NONLINEAR ELECTRICAL NETWORKS

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Abstract. We consider electrical conductance networks in which the conductors are non-ohmic, so that current is not given by a linear function of voltage. This generalizes the directed current networks of Orion Bawdon. Using convex functions, we give short proofs of the key results of Christianson and Erickson [1]. We show the well-definedness of the Dirichlet-to-Neumann map for arbitrary nonlinear conductance networks. We also consider the dual notion of a nonlinear resistance network, and show that the Neumann-to-Dirichlet map is well-defined for connected graphs. In the process, we demonstrate existence and partial uniqueness results for both problems.

Contents

1. Introduction

The concept of a directed current network was introduced by Orion Bawdon around 2005, to model networks including components like diodes[1]. In 2007, Kari Christianson and Lindsay Erickson [1] established basic results about these networks that had been conjectured by Bawdon.

In this paper I present much simpler proofs of their important results, in a much more general setting. I consider general conductors for which the current flow is given as a function of the voltage drop, subject to some natural constraints. I use a basic argument involving convex functions to establish the existence of solutions, and to characterize the solution set. I also show that the solution is unique under some slight additional assumptions. Moreover, I demonstrate that the Dirichlet-to-Neumann map always exists.

I then carry out the exact same arguments in a dual situation, where voltage is a function of current. In this case, I show that the Neumann problem has at least one solution, and that the Neumann-to-Dirichlet map usually exists.

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[1]Confusingly enough, the term "directed network" had previously been used by Jaime Lust and Dominic DiPalantino to refer to a completely different sort of network, a linear network in which the current that flowed from one node to another could differ from the current flowing in the opposite direction. This has very little in common with anything considered here.

2. Nonlinear Conductance Networks

Definition 2.1. *A* graph with boundary *is a triple* $\Gamma = (V, \partial V, E)$*, where* V *is a set of vertices,* $\partial V \subseteq V$ *is a nonempty set of boundary vertices, and* $E \subseteq V \times V$ *is a symmetric, irreflexive relationship on* V*. That is,* $(i,i) \notin E$ *for every* $i \in V$*, and* $(i,j) \in E \iff (j,i) \in E$ *for every* $i, j \in V$*. The* interior nodes *are the nodes in* $\operatorname{int} V = V \setminus \partial V$*.*

Definition 2.2. *A* nonlinear conductance network *is a pair* (Γ, γ)*, where* Γ *is a graph with boundary, and* γ *is a map which assigns to each* $(i,j) \in E$ *a conductance function* $\gamma_{ij} : \mathbb{R} \to \mathbb{R}$*, subject to the following constraints:*

- $\gamma_{ij}(x) = -\gamma_{ji}(-x)$
- $\gamma_{ij}(0) = 0$
- $\gamma_{ij}(x)$ *is a weakly increasing function of* x*, i.e.,* $\gamma_{ij}(x_1) \geq \gamma_{ij}(x_2)$ *if* $x_1 \geq x_2$*.*
- $\gamma_{ij}(x)$ *is a continuous function of* x*.*

The interpretation is that $\gamma_{ij}(u(i) - u(j))$ is the current moving from i to j, if the voltages at i and j are $u(i)$ and $u(j)$.

The restriction to symmetric graphs is not as limiting as it would be in the framework of directed networks. In fact, nonlinear conductance networks generalize directed networks, since the conductance function can be chosen to be of the form $\gamma_{ij}(x) = c_{ij}x$ if $x \geq 0$ and $\gamma_{ij}(x) = 0$ otherwise, for some constant c.

Definition 2.3. *A* potential function *on a network* (Γ, γ) *is a function* $u : V \to \mathbb{R}$*. The* current from node i to j *is*

$$I_{ij}(u) = \gamma_{ij}(u(i) - u(j)).$$

The total current coming out of $i \in V$ *is given by*

$$I_i(u) = \sum_{(i,j)\in E} I_{ij}(u).$$

. A potential function u *is* harmonic *if* $I_i(u) = 0$ *for all* $i \in \operatorname{int} V$*.*

Note that $I_{ij} = -I_{ji}$. Harmonicity corresponds to Kirchhoff's Current Law being satisfied by the network.

