L-Functions: A Crash Course

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Introduction: The Riemann Zeta Function

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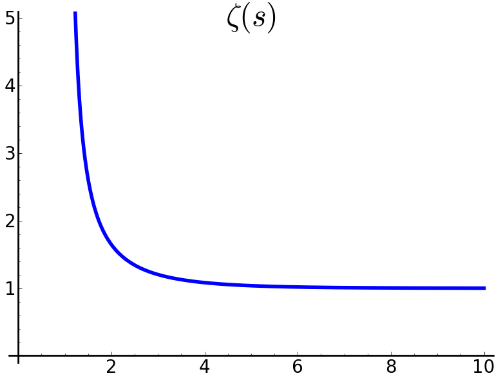

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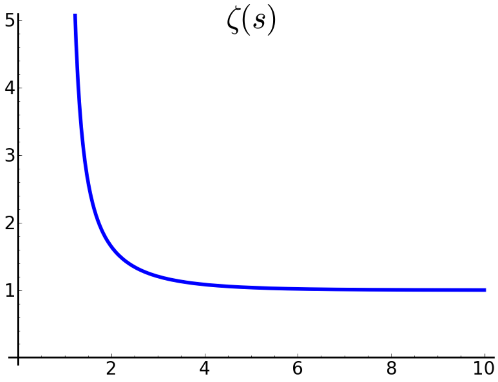

This Dirichlet series converges absolutely for any complex s with $\Re(s) > 1$

The Euler product

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Theorem (Euler 1737)

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p} \frac{1}{1-p^{-s}}$$

where the product is taken over all primes, and the product converges absolutely for $\Re(s) > 1$.

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"Proof"

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{-s} &= 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + \dots \\
&= \left(1^{-s} + 2^{-s} + (2^2)^{-s} + \dots\right)\left(1^{-s} + 3^{-s} + (3^2)^{-s} + \dots\right)\cdots \\
&= \left[(2^{-s})^0 + (2^{-s})^1 + (2^{-s})^2 \dots\right]\left[(3^{-s})^0 + (3^{-s})^1 + (3^{-s})^2 \dots\right]\cdots \\
&= \left(\frac{1}{1-2^{-s}}\right)\left(\frac{1}{1-3^{-s}}\right)\left(\frac{1}{1-5^{-s}}\right)\cdots
\end{aligned}$$

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So

$$\begin{aligned} &(1-2\cdot 2^{-s})\zeta(s) = \zeta(s) - 2\cdot 2^{-s}\zeta(s) \\ &= \left(1^{-s}+2^{-s}+3^{-s}+\ldots\right) - \left(2\cdot 2^{-s}+2\cdot 4^{-s}+2\cdot 6^{-s}+\ldots\right) \\ &= 1^{-s}-2^{-s}+3^{-s}-4^{-s}+\ldots \\ &= \sum_{n=1}^{\infty}(-1)^{n+1}n^{-s}. \end{aligned}$$

Extending $\zeta(s)$ to a Larger Domain

$\sum_{n=1}^{\infty}(-1)^{n+1}n^{-s}$ converges (conditionally) on the strip $0 < \Re(s) \leq 1$, so we can use it to define $\zeta(s)$ on the entire right half plane.

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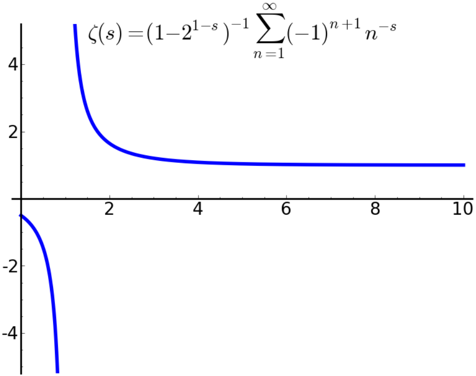

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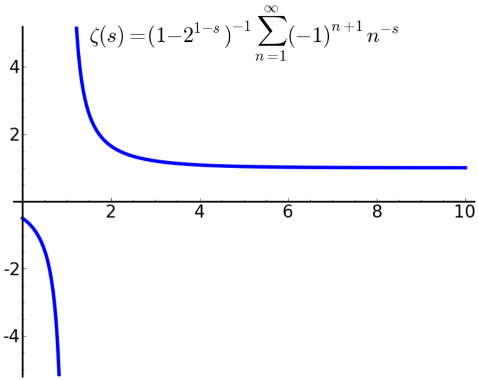

We can see $\zeta(s)$ clearly has a pole at $s = 1$.

