1. Basic Definitions and Conventions

When refering to a *graph*, we mean an undirected graph G with vertex set V broken down into interior vertices (V_{int}) and boundary vertices (∂V) where

(1) X is connected, finite, and has no loops
(2) There exists disjoint subsets, ∂V and V_{int}, of V where ∂V is nonempty and $V = \partial V \cup V_{int}$
(3) When identifying vertices, all ∂V labels come before V_{int}.

When drawing such graphs, boundary nodes are represented by a solid black dot and interior nodes by a open dot. Due to the inability to project three dimensional graphs onto paper, we will often draw the same node more than once. If a node is drawn multiple times, it will be denoted by the same number. The following graph in Figure 1 has multiple labelled vertices.

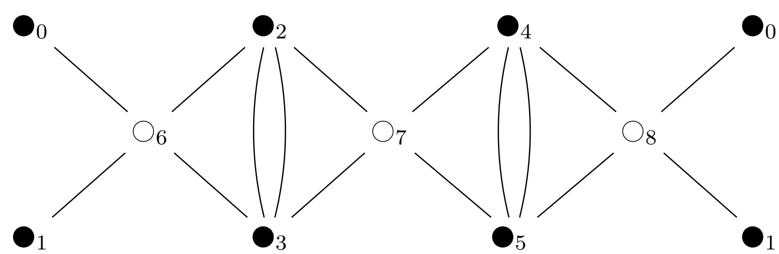


Figure 1. Blah for now

A *conductivity* on a graph G is a function γ which assigns to each edge e a positive real number $\gamma(e)$. Thus, a *resistor network*, $\Gamma(G, \gamma)$, is a graph G with a conductivity function γ [5]. The term *resistor network* is standard for a graph with resistors as edges. The conductance of a resistor is defined as the reciprocal of the resistance.

Suppose $\Gamma = (G, \gamma)$ is a resistor network with n vertices $(v_1, \dots, v_n)$. Let

$$\gamma_{ij} = \sum_{\substack{\text{all edges } e \\ \text{joining } v_i \text{ to } v_j}} \gamma(e)$$

and $\gamma_{ij} = 0$ if there is no edge between v_i and v_j. If n is the number of nodes in the network (G, γ). then *Kirchhoff Matrix* is defined as

$$K_{ij} = \begin{cases} \gamma_{ij} & i \neq j \\ -\sum_{j \neq i} \gamma_{ij} & i = j \end{cases}$$

We often write the Kirchhoff Matrix in its block form. Let's assume that G has m boundary vertices, then

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

where A is an $m \times m$ matrix and C is an $n - m \times n - m$ matrix. From [5], it is shown that C is an invertible matrix. The definition of the *Kirchhoff matrix* leads to the following characteristics:

(1) The off-diagonal entries are positive or 0 ($\gamma_{ij} \geq 0$ for all $i \neq j$
(2) The row sums equal 0.
(3) K is symmetric.

Because of the symmetry of the Kirchhoff matrix, only the γ_{ij}'s in the upper triangle need to be found in order to define the entire K matrix. Moreove, there is a 1-to-1 correspondence between any matrix that satisfies the above characteristics and a Kirchhoff matrix of a network. (This will especially be useful later on in the paper.)

We now define Λ ($m \times m$) to be the *response matrix* (G, γ):

$$\Lambda = A - BC^{-1}B^T = K/C.$$

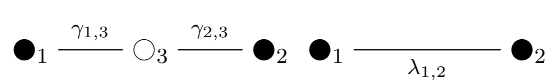
Λ can be interpreted as the Schur complement of C in K. This new matrix, Λ, has the same characteristics as the Kirchhoff matrix:

(1) The off-diagonals are greater than or equal to 0 ($\lambda_{ij} \geq 0$ for $i \neq 0$)
(2) The row sums are 0
(3) Λ is symmetric

Hence, Λ is a Kirchhoff matrix for another resistor network. This is easily shown.