Definition 2.4. *The* Dirichlet problem *on a network* (Γ, γ) *is defined as follows: given* $\phi : \partial V \to \mathbb{R}$*, extend* ϕ *to a harmonic function* u *defined on all of* V*.*

Our goal is to show that this problem always admits a solution, and moreover, $I_{ij}(u)$ is uniquely determined by ϕ.

3. Convexity

The main tool used to establish these results is convexity. In this section we review some basic definitions and facts about convex functions.

Definition 3.1. *A function* $f : \mathbb{R}^k \to \mathbb{R}$ *is* convex *if*

$$tf(x) + (1-t)f(y) \geq f(tx + (1-t)y)$$

for every $x, y \in S$*,* $t \in [0,1]$*.*

This means that the line segment between any two points on the graph of f is above or on the graph. Equivalently, the set of points above f, $\{(x,y) : y \geq f(x)\}$ is convex.

The following properties of convex functions will be useful later:

Lemma 3.2. *If* f *and* g *are convex, so is their sum* $f + g$*.*

(Proof left to reader)

Lemma 3.3. *If* $f : \mathbb{R}^1 \to \mathbb{R}$ *is continuously differentiable, and* $f'(x)$ *is weakly increasing, then* f *is convex.*

Proof. Suppose that $tf(x) + (1-t)f(y) < f(tx + (1-t)y)$ for some t, x, y. Without loss of generality, $x < y$ and $0 < t < 1$. Then

$$\frac{f(tx+(1-t)y)-f(x)}{(tx+(1-t)y)-x} = \frac{f(tx+(1-t)y)-f(x)}{(1-t)(y-x)} > \frac{f(y)-f(tx+(1-t)y)}{(y-x)t} = \frac{f(y)-f(tx+(1-t)y)}{y-(tx+(1-t)y)}$$

so by the mean value theorem $f'(a) > f'(b)$ for some $a \leq tx + (1-t)y \leq b$ contradicting the assumption on f'. □

Lemma 3.4. *If $f\ \mathbb{R}^k \to \mathbb{R}$ is convex and continuously differentiable, and $x_0 \in \mathbb{R}^k$ is a critical point of f, then x_0 is a global minimum, i.e., $f(x) \geq f(x_0)$ for all x.*

Proof. Suppose for the sake of contradiction that $f(x) < f(x_0)$. Then for $t \in [0,1]$ we have

$$f(tx + (1-t)x_0) \leq tf(x) + (1-t)f(x_0),$$

so that

$$0 = \lim_{t\to 0} \frac{f(x_0 + t(x-x_0)) - f(x_0)}{t} \leq \lim_{t\to 0} \frac{tf(x) + (1-t)f(x_0) - f(x_0)}{t} = f(x) - f(x_0) < 0,$$

a contradiction. □

Of course, the converse is true - any global minimum of f will be a critical point.

Lemma 3.5. *If f is a convex function and $f(x) \geq B$ for all x, then the set $S = \{x\ :\ f(x) = B\}$, which is empty or the set of global minima, is convex.*

Proof. If $x, y \in S$, then $B \leq f(tx+(1-t)y) \leq tf(x)+(1-t)f(y) = tB+(1-t)B = B$, so $tx+(1-t)y \in S$. □

It follows then that the critical points of a continuously differentiable convex function form a convex set.

4. Pseudopower

Definition 4.1. *For u a potential function on (Γ, γ) and $(i,j) \in E$, the* pseudopower along (i,j) *is given by $p_{ij}(u(i) - u(j))$, where*

$$p_{ij}(x) = \int_0^x \gamma_{ij}(x)\, dx.$$

The total pseudopower *of u is the sum*

$$p(u) = \sum_{(i,j)\in E} p_{ij}(u(i) - u(j)).$$

For the case of traditional linear resistance networks and the directed case, we have

$$p_{ij}(u_i - u_j) = \frac{1}{2}(u_i - u_j) f_{ij}(u_i - u_j),$$

so the pseudopower is just half the power being dissipated along an edge. In the more general case, however, this interpretation disappears.

