Factoring peak polynomials
University of Washington Mathematics REU 2014

Matthew Fahrbach
University of Kentucky
June 30, 2014
Outline

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   - Enumeration theorems

2. Results
   - Complex zeros of $p(S; n)$
   - Positivity conjecture
   - Polynomials for specific peak sets

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Permutations

**Definition**

A **permutation** $\pi = \pi_1 \pi_2 \ldots \pi_n$ in the symmetric group $\mathcal{S}_n$ is a bijection from the set $\{1, 2, \ldots, n\}$ to itself.

**Graph of $\pi = 4 1 3 2$**
An index $i$ is a peak of a permutation $\pi$ if $\pi_{i-1} < \pi_i > \pi_{i+1}$.
Peak set

Definition

The peak set $P(\pi)$ of a permutation $\pi$ is the set of all peaks in $\pi$.

If $\pi = 2 \ 8 \ 4 \ 3 \ 5 \ 1 \ 6 \ 9 \ 7 \in \mathcal{S}_9$, then

$$P(\pi) =$$
The **peak set** $P(\pi)$ of a permutation $\pi$ is the set of all peaks in $\pi$.

If $\pi = 2 \ 8 \ 4 \ 3 \ 5 \ 1 \ 6 \ 9 \ 7 \in S_9$, then

$$P(\pi) = \{2, 5, 8\}.$$
Permutations with a given peak set

Definition

Given any finite set $S$ of positive integers, let

$$\mathcal{P}(S; n) = \{\pi \in \mathfrak{S}_n : P(\pi) = S\}.$$ 

Permutations in $\mathfrak{S}_3$ whose peak set is $\{2\}$:

$$\begin{align*}
1 &\ 2 &\ 3 \\
1 &\ 3 &\ 2 \\
2 &\ 1 &\ 3 \\
2 &\ 3 &\ 1 \\
3 &\ 1 &\ 2 \\
3 &\ 2 &\ 1
\end{align*}$$

$$\mathcal{P}(\{2\}; 3) = \{1\ 3\ 2,\ 2\ 3\ 1\}$$
Theorem (Billey, Burdzy, and Sagan - 2013)

For $n \geq 1$ we have

$$\#P(\emptyset; n) = 2^{n-1}.$$ 

The peak set with a no elements is the base case for some of our inductive arguments.
Main enumeration theorem

**Theorem (Billey, Burdzy, and Sagan - 2013)**

If \( S = \{i_1 < i_2 < \cdots < i_s\} \), \( S_1 = S \setminus \{i_s\} \), and \( S_2 = S_1 \cup \{i_s - 1\} \), then

\[
\#\mathcal{P}(S; n) = p(S; n)2^{n - \#S - 1},
\]

where \( p(S; n) \) is a polynomial depending on \( S \) of degree \( i_s - 1 \) given by

\[
p(S; n) = \binom{n}{i_s - 1} p(S_1; i_s - 1) - 2p(S_1; n) - p(S_2; n).
\]

Moreover, \( p(S; i_s) = 0 \).
The peak set with a single element is the base case for some of our inductive arguments.
Example: Probability that $P(\pi) = \{50\}$ if $\pi \in S_{100}$

$$\#P(\{50\}; 100) = \left( \left( \frac{100 - 1}{50 - 1} \right) - 1 \right) 2^{100-\#\{50\}-1} \approx 1.536 \times 10^{58}$$
Peak set with a constant element

Example: Probability that $P(\pi) = \{50\}$ if $\pi \in S_{100}$

$$\#P(\{50\}; 100) = \left( \binom{100 - 1}{50 - 1} - 1 \right) \times 2^{100-\#\{50\}-1} \approx 1.536 \times 10^{58}$$

$$\frac{\#P(\{50\}; 100)}{100!} \approx 1.713 \times 10^{-100}$$
We used Sage to:

- Compute $\#\mathcal{P}(S; n)$ using alternating permutations and the inclusion-exclusion principle
- Sample values to interpolate the peak set polynomial
- Factor and find the complex zeros of $p(S; n)$
Example: Zeros of factored peak polynomials

\begin{align*}
p(\{3, 7\}; n) &= \frac{1}{80} n^2(n - 3)(n - 7)(n^2 - \frac{25}{3}n + \frac{62}{3}) \\
p(\{6\}; n) &= \frac{1}{120} (n - 6)(n^4 - 9n^3 + 31n^2 - 39n + 40) \\
p(\{4, 6, 9\}; n) &= \frac{5}{2016}n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-9)
\end{align*}
All peaks are roots

**Theorem**

*If $S = \{i_1 < i_2 < \cdots < i_s\}$, then all $i \in S$ are zeros of $p(S; n)$.***

**Proof sketch.**

Induct on the peak sets whose maximum element is $i_s$. 

Matthew Fahrbach  
Factoring peak polynomials
Where are the remaining zeros of a peak polynomial?

