{u}_q$ if and only if $u_p - u_q \geq \tilde{u}_p - \tilde{u}_q$ if and only if $\gamma_{pq}(u_p - u_q) - \gamma_{pq}(\tilde{u}_p - \tilde{u}_q) \geq 0$. Thus, each term in the sum is nonnegative, so $g' \leq 0$, and g is decreasing. Thus, $\|w - \tilde{w}\|_\alpha = g^{1/\alpha}$ is decreasing.

The cases for $\alpha = 1$ and $\alpha = \infty$ follow by taking $\alpha \to 1^+$ and $\alpha \to \infty$. □

Theorem 20 (Uniqueness)**.** *There is at most one solution to*

$$w' = \theta - \pi_I(K(\phi, w)), \quad w(0) = w_0.$$

Proof. If w and $\tilde{w}$ are two solutions, then $\|w - \tilde{w}\|_\alpha$ is decreasing. But $\|w(0) - w(0)\|_\alpha = 0$, so $w(t) - \tilde{w}(t) = 0$ for all t. □

Suppose we have two solutions w and $\tilde{w}$ with different initial data. Since $\|w - \tilde{w}\|_\alpha$ is decreasing for all α, it has a limit as $t \to \infty$. Since this is true for all values of α, it would be reasonable to suppose

Theorem 21. *Suppose that w and $\tilde{w} : [0, \infty) \to \mathbb{R}^V$ satisfy*

$$w' = \theta - K(\phi, w), \quad \tilde{w}' = \theta - K(\phi, \tilde{w}).$$

Then $\lim_{t \to \infty}(w(t) - \tilde{w}(t))$ exists.

To prove this, it suffices to prove the more general

Proposition 22. *If $u : [0, \infty) \to \mathbb{R}^d$ is continuous and $\lim_{t \to \infty} \|u(t)\|_\alpha$ exists for all α, then $\lim_{t \to \infty} u(t)$ exists.*

Proof. We first consider the case where $u : [0, \infty) \to A$, where $A = \{x = (x_1, \dots, x_n) \in \mathbb{R}^d : x_1 \geq x_2 \geq \cdots \geq x_d \geq 0\}$. Suppose $\lim_{t \to \infty} \|u(t)\|_\alpha$ exists for all α, and I will show $\lim_{t \to \infty} u(t) = u_0$ for some $u_0 \in A$.

Since $\|u(t)\|_\infty$ converges, there is an M with $\|u(t)\|_\infty \leq M$ for all t. Let $S = A \cap \{x : \|x\|_\infty \leq M\}$ which is compact. Since $u(n) \in S$, there is a subsequence $\{u(n_k)\}$ converging to a $u_0 \in S$. Then

$$\lim_{t \to \infty} \|u(t)\|_\alpha = \lim_{n \to \infty} \|u(n_k)\| = \|u_0\|_\alpha.$$

Suppose for the sake of contradiction that $u(t)$ does not converge to u_0. Then there is an $\epsilon > 0$ and a sequence $t_n \to \infty$ with $\|u(t_n) - u_0\| \geq \epsilon$ for all n. But then by compactness of S, there is a subsequence $\{t_{n_k}\}$ such that $u(t_{n_k}) \to v \in S$. If we can prove that $v = u$, then we will have a contradiction and the proof will be complete.

By definition of A, $u_1 = \|u\|_\infty$ and $v_1 = \|v\|_\infty$, so $u_1 = v_1$. For $\alpha \neq \infty$,

$$\|u - u_1 e_1\|_\alpha = (\|u\|_\alpha^\alpha - u_1^\alpha)^{1/\alpha} = (\|v\|_\alpha^\alpha - v_1^\alpha)^{1/\alpha} = \|v - v_1 e_1\|_\alpha ,$$

where e_1 is the standard basis vector for $\mathbb{R}^d$. Then, taking $\alpha \to \infty$, we have $\|u - e_1 u_1\|_\infty = \|v - e_1 v_1\|_\infty$. By construction, $u_2 = \|u - e_1 u_1\|_\infty$ and $v_2 = \|v - e_1 v_1\|_\infty$. Repeating the argument yields $u_3 = v_3$ and so on, so that $u = v$.

Now I remove the restriction that u takes values in A. Suppose $u : [0, \infty) \to \mathbb{R}^d$ is continuous and $\lim_{t\to\infty} \|u(t)\|_\alpha$ exists for all α. For σ in the symmetric group S_d and $\tau \in \{-1, 1\}^d$, let

$$A_{\sigma,\tau} = \{x \in \mathbb{R}^d : |x_{\sigma(1)}| \geq \cdots \geq |x_{\sigma(d)}| \geq 0,\ \tau(k) x_k \geq 0 \text{ for } k = 1, \ldots, d\}.$$

Then $A_{\sigma,\tau}$ differs from A by a reflection and permutation of coordinates, and $\mathbb{R}^d = \bigcup_{\sigma,\tau} A_{\sigma,\tau}$. For any $u \in \mathbb{R}^d$, we can find an $\overline{u} \in A$ such that $\|\overline{u}\|_\alpha = \|u\|_\alpha$ for all α, by changing the signs and permuting the order of the coordinates of u. If $\overline{u}(t)$ is the point in A corresponding to $u(t)$, then by the previous case, $\lim_{t\to\infty} \overline{u}(t) = u_0$ for some $u_0 \in A$. (The reader may verify that $\overline{u}$ is continuous.)