The inverse problem associated with resistor networks is given a response matrix, Λ, and graph, G, find the conductivities (i.e. the entries in the Kirchhoff matrix). This simple definition of the inverse problem leads to the term *recoverability.* The idea is that with any response matrix there exist unique conductivities on G that generated Λ. Prior research shows that given certain characteristics of the graph, there is recoverability. Hence, we can find theoretically find the conductivities with the response matrix. When there are non-unique conductivities corresponding to the response matrix, we say that the graph is *n to 1* (The phrase coined in [2]) where $n > 1$. In this sense with a given Λ there exists n different conductivities that created the response matrix. The simpliest example of such graphs is the "series" connection (View Figure 2). In this case, we can show that Λ is a 2×2 matrix. Then $\lambda_{1,2} = \frac{\gamma_{1,3}\gamma_{2,3}}{\gamma_{1,3}+\gamma_{2,3}}$. Because we can not break $\lambda_{1,2}$ down into two components, $\gamma_{1,3}$ and $\gamma_{2,3}$ can be any value such that this ratio holds. Thus, there is ∞ *to 1* different conductivities that produce the response matrix. The more interesting cases occur when n is a finite number greater than one.

FIGURE 2. The graphs of the Kirchhoff and response matrices for series configurations



2. Motivation

The concept of *n to 1* graphs where n is a finite number has been of great interest. In [3], the 2-to-1 graphs were explored thoroughly (and 2^n) using n-gon in n-gon graphs. They established the existance of at least 2^n-to-1 graphs. It wasn't until the work by [1] and [2], who showed the existance of 3-to-1 graphs, that n-to-1 graphs were possible (n is finite). However, the creation of these graphs and their underlying structure remains relatively limited. Moreover, we will take a different approach to solving n-to-1 graphs then previously thought. In this paper, we explore the construction of n-to-1 graphs (when n does not equal a power of 2) and certain properties that arise.

3. Preliminary Notions

3.1. Star-K Transformation. The concept of Star-K Transformation and the enforcement of the quadrilateral rule are essential to constructing n-to-1 graphs. We define a *star* as a graph in which there exists no interior to interior edges and no boundary to boundary edges. An n-star (for $n > 1$) is defined as n boundary vertices connected to a single interior vertex. The K transformation takes the star and creates a complete graph with the interior vertex removed. We will denote K_n as a complete graph with n boundary vertices (This is not to be confused with the Kirchhoff matrix K). Thus the *Star-K Transformation* takes a n-star to a K_n.

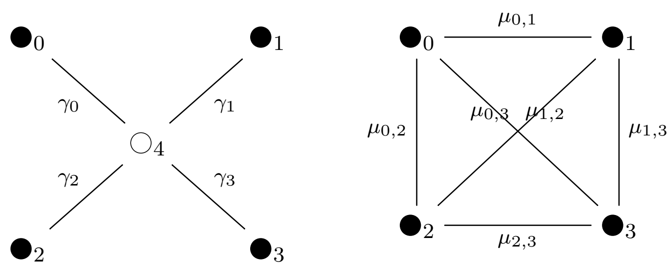


Figure 3. 4-star transformed to a K_4

The importance of this Transformation lies in understanding what it does to the Kirchhoff matrix. By eliminating interior vertices, we are doing row reductions to reduce B to the 0 matrix. Once all interior vertices have been "row reduced", the upper left-hand matrix is $A - BC^{-1}B^T$, which is defined as the response matrix, Λ.

Knowing this response matrix, we understand the relationship between γ_i and μ_{ij}. It is shown that

$$\frac{\gamma_0\gamma_1}{\sigma} = \mu_{0,1}$$

where $\sigma = \sum_i \gamma_i$. Similarly, $\mu_{1,3} = \frac{\gamma_1\gamma_3}{\sigma}$.

3.2. Quadrilateral Rule. The quadrilateral rule derives from the 4-star and K_4 (Figure 3) and the Determinant Connection Theorem found in [5]. The basic idea is that in the 4-star there exists no 2-connection between any 4 boundary vertices. Because of this and the Determinant Connection Theorem, we know that the determinant of any 2×2 off-diagonal sub-matrix (γ_{ii} is not an entry in the sub-matrix) of the Kirchhoff matrix is 0. In the K_4 graph, it is equvialent to $det[2 \times 2] = 0$ or

$$\mu_{0,1}\mu_{2,3} = \mu_{1,3}\mu_{0,2} = \mu_{0,3}\mu_{1,2}$$

In more generality for larger stars, the quadrilateral rule is stated as

$$\mu_{ij}\mu_{kl} = \mu_{ik}\mu_{jl} \quad \text{for all } i \neq j \neq k \neq l. \tag{1}$$

The quadrilateral rule leads to an important theorem developed in [4] regarding computation of the γ from the response matrix.