Lemma 4.2. *The total pseudopower is a convex, continuously differentiable function of x, bounded below by 0.*

Proof. Since $\gamma_{ij}(0) = 0$ and γ_{ij} is weakly increasing, $p_{ij}(x) \geq 0$ for all x, with equality if (but not only if) $x = 0$. Therefore, the total pseudopower $p(u)$ is also bounded below by 0. Also, each $p_{ij}(x)$ is a convex, continuously differentiable function of x, and so the total pseudopower is also continuously differentiable. It is convex because for $t \in [0,1]$ and $u, w :\ V \to \mathbb{R}$,

$$p(tu + (1-t)w) = \sum_{(i,j)\in E} p_{ij}(tu(i) - tu(j) + (1-t)w(i) - (1-t)w(j)) \leq$$

$$\sum_{(i,j)\in E} (tp_{ij}(u(i) - u(j)) + (1-t)p_{ij}(w(i) - w(j))) = tp(u) + (1-t)p(w),$$

using the convexities of the individual p_{ij}. □

The usefulness of pseudopower lies in the following lemma:

Lemma 4.3. *When the boundary values of u are fixed to equal ϕ, the critical points of $p(u)$ are the solutions to the Dirichlet Problem with boundary potential ϕ.*

Proof. Suppose that u is a critical point. Then for $i \in \operatorname{int} V$,

$$0 = \frac{\partial p(u)}{\partial u(i)} = \sum_{j,\ (i,j)\in E} \frac{\partial p_{ij}(u(i) - u(j))}{\partial u(i)} = \sum_{j,\ (i,j)\in E} \gamma_{ij}(u(i) - u(j)) = I_i(u).$$

□

Theorem 4.4. *For any nonlinear conductance network (Γ, γ), any boundary potential $\phi : \partial V \to \mathbb{R}$, the solutions to the Dirichlet problem with boundary potential ϕ form a convex subset of $\mathbb{R}^{|V|}$.*

Proof. This follows immediately from Lemmas 3.4, 3.5, and 4.3. □

5. Existence

To prove that the Dirichlet problem always has a solution, we just need to show that for fixed boundary potential ϕ, the minimal value of $p(u)$ is attained somewhere. (In what follows, $p(u)$ will basically be seen as a function of the non-boundary voltages). Let m and M be the minimum and maximum values of ϕ, respectively. Let K be the set of u which are bounded between m and M. This set K is a hypercube, so it is compact, and p attains a minimum on K, because it is continuous and bounded below by 0. Let the minimum value be B.

Lemma 5.1. *If u has boundary voltages ϕ, then $p(u) \geq B$.*

Proof. Construct a new potential function u' by

- $u'(x) = u(x)$ if $m \leq u(x) \leq M$
- $u'(x) = M$ if $u(x) > M$
- $u'(x) = m$ if $u(x) < m$

for $x \in \operatorname{int} V$. Then clearly $u' \in K$, so that $p(u') \geq B$, and also, for any $(i,j) \in E$, $u'(i) - u'(j)$ has the same sign as $u(i) - u(j)$ but is not greater in magnitude. Therefore, $p_{ij}(u'(i) - u'(j)) \leq p_{ij}(u(i) - u(j))$, since $p_{ij}(x)$ is weakly increasing for $x \geq 0$ and weakly decreasing for $x \leq 0$. It follows that $p(u) \geq p(u') \geq B$. □

Theorem 5.2. *The Dirichlet problem always has at least one solution.*

Proof. By the previous lemma, B is a global lower bound for $p(u)$. Therefore, any $u_0 \in K$ which attains the value B will be a global minimum, and therefore a solution to the Dirichlet problem by Lemmas 3.4 and 4.3. Such a u_0 exists because K is compact. □

6. Currents Determined

Although there may be multiple solutions to the Dirichlet problem, we show in this section that the currents are completely determined.

Theorem 6.1. *If u and w are two solutions to the Dirichlet problem for the same boundary potential ϕ, then $I_{ij}(u) = I_{ij}(w)$ for every $(i,j) \in E$.*

Proof. Define a relation R on V by $i\ R\ j$ if $(i,j) \in E$ and $I_{ij}(w) > I_{ij}(u)$. We want R to be the empty relation. Note that $i\ R\ j$ implies $\gamma_{ij}(w(i) - w(j)) > \gamma_{ij}(u(i) - u(j))$, so that $w(i) - w(j) > u(i) - u(j)$.