Recall that the degree of the polynomial is $m - 1$, where $m = \max S$. We have the most unknown zeros when the peak set contains a single element.
Complex zeros of a single peak

$p(\{25\}; n) = \left(\frac{n - 1}{25 - 1}\right) - 1$
Complex zeros of a single peak

\[ p(\{50\}; n) = \left( \frac{n - 1}{50 - 1} \right) - 1 \]
Complex zeros of a single peak

\[ p(\{75\}; n) = \left( \frac{n - 1}{75 - 1} \right) - 1 \]
Our motivation for studying zeros comes from the following conjecture.

**Conjecture (Billey, Burdzy, and Sagan - 2013)**

Let $m = \max S$ and $c^S_k$ be the coefficient of $\binom{n-m}{k}$ in the expansion

$$p(S; n) = \sum_{k=0}^{m-1} c^S_k \binom{n-m}{k}.$$ 

Each coefficient $c^S_k$ is a positive integer for all $0 < k < m$ and all admissible sets $S$. 
Note that we can sample values of $p(S; n)$ using the main enumeration theorem.

$$
\#\mathcal{P}(S; n) = p(S; n)2^{n-\#S-1} \implies p(S; n) = \frac{\#\mathcal{P}(S; n)}{2^{n-\#S-1}}
$$

Example: Sample value of $p(\{2, 5\}; 6)$

We calculate $\#\mathcal{P}(\{2, 5\}; 6)$ using a computer, so then

$p(\{2, 5\}; 6) = \frac{\#\mathcal{P}(\{2, 5\}; 6)}{2^6-\#S-1} = \frac{80}{8} = 10$.
Positivity conjecture

Note that we can sample values of $p(S; n)$ using the main enumeration theorem.

$$
\#P(S; n) = p(S; n)2^{n-\#S-1} \implies p(S; n) = \frac{\#P(S; n)}{2^{n-\#S-1}}
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Example: Sample value of $p(\{2, 5\}; 6)$

We calculate $\#P(\{2, 5\}; 6)$ using a computer, so then

$$
p(\{2, 5\}; 6) = \frac{\#P(\{2, 5\}; 6)}{2^{6-2-1}} = \frac{80}{8} = 10.
$$
Example: Positivity conjecture

If $S = \{2, 5\}$ then $\deg p(S; n) = 4$, and we can interpolate $p(S; n)$ by sampling 5 points.

<table>
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<th>0 = p(S; 5)</th>
<th>10 = p(S; 6)</th>
<th>35 = p(S; 7)</th>
<th>84</th>
<th>168</th>
<th>300</th>
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Example: Positivity conjecture

If $S = \{2, 5\}$ then $\deg p(S; n) = 4$, and we can interpolate $p(S; n)$ by sampling 5 points.

\[
p(S; n) = 10 \binom{n - 5}{1} + 15 \binom{n - 5}{2} + 9 \binom{n - 5}{3} + 2 \binom{n - 5}{4}
= \frac{1}{12} n(n - 1)(n - 2)(n - 5)
\]
Conjecture

If $S$ is admissible, then $p(S; n)$ does not have any zeros whose real part is greater than $\max S$.

The conjecture above implies the truth of the positivity conjecture, because it implies that $p(S; n)$ and all of its derivatives are positive after $m = \max S$. The forward differences $c_k^S$ are discrete analogs of the derivates of $p(S; n)$. 
Odd differences

The difference between consecutive peaks of $S$ determines the zeros of $p(S; n)$.

**Example: Odd differences**

$$p(\{2, 7\}; n) = \frac{1}{180} n(n - 1)(n - 2)(n - 7)(n^2 - \frac{19}{2} n + 27)$$

$$p(\{3, 5, 8\}; n) = \frac{1}{120} n(n - 1)(n - 2)(n - 3)(n - 4)(n - 5)(n - 8)$$
Final difference of 3

**Theorem**

If $S = \{i_1 < i_2 < \cdots < i_s < i_s + 3\}$, then

$$p(S; n) = \frac{p(S_1; i_s + 1)}{2(i_s + 1)!} \left(n - (i_s + 3)\right) \prod_{i=0}^{i_s} (n - i).$$

Note that $p(S_1; n)$ may be chaotic, but the zeros of $p(S; n)$ are well-behaved by forcing $i_s + 3$ to be a peak.
Run of adjacent peaks

**Theorem**

If $S = \{m, m + 3, \ldots, m + 3k\}$ with $k \geq 1$, then

$$p(S; n) = \frac{(m - 1)(n - (m + 3k))}{2(m + 1)!(12^{k-1})^{m+3(k-1)}} \prod_{i=0}^{m+3(k-1)} (n - i).$$

**Example**

If $S = \{3, 6, 9\}$, then

$$p(S; n) = \frac{2n(n - 1)(n - 2)(n - 3)(n - 4)(n - 5)(n - 6)(n - 9)}{2(4)!(12)}.$$
Summary

1. Permutations with a given peak set can be enumerated by a unique polynomial that is recursively defined.
2. We proved that all peaks in a peak set are zeros of its corresponding peak polynomial.
3. Odd gaps between adjacent peaks determines some of the zeros of the peak polynomial.
4. We know the peak polynomial for peak sets of the form \{m, m + 3, \ldots, m + 3k\}.
Questions
Acknowledgements

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- Jim Morrow, University of Washington
- Sara Billey, University of Washington
- Alan Talmage, Washington University in St. Louis