Theorem 3.1. *A network on a complete graph (K_n) is response-equivalent to a star if and only if its conductivities satisfy Equation 1.*

In particular, [4] discovered a method of recovering the γ's given all sides of the quadrilateral. Define

$$\alpha_i = \sqrt{\frac{\mu_{ij}\mu_{ik}}{\mu_{jk}}}.$$

Then we can formally compute γ_i's by

$$\gamma_i = \alpha_i \sum_j \alpha_j. \tag{2}$$

We must be careful when applying this. **The μ_{ij}'s needed for the calculation do not necessarily come directly from the response matrix.** In some cases, the response matrix has the sum of multiple edges in the quadrilateral. Hence in order to compute the γ_i we must be able to calculate each edge in the quadrilateral separately.

3.3. **Parameterizing the Response Matrix.** This remarkable formula allows us to parameterize the response matrix so that it preserves the quadrilateral rule. A closer examination of Figure 3 reveals how we can choose λ_{ij} and create a valid graph with conductivities. Assume that the μ's for $\mu_{0,1}$ and $\mu_{2,3}$ are fixed. We are now able to select any positive values for $\mu_{0,2}$ and make $\mu_{1,3} = \frac{\mu_{0,1}\mu_{2,3}}{\mu_{0,2}}$ satisfy the quadrilateral rule. Moreover, we can choose a positive value for $\mu_{0,3}$ (independent of $\mu_{0,2}$) and make $\mu_{1,2} = \frac{\mu_{0,1}\mu_{2,3}}{\mu_{0,3}}$. By doing this, we can recover the corresponding γ's by Equation 2. Thus, we can construct a "response matrix" which corresponds to real conductivities on a 4-star.

To avoid confusion with the response matrix and the Kirchhoff matrix, we will defined new terminology.

Definition 3.1. *R-MultiGraph and R Matrix*
The R-MultiGraph is the graph of the star after performing the Star-K Transformation. In Figure 3, the complete graph is the R-MultiGraph. We use the term multi-graph to describe R because as we will see later multiple edges will be allowed in the R-MultiGraph. The R-Matrix is a matrix that stores the values of the μ's or the "conductivities" on the R-MultiGraph. Because multiple edges are allowed in the R-MultiGraph, the entries in the R-Matrix often will be sets containing the μ's. If a multiple edge occurs in the R-MultiGraph, the R-Matrix separates the multiple edges storing both values as the entry in the matrix. This slightly differs from the response matrix since a multiple edge in the R-MultiGraph results is a sum of the μ's in the response matrix between any two vertices. When there is only a single edge in the R-MultiGraph, the response matrix and the R-Matrix will contain the same values. All off-diagonal entries in the R-Matrix must be positive and the diagonal entries are the same as the response matrix. In addition, the R-Matrix like the response is symmetric.

The R-Matrix is a useful tool because in order to recover the conductivities we must know all the sides on the quadrilateral. In order to better understand this terminology, we will provide some examples in the next section.

3.4. **Connecting Multiple Stars.** Although we can connect different n-stars to each other, we will focus on 4-stars. The following figures illustrate different connections of multiple 4-stars and their corresponding R-MultiGraphs.

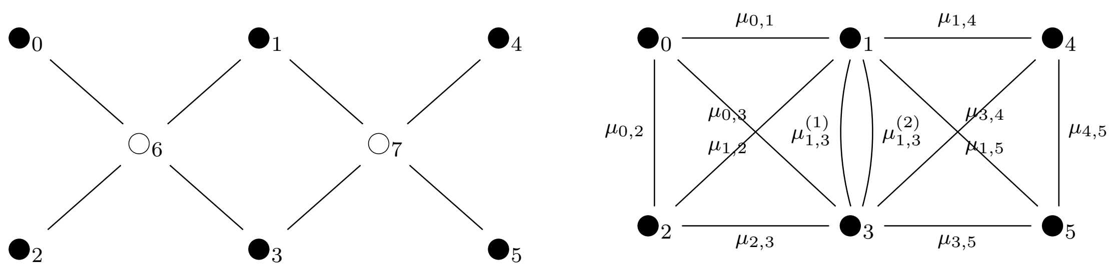


Figure 4. Connection of Multiple 4-Stars by Parallel Sides

The response matrix in Figure 4 would be the following:

$$\Lambda = \begin{bmatrix} -\sum \mu_i & \mu_{0,1} & \mu_{0,2} & \mu_{0,3} & 0 & 0 \\ \mu_{0,1} & -\sum \mu_i & \mu_{1,2} & \mu_{1,3}^{(1)} + \mu_{1,3}^{(2)} & \mu_{1,4} & \mu_{1,5} \\ \mu_{0,2} & \mu_{1,2} & -\sum \mu_i & \mu_{2,3} & 0 & 0 \\ \mu_{0,3} & \mu_{1,3}^{(1)} + \mu_{1,3}^{(2)} & \mu_{2,3} & -\sum_i \mu_i & \mu_{3,4} & \mu_{3,5} \\ 0 & \mu_{1,4} & 0 & \mu_{3,4} & -\sum \mu_i & \mu_{4,5} \\ 0 & \mu_{1,5} & 0 & \mu_{3,5} & \mu_{4,5} & -\sum \mu_i \end{bmatrix},$$

whereas the R-Matrix is

$$\begin{bmatrix} -\sum \mu_i & \mu_{0,1} & \mu_{0,2} & \mu_{0,3} & 0 & 0 \\ \mu_{0,1} & -\sum \mu_i & \mu_{1,2} & \{\mu_{1,3}^{(1)}, \mu_{1,3}^{(2)}\} & \mu_{1,4} & \mu_{1,5} \\ \mu_{0,2} & \mu_{1,2} & -\sum \mu_i & \mu_{2,3} & 0 & 0 \\ \mu_{0,3} & \{\mu_{1,3}^{(1)}, \mu_{1,3}^{(2)}\} & \mu_{2,3} & -\sum_i \mu_i & \mu_{3,4} & \mu_{3,5} \\ 0 & \mu_{1,4} & 0 & \mu_{3,4} & -\sum \mu_i & \mu_{4,5} \\ 0 & \mu_{1,5} & 0 & \mu_{3,5} & \mu_{4,5} & -\sum \mu_i \end{bmatrix},$$

Note the distinction between the two. In row 1 column 4, the response matrix has the sum of the μ's whereas the R-Matrix has in entry 1,4 two values of μ. We can see that in the R-MultiGraph we get two edges from vertices 1 and 3, so the R-Matrix and Λ will only differ by this entry.

Figure 3.4 will often be used in creating n-to-1 graphs. Later we will refer to this type of connection of multiple stars as a *inversion.* Note that when you see an R-MultiGraph like Figure 3.4 it comes from that resistor network graph and is simply a quadrilateral.

4. Positivity and Polynomials

As shown previously, the structure of the graphs give rise to polynomials formed from linear combinations of products of $(C_j - x)^k$ and x^k for $k = 1, \ldots, n$ where the coefficients need to be positive. We also proved that the positivity of f_j's prior to the end of the arms, those before multiplication occurs, don't affect the ending $\sigma(x)$. For instance, the λ's which ensures positivity of these f_j don't appear in the ending calculations for $\sigma(x)$. *As a result, we can solve the solutions to $\sigma(x)$ first and then choose these λ's larger than all solutions to*

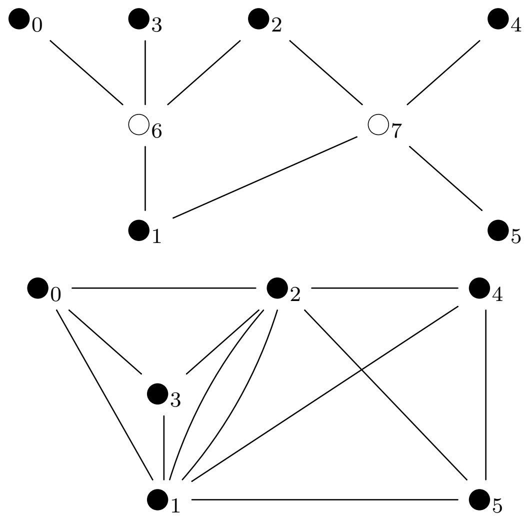


FIGURE 5. Connection of Multiple 4-Stars by Diagonal Side

$\sigma(x)$ for the prior f_j's.