Let $>$ be the transitive closure of R, i.e., $i > j$ if there are some $n \geq 1$, $a_0 = i, a_1, \ldots, a_n = j$ such that $a_k\ R\ a_{k+1}$, for $0 \leq k < n$. If $i > j$, then

$$(w(i) - w(j)) - (u(i) - u(j)) = \sum_{k=0}^{n-1} (w(a_k) - w(a_{k+1})) - (u(a_k) - u(a_{k+1})) > 0. \tag{1}$$

In particular, then $i > i$ can never be true, so $>$ is irreflexive. Therefore, it is a strict partial order (an irreflexive, transitive relation). Define $i \geq j$ if $i > j$ or $i = j$. Then $\geq$ is a partial order.

We want to show that $i > j$ never happens for any i, j. Suppose that $i > j$. Since V is a finite set, we can find a maximal $i' \in V$ such that $i \geq i$ and a minimal $j' \in E$ such that $j \geq j'$. That is, $i' \geq i > j \geq j'$, and there are no i'' or j'' with $i'' > i'$ or $j' > j''$.

It turns out that both i' and j' must be on the boundary. This is true for i' because

$$I_{i'}(w) = \sum_{k,\ (i',k)\in E} I_{i'k}(w) > \sum_{k,\ (i',k)\in E} I_{i'k}(u) = I_{i'}(u),$$

where the central inequality follows from the fact that at least one k satisfies $i' \ R\ k \iff I_{i'k}(w) > I_{i'k}(u)$, but no k satisfies $k\ R\ i' \iff I_{ki'}(w) > I_{ki'}(u) \iff I_{i'k}(w) < I_{i'k}(u)$. A similar argument holds for j'.

Then by Equation (1) we have $0 = (\phi(i') - \phi(j')) - (\phi(i') - \phi(j')) = (w(i') - w(j')) - (u(i') - u(j')) > 0$, a contradiction. So no i, j satisfy $i > j$, and therefore, no i, j satisfy $i\ R\ j$, and so $I_{ij}(w) \leq I_{ij}(u)$ for all i, j. Of course this implies $I_{ij}(w) = I_{ij}(u)$. □

This result allows us to describe what the set of solutions will look like. If f is a continuous weakly increasing function, then the set $\{x\ :\ f(x) = a\}$ for some a is either a closed interval or a closed half-line. Therefore, the set of solutions $V_\phi = \{u\ :\ u \text{ is a solution to the Dirichlet problem with boundary values } \phi\}$ can be defined using equations of the form

$$u(i) = \phi(i),$$

$$u(i) - u(j) \leq \alpha_{ij}$$

for various α_{ij}. In particular, for the case of the directed current networks considered in [1] and [3], the α_{ij} will always be 0, because of the possible conductance functions in that case. Incidentally, this presents an alternative proof of the convexity of the set of possible solutions for fixed ϕ.

Corollary 6.2. *If (Γ, γ) is a nonlinear conductance network such that every interior node is connected to a boundary node via a path along which the functions γ_{ij} are strictly monotone, then the Dirichlet problem has a unique solution.*

Proof. If (i, j) is an edge such that γ_{ij} is strictly monotone, then by the Theorem 6.1 the voltage difference across (i, j) is determined by the boundary voltages ϕ. Therefore, if $i \in V$ is any node connected to the boundary by such a path, the voltage $u(i)$ is completely fixed by ϕ. □

This establishes the uniqueness results for symmetric directed networks (as defined in [1]) and linear resistance networks.

Corollary 6.3. *The Dirichlet-to-Neumann map is well-defined for arbitrary nonlinear conductance networks.*

Proof. By Theorem 6.1, it is clear that $I_i(u)$ depends only on ϕ, for any $i \in V$, including $i \in \partial V$. □

7. Nonlinear Resistance Networks

In what follows, we consider the dual situation, where voltage along each edge is given by a monotone, continuous function of current. By dual arguments, we will establish the following:

- Given boundary currents adding to 0, there is some solution to the Neumann problem.
- The voltage drops in the solution are uniquely determined.
- The set of solutions is a convex set.