Let $v_1 = u_0$, and let $v_2, \ldots, v_m$ be all the points which differ from v_1 by sign changes and permutations of the coordinates. We can choose $\epsilon > 0$ such that the open balls $B(v_j, \epsilon)$ are disjoint, and for each j,

$$B(v_j, \epsilon) \subset \bigcup_{\substack{\sigma,\tau \\ v_j \in A_{\sigma,\tau}}} A_{\sigma,\tau}.$$

There is an M such that $|\overline{u}(t) - u_0| < \epsilon$ for all $t \geq M$. This implies that $u([M, \infty))$ is contained in $\bigcup_{j=1}^m B(v_j, \epsilon)$, but since $u([M, \infty))$ is connected it must be contained in only one of the balls $B(v_j, \epsilon)$. If $t \geq M$, and $u(t) \in A_{\sigma,\tau}$, then $v_j \in A_{\sigma,\tau}$, and since $A_{\sigma,\tau}$ is congruent to A, we have $\|u(t) - v_j\|_\infty = \|\overline{u}(t) - u_0\|_\infty$. Thus, $\|u(t) - v_j\|_\infty \to 0$ as $t \to \infty$. $\square$

Proposition 23 (Second difference estimate)**.** *Suppose*

$$\frac{d}{dt}\pi_I(u) = \theta - \pi_I(K(u)), \quad \frac{d}{dt}\pi_I(v) = \zeta - \pi_I(K(v)), \quad \pi_B(u) = \pi_B(v).$$

Then

$$\|u(t) - v(t)\|_\infty \leq \big(1 + \|u(0) - v(0)\|_\infty\big) \exp \int_0^t \|\theta(\tau) - \zeta(\tau)\|_\infty \, d\tau.$$

Proof. Fix $\alpha \in (1, \infty)$, and let $f(x) = |x|^\alpha$. Let

$$g(t) = \|u(t) - v(t)\|_\alpha^\alpha = \sum_{p \in V} f(u_p - v_p) = \sum_{p \in I} f(u_p - v_p)$$

since $\pi_B(u) = \pi_B(v)$. Then

$$\begin{aligned} g' &= \sum_{p \in I} f'(u_p - v_p)(u_p' - v_p') = \sum_{p \in I} f'(u_p - v_p)(\theta_p - (K(u))_p - \zeta + (K(v))_p) \\ &= -\sum_{p \in V} f'(u_p - v_p)((K(u))_p - (K(v))_p) + \sum_{p \in I} f'(u_p - v_p)(\theta_p - \zeta_p) \end{aligned}$$

since $\pi_I(K(v)) = 0$ and $f'(u_p - v_p) = 0$ for $p \in B$. By the same reasoning as in the other lemma, the first term is less than or equal to zero. Thus,

$$g' \leq \sum_{p \in I} f'(u_p - v_p)(\theta_p - \zeta_p) \leq \|\theta - \zeta\|_\infty \sum_{p \in I} f'(u_p - v_p)$$

Note

$$f'(x) = \alpha |x|^{\alpha - 1} \leq \alpha \max(1, |x|^\alpha) \leq \alpha(1 + f(x)),$$

so that

$$g' \leq \|\theta - \zeta\|_\infty \sum_{p \in I} \alpha(1 + f(u_p - v_p)) = \alpha \|\theta - \zeta\|_\infty (|I| + g).$$

Thus, $g'/(|I| + g) \leq \alpha(\|\theta - \zeta\|_\infty)$. Integrate from 0 to t:

$$\log(|I| + g(t)) - \log(|I| + g(0)) = \int_0^t \frac{g'(\tau)}{|I| + g(\tau)} \, d\tau \leq \alpha \int_0^t \|\theta(\tau) - \zeta(\tau)\|_\infty \, d\tau.$$

Let $h(t)$ be the integral on the right. Then by exponentiation,

$$|I| + g(t) \leq (|I| + g(0)) e^{\alpha h(t)}.$$

Recalling $g(t) = \|u(t) - v(t)\|_\alpha^\alpha$, we have

$$\|u(t) - v(t)\|_\alpha^\alpha \leq |I| + g(t) \leq (|I| + \|u(0) - v(0)\|_\alpha^\alpha) e^{\alpha h(t)}.$$

Raising this to the $1/\alpha$ and applying Minkowski's inequality,

$$\|u(t) - v(t)\|_\alpha \leq \big(|I| + \|u(0) - v(0)\|_\alpha^\alpha\big)^{1/\alpha} e^{h(t)} \leq \big(|I|^{1/\alpha} + \|u(0) - v(0)\|_\alpha\big) e^{h(t)}.$$

Taking $\alpha \to \infty$,

$$\|u(t) - v(t)\|_\infty \leq \big(1 + \|u(0) - v(0)\|_\infty\big) e^{h(t)}. \qquad \square$$

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