We can now turn our focus to understanding $\sigma(x)$. The first thing to note is that $\sigma = p(x) + f_0 = p(x) + x$ where $p(x)$ is a polynomial formed from linear combinations of products of x^k and $(C_j - x)^k$ for $k = 1, \ldots, n$ where the coefficients are positive. Because f_0 will always appear in σ, the construction of the arms only depend on the function $p(x)$. Hence, the ultimate goal is to find $p(x)$ as linear combinations of products of $(C_j - x)^k$ and x^k, and satsifies some additional properties. In order for this graph to have valid conductivities, these properties need to be true of $\sigma(x)$:

(1) The solutions to $\sigma(x) = \lambda_{0,1}$ need to be positive (strictly larger than 0) for some positive $\lambda_{0,1}$.
(2) All the C_j need to be larger than all the solutions to $\sigma(x)$ to ensure that a particular f_j is positive. This implies that the C_j's must also be positive.
(3) The function $\sigma(x)$ must be of the form $p(x) + x$, where $p(x)$ is a linear combinations of products of $(C_j - x)^k$ and x^k, where all the coefficients must be positive.

The goal is to find a $p(x)$ and hence σ with the properties and then construct the corresponding network graph.

Consider the first property of σ, where $\sigma(x) = \lambda_{0,1}$ must have all positive roots for some positive $\lambda_{0,1}$. For now, we will assume $\lambda_{0,1} = 0$ and find a σ which satisfies the three properties. We will then use a simple trick to get $\lambda_{0,1}$ positive by shifting the entire polynomial up a fixed number. Without loss of generality, we will construct polynomials with roots between $(0, 1)$. Therefore, the $C_j \geq 1$ for all j in order to satsify condition (2): the C_j's need to be larger than all solutions σ. We will see that letting the $C_j = 1$ for all j produces a widely studied class of polynomials. As a result, the rest of this section we will take $C_j = 1$ for all j. This topic of constructing polynomials with certain root properties has been extensively explored, particularly roots between $(-1,1)$. There exists many polynomials with this property including Chebyshev and other Jacobi polynomials. You can do this process with any of these polynomials; however for this paper, we will use Legendre polynomials as our basis for constructing σ.

4.1. **Legendre Polynomials.** Standard Legendre polynomials have roots between $(-1,1)$, so we will start by constructing these polynomials and shift accordingly. Using Bonnet's recursion formula for Legendre polynomials, $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$, where $P_0(x) = 1$ and $P_1(x) = x$, you can construct an explicit calculation for the Legendre polynomial,

$$P_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \left(\frac{1+x}{2}\right)^{n-k} \left(\frac{1-x}{2}\right)^k.$$

We will shift the polynomials to have roots between $(0,1)$ by setting $x \mapsto 2x-1$. Thus the "shifted" Legendre polynomials with roots between 0 and 1 are

$$P_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 x^k (1-x)^{n-k}. \tag{3}$$

Note that the Legendre polynomials are in the "basis" that we want (i.e. x^k and $(1-x)^k$). However, property (3) fails for σ as some of the coefficients are negative. Unfortunately, we can't directly use Legendre polynomials, but they do have the property that are the roots are positive and $C_j = 1$ is bigger than all the roots for all j. For now, we will use these polynomials as acting like σ; thus denote

$$\tilde{\sigma}(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 x^k (1-x)^{n-k}.$$

The first few "shifted" Legendre polynomials are:

$$\begin{aligned}
n = 0: &\quad P_0(x) = 1 \\
n = 1: &\quad P_1(x) = 1 - 2x \\
n = 2: &\quad P_2(x) = (1-x)^2 - 4x(1-x) + x^2 \\
n = 3: &\quad P_3(x) = (1-x)^3 - 9x(1-x)^2 + 9x^2(1-x) - x^3 \\
n = 4: &\quad P_4(x) = (1-x)^4 - 16x(1-x)^3 + 36x^2(1-x)^2 - 16x^3(1-x) + x^4
\end{aligned}$$

The benefit of Legendre polynomials are that the roots are now restricted to be between $(0,1)$ and they come from linear combinations of products of x^k and $(C_j - x)^k$ where $C_j \geq 1$. We now try to satisfy the property that the coefficients of these linear combinations need to be positive. In order to solve this, we introduce a new type of polynomial known as **Bernstein Polynomials**.

4.2. **Bernstein Basis Polynomials.** The Bernstein Basis Polynomials of degree n are defined as

$$b_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, \ldots, n.$$

The Bernstein basis polynomials of degree n form a basis for the vector space of polynomials of degree less than or equal to n. Bernstein basis polynomials have a multitude of properties useful in solving our current problem.