Definition 7.1. *A* nonlinear resistance network *is a pair (Γ, ρ), where Γ is a graph with boundary, and ρ is a map which assigns to each $(i, j) \in E$ a function $\rho_{ij} :\ \mathbb{R} \to \mathbb{R}$ such that*

- $\rho_{ij}(-x) = -\rho_{ji}(x)$.
- $\rho_{ij}(0) = 0$.
- *For every $(i, j) \in E$, $\rho_{ij}(x)$ is a continuous and weakly increasing function of x.*

This definition is exactly the same as Definition 2.2, but the interpretation is different: now ρ_{ij} expresses the voltage difference as a function of current, not vice versa.

Definition 7.2. *A* current function *on a nonlinear resistance network* (Γ, ρ) *is a function* $I: E \to \mathbb{R}$ *such that* $I(i,j) = -I(j,i)$*, and for all* $j \in \operatorname{int} V$

$$\sum_{k,\ (j,k)\in E} I(j,k) = 0.$$

Current functions form a vector space W. Note that if I is any current function, then

$$\sum_{j\in\partial V} \sum_{(k,j)\in E} I(j,k) = \sum_{j} \sum_{k,\ (j,k)\in E} I(j,k) = \sum_{(j,k)\in E} I(j,k) = \frac{1}{2}\left(\sum_{(j,k)\in E} I(j,k) - I(k,j)\right) = 0. \tag{2}$$

In other words, the sum of the boundary currents must be zero.

Definition 7.3. *A current function* I *satisfies* Kirchhoff's Voltage Law *if there is a function* $u: V \to \mathbb{R}$ *such that for every* $(i,j) \in E$

$$\rho_{ij}(I(i,j)) = u(i) - u(j)$$

Definition 7.4. *Given a function* $\phi: \partial V \to \mathbb{R}$ *with* $\sum_{j\in\partial V} \phi(j) = 0$*, the* Neumann problem *is to find a current function* I *satisfying the Kirchhoff Voltage Law, and satisfying*

$$\sum_{k,\ (j,k)\in E} I(j,k) = \phi(j)$$

for $j \in \partial V$*. The* $\phi(i)$ *are called* boundary currents.

Note that if $\sum_{j\in\partial V} \phi(j) \neq 0$, then there could not be any solution, by Equation (2).

8. Pseudopower

Definition 8.1. *For a fixed current function* I*, for each edge* $(i,j) \in E$*, the* pseudopower along (i,j) *is* $p_{ij}(I(i,j))$*, where*

$$p_{ij}(x) = \int_0^x \rho_{ij}(t)\, dt.$$

The total pseudopower *of* I *is*

$$p(I) = \sum_{(i,j)\in E} p_{ij}(I(i,j)).$$

Lemma 8.2. *The total pseudopower* $p(I)$ *is a convex, continuously differentiable function of* $I \in W$*, and is bounded below by* 0.

Proof. The proof is entirely analogous to the proof of Lemma 4.2. □

Let W_ϕ be the set of all current functions having a given boundary current ϕ.

Lemma 8.3. *If* $\sum_{i\in\partial V} \phi(i) = 0$*, and* Γ *is connected, then* W_ϕ *is nonempty.*

Proof. We need a current function for which the total current flowing out of a node i is $\phi(i)$ if $i \in \partial V$ and 0 otherwise. Therefore, without loss of generality, we can turn all interior nodes of Γ into boundary nodes. So only consider graphs without interior nodes. Then proceed by induction on the number of nodes. The base case, where $|V| = 0$, is trivial.

Otherwise, take any node $x \in V$, and then take y to be as far removed from x as possible. Then y is not an articulation point of Γ. Let z be a neighbor of y. Let Γ' be the graph obtained by deleting y. Let $\phi': V \setminus \{y\} \to \mathbb{R}$ be given by $\phi'(i) = \phi(i)$ unless $i = z$, and $\phi'(z) = \phi(z) + \phi(y)$. Note that $\sum_{i\in V\setminus\{y\}}$ is still 0.

Then by induction, there is some $I \in W_{\phi'}$. Construct $I' \in W_\phi$ by taking:

- $I'(i,j) = I(i,j)$ if $i, j \neq y$
- $I'(y,i) = -I'(i,y) = 0$ if $i \neq z$
- $I'(y,z) = -I'(z,y) = \phi(y)$.

Then clearly, the total current flowing out of i is $\phi'(i) = \phi(i)$ if $i \neq z, y$, the total current flowing out of y is $I'(y,z) = \phi(y)$, and the total current flowing out z is $\phi'(z) + I'(z,y) = \phi(z)$. So $I' \in W_\phi$. □

We assume henceforth that Γ is connected.

Definition 8.4. *If C is an oriented cycle in Γ, the* current function along C *is given by $I_C(i,j) = 1$ if $(i,j) \in C$, $I_C(i,j) = -1$ if $(j,i) \in C$, and $I_C(i,j) = 0$ otherwise.*

In other words, I_C is the current function where 1 unit flows along C and nothing else happens. Note that the total current flowing out of *every* node is zero.

Lemma 8.5. *If $I \in W_\phi$ is a critical point of p on W_ϕ then I is a solution to the Neumann problem with boundary currents ϕ.*

Proof. Take any oriented cycle C. Consider the current function $I + \epsilon I_C$. Because the total current flowing out of every node in I_C is 0, $I + \epsilon I_C \in W_\phi$. Since I was a critical point, we must have

$$0 = \frac{\partial p(I + \epsilon I_C)}{\partial \epsilon}\Bigg|_{\epsilon=0} = \sum_{(i,j)\in C} p'_{ij}(I(i,j)) - \sum_{(i,j)\in C)} p'_{ji}(I(j,i)) = 2 \sum_{(i,j)\in C} \rho_{ij}(I(i,j)).$$

Since this is true for any cycle, we can choose $u : V \to \mathbb{R}$ such that $u(i) - u(j) = \rho_{ij}(I(i,j))$ for all $(i,j) \in E$. Therefore, I satisfies the Kirchhoff Voltage Law. □

Theorem 8.6. *For boundary currents ϕ, $I \in W_\phi$ is a solution to the Neumann problem if it is a global minimum of the total pseudopower p on W_ϕ.*

Proof. This follows directly from Lemmas 3.4 and 8.5 □

9. Existence

We still need to show that a global minimum actually occurs.

Definition 9.1. *A current function I is* acyclic *if there does not exist an oriented cycle C such that $I(i,j) > 0$ for all $(i,j) \in C$.*

For example, if there are some i, j, k such that $I(i,j), I(j,k), I(k,i) > 0$, then I is not acyclic.
Let $K_\phi \subseteq W_\phi$ be the set of all acyclic current functions. Clearly, K_ϕ is closed. It is also bounded:

Lemma 9.2. *If $I \in K_\phi$, then for all $(i,j) \in E$,*

$$I(i,j) \leq \sum_k |\phi_k|$$

Proof. Without loss of generality, $I(i,j) > 0$. Let S be the set of vertices upstream from i. This is meaningful, because I is acyclic. To be more precise, $v \in S$ iff there is a chain of vertices $v = c_1, c_2, \ldots, c_r = v_i$ such that $(c_l, c_{l+1}) \in E$ and $I(c_l, c_{l+1}) > 0$. Then S does not contain j, since I is acyclic. The total current flowing from S to $V \setminus S$ is equal to the total boundary current along the boundary of S, so it is at most $\sum_k |\phi_k|$. Also, if $(p,q) \in E$, $p \in S$, and $q \notin S$, then $I(p,q) \geq 0$, since otherwise the chain from p to i could be extended to make a chain from q to i, contradicting $q \notin S$. In other words, there is no edge along which current is flowing from $V \setminus S$ to S. Therefore, $I(i,j)$ is bounded from above by the total amount of current flowing from S to $V \setminus S$, which as already noted is bounded by $\sum_k |\phi_k|$. □

It turns out that the minimum value of pseudopower, if it occurs, must occur on K_ϕ.