Property 4.1. *The Bernstein basis of polynomials form a partition of unity. Thus,*

$$\sum_{k=0}^{n} b_{k,n} = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1.$$

Proof. Consider that $(1-x+x)^n = 1$. Applying binomial formula, we get that

$$1 = (1-x+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}.$$

□

We will use this fact to make the coefficients of the Legendre polynomials positive while simultaneously shifting the polynomial upward making $\lambda_{0,1} > 0$.

Property 4.2. *Let f be a continuous function on the interval $[0,1]$ and $B_n(f)$ be the polynomial on $[0,1]$ such that*

$$B_n(f) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

Then $B_n(f)$ tends uniformly to $f(x)$ as $n \to \infty$.

As a result, the Bernstein basis polynomials can be used to prove the Weierstrass approximation theorem.

Recall that, $\sigma(x) = p(x) + x$, and that we need to find $p(x)$ in order to construct our graphs. So far, we have written $\tilde{\sigma}(x)$ in terms of Legendre polynomials. Now using Property (4.2), we can write x in terms of the Bernstein basis polynomials. Moreover if $n \geq 1$ and since Bernstein polynomials form a basis, x will be exactly given by the formula: $B_n(f) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^k f\left(\frac{k}{n}\right)$. Therefore,

$$x = \sum_{k=0}^{n} \frac{k}{n} \binom{n}{k} x^k (1-x)^{n-k}, \quad \text{for } n \geq 1. \tag{4}$$

Next, we will find $p(x) = \sigma(x) - x$. So far, we have assumed that $\lambda_{0,1} = 0$ and that $\tilde{\sigma}$ is a Legendre polynomial with negative coefficients. First, we will construct a pre-$p(x)$ function, which doesn't quite satisfy the properties we need. Denote this function by $\tilde{p}(x) = \tilde{\sigma}(x) - x$. Using Equation (3) and (4), we get that

$$\tilde{p}(x) = \sum_{k=0}^{n} \left((-1)^k \binom{n}{k} - \frac{k}{n} \right) \binom{n}{k} x^k (1-x)^{n-k}. \tag{5}$$

Next, the following claim will ensure positivity of the coefficients of $p(x)$ and show the existence of a positive $\lambda_{0,1}$ for σ that has n positive roots.

Proposition 4.1. *Let $C \in \mathbb{R}$ and $p_n(x) = \sum_{k=0}^{n} a_k \binom{n}{k} x^k (1-x)^{n-k}$, a polynomial of degree n written in terms of the Bernstein basis polynomials. Then*

$$p_n(x) + C = \sum_{k=0}^{n} (a_k + C) \binom{n}{k} x^k (1-x)^{n-k}.$$

In other words, adding a constant, C, to every coefficient, a_k, shifts the polynomial up by C.

Proof. We see that

$$\sum_{k=0}^{n} (a_k + C) \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^{n} a_k \binom{n}{k} x^k (1-x)^{n-k} + C \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}.$$

Applying Property (4.1), the partition of unity, we get that

$$\begin{aligned}
\sum_{k=0}^{n} a_k \binom{n}{k} x^k (1-x)^{n-k} + C \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} &= \sum_{k=0}^{n} a_k \binom{n}{k} x^k (1-x)^{n-k} + C \\
&= p_n(x) + C.
\end{aligned}$$

$\square$

This proposition allows us to add a number to every coefficient to make

$$\tilde{p(x)} = \sum_{k=0}^{n} \left((-1)^k \binom{n}{k} - \frac{k}{n} \right) \binom{n}{k} x^k (1-x)^{n-k}$$

have all positive coefficients. As a sufficient condition, the smallest the coefficient of $\tilde{p}(x)$ is $\binom{n}{\lceil \frac{n}{2} \rceil} + 1$. Hence adding this to every coefficient guarantees positivity of every coefficient of $\tilde{p}(x)$. Moreover by Proposition 4.1, adding a positive number to every coefficient shifts the polynomial up by that constant. Denote this new polynomial as

$$\begin{aligned}
p(x) &= \tilde{p}(x) + \binom{n}{\lceil \frac{n}{2} \rceil} + 1 \\
&= \sum_{k=0}^{n} \left((-1)^k \binom{n}{k} - \frac{k}{n} + \binom{n}{\lceil \frac{n}{2} \rceil} + 1 \right) \binom{n}{k} x^k (1-x)^{n-k}.
\end{aligned}$$