Lemma 9.3. *For $I \in W_\phi$, there is some $I' \in K_\phi$, such that $p(I') \leq p(I)$*

Proof. If I is acyclic then we are done, so suppose there is some cycle C such that I has positive currents along C. Let μ be the smallest current of I along C, and let $I_1 = I - \mu I_C$, which is still a current function in W_ϕ. Now decreasing the magnitude of a current along an edge does not increase the pseudopower there, so $p(I_1) \leq p(I)$. If $I_1 \in K_\phi$ then we are done. Otherwise, perform the same operation on I_1, and continue until we have a current function in K_ϕ. This process always terminates, because at each step we strictly increase the number of edges on which current vanishes. □

Theorem 9.4. *The Neumann problem has a solution if $\sum_{k\in\partial V} \phi_k = 0$ and Γ is connected.*

Proof. By Lemma 9.3(!) K_ϕ is nonempty. It is also bounded (by Lemma 9.2) and closed, so p attains a minimum on it, at some point I_0. Then for any $I \in W_\phi$, $p(I) \geq p(I') \geq p(I_0)$ for some $I' \in K_\phi$, by Lemma 9.3. Therefore, I_0 is a global minimum of p, so by Theorem 8.6 I_0 is a solution to the Neumann problem with boundary currents ϕ. □

10. Voltages Determined

Now, we show that for given boundary currents ϕ, the voltage differences across each edge are uniquely determined. Suppose we have two solutions to the Neumann problem, I and J. Define an auxiliary directed graph on V, in which there is an edge from i to j if $(i,j) \in E$ and $\rho_{ij}(I(i,j)) < \rho_{ij}(J(i,j))$.

Lemma 10.1. *If S is any set of vertices in the graph, it cannot be the case that every edge from S to $V \setminus S$ is in the auxiliary graph, unless there are no edges between S and $V \setminus S$ in the original graph.*

Proof. If there is an edge from i to j, then $I_{ij} < J_{ij}$. So if every edge from S to $V \setminus S$ is in the auxiliary graph, then the current flowing from S to $V \setminus S$ is greater in J than in I, but it should be fixed by the boundary currents. □

Lemma 10.2. *There are no edges in the auxiliary graph.*

Proof. Suppose that (i,j) is an edge in the auxiliary graph. Let S be the set of all vertices that can reach i by traveling along 0 or more edges of the original graph, without ever going against the direction of the auxiliary graph. If $x \in S$ but $y \notin S$, and there is an edge from x to y, there must be an edge in the auxiliary graph from x to y, or else we could travel from y to i via x, contradicting $y \notin S$. So every edge from S to $V \setminus S$ is in the auxiliary graph. By Lemma 10.1 there can be no edges from S to $V \setminus S$. Thus $j \in S$. So we can travel from j to i without traveling against the arrows. This produces a cycle C such that as we travel around C, we never travel against the arrow, and at at least one point (where we move from i to j), we travel with an arrow. Therefore, the voltage drops along C never decrease, and increase in at least one point, as we switch from I to J. But this is impossible, since the sum of the voltage drops along a cycle should be 0 in both I and J. □

Theorem 10.3. *For fixed boundary currents ϕ, all solutions to the Neumann problem have the same voltage drop across any given edge.*

Proof. Since there are no edges in the auxiliary graph, $\rho_{ij}(I(i,j)) \geq \rho_{ij}(J(i,j))$, for every edge (i,j). Switching i and j, we also have

$$-\rho_{ij}(I(i,j)) = \rho_{ji}(I(j,i)) \geq \rho_{ji}(J(j,i)) = -\rho_{ij}(J(i,j))$$

So $\rho_{ij}(I(i,j)) = \rho_{ij}(J(i,j))$ for all edges. □

Corollary 10.4. *For fixed boundary currents ϕ satisfying $\sum_{k \in \partial V} \phi(k) = 0$, the space of solutions to the Neumann problem is a convex subset of W_ϕ.*

Proof. If I and J are two solutions, and $t \in [0,1]$, then $I' = tI + (1-t)J$ is certainly a current function in W_ϕ. For $(i,j) \in E$, the function ρ_{ij} is weakly monotone, and so the preimage of any point is an interval, necessarily convex. Therefore,

$$\rho_{ij}(\alpha) = \rho_{ij}(\beta) \Rightarrow \rho_{ij}(t\alpha + (1-t)\beta) = \rho_{ij}(\alpha) = \rho_{ij}(\beta).$$