By this construction of $p(x)$, all the $C_j \geq 1$, $p(x)$ has all positive coefficients and $p(x)$ is linear combinations of products of $(C_j - x)^k$ and x^k. This allows us to construct $\sigma(x)$. Therefore, let

$$\begin{aligned}
\sigma(x) &= \tilde{\sigma}(x) + \binom{n}{\lceil \frac{n}{2} \rceil} + 1 \\
&= \tilde{p}(x) + \binom{n}{\lceil \frac{n}{2} \rceil} + 1 + x \\
&= p(x) + x \\
&= \sum_{k=0}^{n} \left((-1)^k \binom{n}{k} - \frac{k}{n} + \binom{n}{\lceil \frac{n}{2} \rceil} + 1 \right) \binom{n}{k} x^k (1-x)^{n-k} + x.
\end{aligned}$$

This $\sigma(x)$ has all the properties necessary. For instance because $\sigma(x) = \tilde{\sigma}(x) + \binom{n}{\lceil \frac{n}{2} \rceil} + 1$ and $\tilde{\sigma}$ has n zeroes between $(0,1)$, then if $\sigma(x) = \binom{n}{\lceil \frac{n}{2} \rceil} + 1$ it will have exactly n solutions between $(0,1)$. As result, we showed there exists a positive $\lambda_{0,1}$, namely $\binom{n}{\lceil \frac{n}{2} \rceil} + 1$, such that the solutions are all positive and between $(0,1)$. Putting this together with the properties of $p(x)$- the coefficients being positive, C_j greater than 1, and $p(x)$ linear combinations of products of x^k and $(C_j - x)^k$ and x^k- we see that all the solutions of $\sigma(x) = \binom{n}{\lceil \frac{n}{2} \rceil} + 1$ are less than $C_j = 1$ for all j. Therefore, this σ satisfies all the properties needed.
The following are the first few σ and $p(x)$ given by the above formula:

$$
\begin{aligned}
n = 1: \quad & p(x) = 3(1-x) \\
& \sigma(x) = 3(1-x) + x \\
n = 2: \quad & p(x) = 4(1-x)^2 + x(1-x) + 3x^2 \\
& \sigma(x) = 4(1-x)^2 + x(1-x) + 3x^2 + x \\
n = 3: \quad & p(x) = 5(1-x)^3 + 2x(1-x)^2 + 19 \\
& \sigma(x) = 5(1-x)^3 + 2x(1-x)^2 + 19 + x \\
n = 4: \quad & p(x) = 8(1-x)^4 + 11x(1-x)^3 + 75x^2(1-x)^2 + 9x^3(1-x) + 7x^4 \\
& \sigma(x) = 8(1-x)^4 + 11x(1-x)^3 + 75x^2(1-x)^2 + 9x^3(1-x) + 7x^4 + x \\
n = 5: \quad & p(x) = 12(1-x)^5 + 29x(1-x)^4 + 206x^2(1-x)^3 + 4x^3(1-x)^2 + 76x^4(1-x) + 9x^5 \\
& \sigma(x) = 12(1-x)^5 + 29x(1-x)^4 + 206x^2(1-x)^3 + 4x^3(1-x)^2 + 76x^4(1-x) + 9x^5 + x
\end{aligned}
$$

In summary, to construct a $\sigma(x) = p(x) + x$, with all the characteristics we need you proceed as followed

(1) Write the shifted Legendre polynomials with roots between $(0,1)$ and call it $\tilde{\sigma}$.
(2) Write x in terms of the Bernstein basis polynomials
(3) Define $\tilde{p}(x) = \tilde{\sigma}(x) - x$ and rewrite into Bernstein basis polynomials
(4) Add $\binom{n}{\lceil \frac{n}{2} \rceil} + 1$ to every coefficient $\tilde{p}(x)$ to make the coefficients positive, while simultaneously shifting the polynomial up by some constant. Then set $p(x) = \tilde{p}(x) + \binom{n}{\lceil \frac{n}{2} \rceil} + 1$.

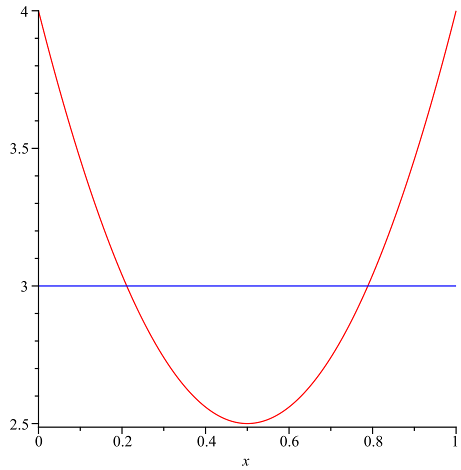
By doing the above steps, you should get

$$p(x) = \sum_{k=0}^{n} \left((-1)^k \binom{n}{k} - \frac{k}{n} + \binom{n}{\lceil \frac{n}{2} \rceil} + 1 \right) \binom{n}{k} x^k (1-x)^{n-k}$$

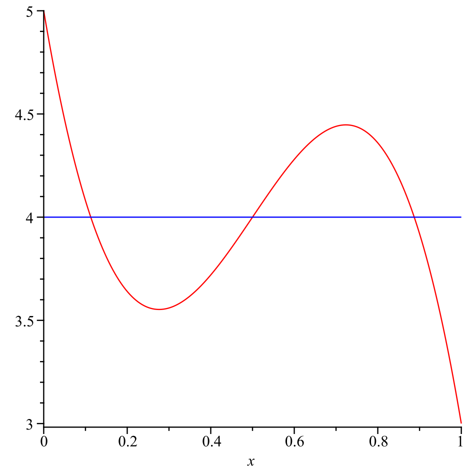
$$\sigma(x) = p(x) + x = \sum_{k=0}^{n} \left((-1)^k \binom{n}{k} - \frac{k}{n} + \binom{n}{\lceil \frac{n}{2} \rceil} + 1 \right) \binom{n}{k} x^k (1-x)^{n-k} + x.$$

By setting $\sigma(x) = \binom{n}{\lceil \frac{n}{2} \rceil} + 1$, you will get n positive solutions where all the f_j's are positive for all n solutions.

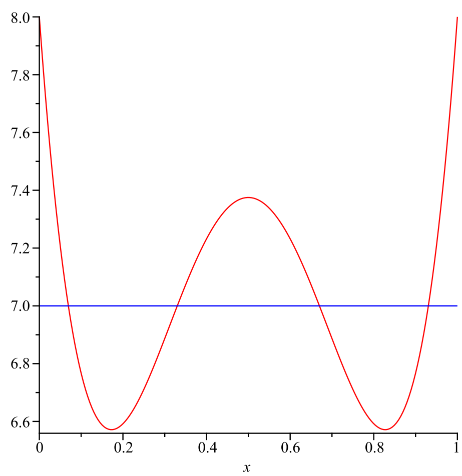
Figure 6. The first few graphs of σ



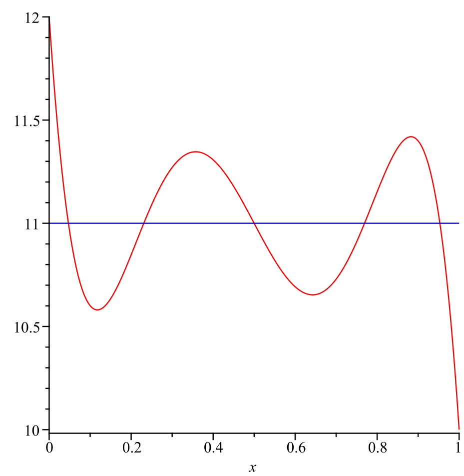
(a) When $n = 2$, $\sigma(x)$ gives two solutions for $\lambda_{0,1} = 3$



(b) When $n = 3$, $\sigma(x)$ gives three solutions for $\lambda_{0,1} = 4$



(c) When $n = 4$, $\sigma(x)$ gives four solutions for $\lambda_{0,1} = 7$



(d) When $n = 5$, $\sigma(x)$ gives five solutions for $\lambda_{0,1} = 11$

References

[1] Ilya Grigoriev, *What We Don't Know About the 3-to-1 Graph*, University of Washington Math REU (2006).

[2] Chad Klumb, *Discrete Unsolvability for the Inverse Problem for Electrical Networks*, University of Washington Math REU (2011).

[3] Jennifer French and Shen Pan, 2^n *to 1 Graphs*, University of Washington Math REU (2004).

[4] Jeffrey Russell, *Star and K Solve the Inverse Problem*, University of Washington Math REU (2003).

[5] Edward Curtis and James Morrow, *Inverse Problems for Electrical Networks*, World Scientific, Singapore, 2000.