In particular, by Theorem 10.3, we have $\rho_{ij}(I(i,j)) = \rho_{ij}(J(i,j))$, and so

$$\rho_{ij}(I'(i,j)) = \rho_{ij}(tI(i,j) + (1-t)J(i,j)) = \rho_{ij}(I(i,j)) = \rho_{ij}(J(i,j)).$$

So I' satisfies the Kirchhoff Voltage Law because I and J do. Thus I' is another solution to the same Neumann problem. □

Corollary 10.5. *The Neumann-to-Dirichlet map is well-defined on any connected nonlinear resistance network. In other words, given boundary currents summing to zero, there is a unique corresponding boundary potential also summing to zero.*

Proof. This follows directly from Theorem 10.3 and the stipulation that the graph be connected. □

If we have a network in which there is a monotone one-to-one correspondence between current and voltage along each edge, then Corollaries 10.5 and 6.3 combine to establish a one-to-one correspondence between boundary currents and boundary voltages (summing to zero). This is the case for the usual linear electrical networks considered by Curtis and Morrow [2], as well as the symmetric networks considered by Christianson and Erickson [1].

Corollary 10.6. *If (Γ, ρ) is a connected nonlinear resistance network, such that every ρ_{ij} is strictly monotone, then the Neumann problem has a unique solution for each boundary current function ϕ summing to 0.*

Proof. Analogous to Corollary 6.2. □

11. Future Work

In this paper we have shown that nonlinear conductance networks have well-defined Dirichlet-to-Neumann maps. This immediately raises the issue of whether the conductance functions can be recovered from the Dirichlet-to-Neumann map, by analogy to the approach taken by Curtis and Morrow [2]. Previously, Joel Nishimura [3] had investigated this issue for "bidirected networks," which are the same as the "symmetric networks" of [1]. Unfortunately, Nishimura was unable to prove a critical conjecture. Moreover, he may have been working under false assumptions from the unpublished work of Orion Bawdon[2].

At any rate, the general case of nonlinear networks is more difficult, because of the ability for a group of conductors to conceal part of the graph of another conductor. Specifically, this comes about because conductance functions are allowed to be bounded. For example, if there is a vertex v such that all but one conductor out of v has a bounded current capacity, then the current flowing through the remaining conductor will be bounded. This makes it impossible to probe the value of its conductivity function beyond a certain range, making recovery impossible in general.

If we restrict the conductivity functions to be unbounded in both directions, there may be some prospects for recovery. For example, in another paper I have shown that with the additional stipulation that the conductivity functions are strictly increasing, critical circular planar graphs are recoverable. It seems likely that this may generalize to the case where the conductivities need only be weakly increasing, but further work is needed.

Another interesting idea would be to consider conductors in which the correspondence between current and voltage is no longer functional in either direction, but is instead given by a relation satisfying some sort of monotonicity and connectedness requirements. For example, it seems natural to require $\gamma \subseteq \mathbb{R} \times \mathbb{R}$ to satisfy the following conditions:

- $(0, 0) \in \gamma$
- γ is connected
- If $(x_1, y_1), (x_2, y_2) \in \gamma$, then (x_1, y_1) is comparable to (x_2, y_2), i.e., either $x_1 \leq x_2$ and $y_1 \leq y_2$, or $x_1 \geq x_2$ and $y_1 \geq y_2$.
- γ is not bounded by any bound on x or y. Together with the connectedness of γ, this amounts to saying that for every x there is some y such that $(x, y) \in \gamma$, and for every y there is some x such that $(x, y) \in \gamma$.

The last requirement is the least natural, but seems necessary for the Dirichlet-to-Neumann and Neumann-to-Dirichlet maps to be defined.

12. References

[1] Christianson, Kari and Lindsay Erickson, "The Dirichlet Problem on Directed Networks", 2007.

[2] Curtis, Edward B. and James A. Morrow, *Inverse Problems for Electrical Networks*, Series on Applied Mathematics - Vol. 13, World Scientific, New Jersey, 2000.

[3] Nishimura, Joel, "The Inverse Problem for Directed Current Electrical Networks", 2005.

[2] For example, Joel Nishimura states in §2 of his paper that "a voltage input produces only one solution," and it is unclear whether he is talking about bidirected graphs or directed graphs in general.