The Completed Zeta Function

Define the *completed zeta function*

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

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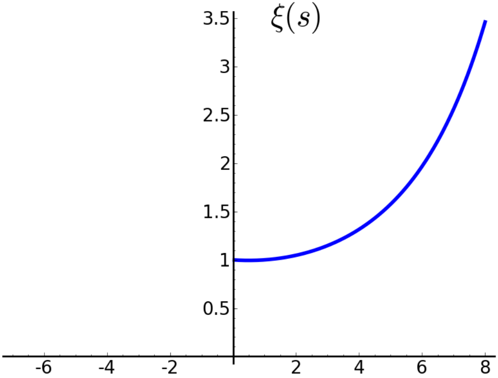

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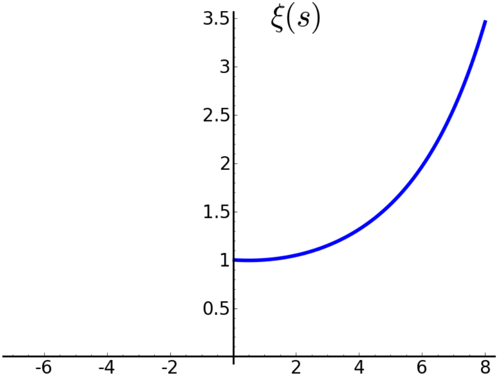

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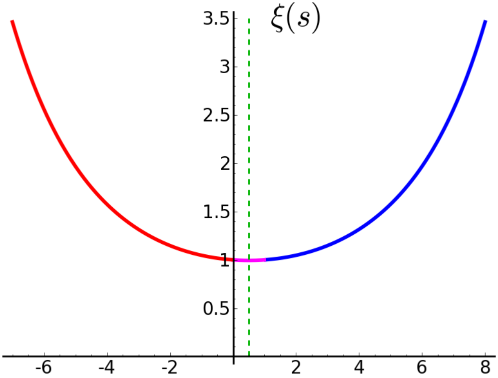

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$\zeta(s)$ Analytically continued to $\mathbb{C}$

So we have

$$zeta(s) = \begin{cases} (1-2^{1-s})^{-1}\sum_{n=1}^{\infty}(-1)^{n+1}n^{-s} & \Re(s) > 0 \\ 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) & \Re(s) \le 0 \end{cases}$$

defined for all $s \in \mathbb{C}$ except $s = 1$.

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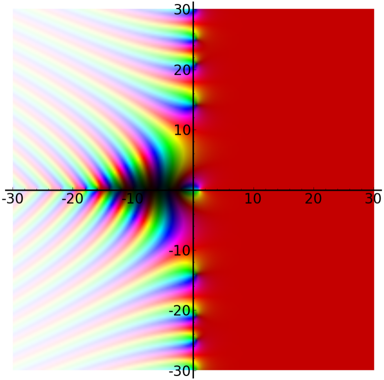

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Conjecture (Riemann Hypothesis)

All nontrivial zeros of ζ are simple and lie on the line $\Re(s) = \frac{1}{2}$.

The Zeros of ζ

The imaginary parts of the first few zeros of $\zeta(s)$ in the upper half plane are

$$\begin{aligned}&14.134725142\ldots\\&21.022039639\ldots\\&25.010857580\ldots\\&30.424876126\ldots\\&32.935061588\ldots\\&37.586178159\ldots\\&40.918719012\ldots\\&43.327073281\ldots\\&48.005150881\ldots\\&49.773832478\ldots\\&52.970321478\ldots\\&56.446247697\ldots\\&59.347044003\ldots\\&60.831778525\ldots\end{aligned}$$

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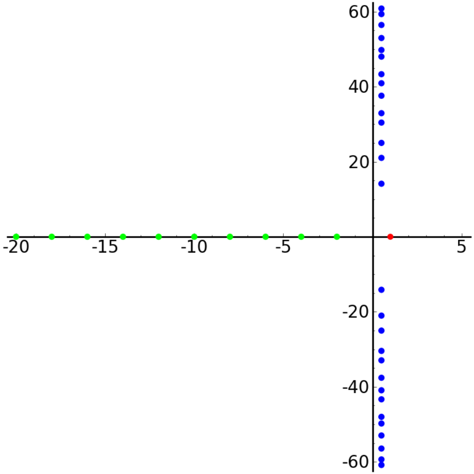

The Explicit Formula for $\zeta(s)$

Consider as a function of $x > 1$ the sum

$$S_\zeta(x, T) = \sum_{|\rho|<T} \frac{x^\rho}{\rho}$$

where ρ runs over nontrivial zeros of $\zeta(s)$.

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where ρ runs over nontrivial zeros of $\zeta(s)$.
According to RH, nontrivial zeros come in pairs and have the form $\rho = \frac{1}{2} \pm i\gamma$, so in the above sum for a single zero pair we have

$$\begin{aligned}\frac{x^\rho}{\rho} + \frac{x^{\overline{\rho}}}{\overline{\rho}} &= \frac{x^{1/2+i\gamma}}{1/2 + i\gamma} + \frac{x^{1/2-i\gamma}}{1/2 - i\gamma} \\ &= \frac{\sqrt{x}}{1/4 + \gamma^2} \left[\cos(\gamma \log x) + 2\gamma \sin(\gamma \log x)\right]\end{aligned}$$

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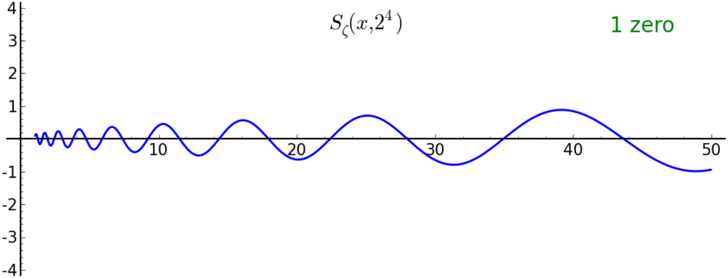

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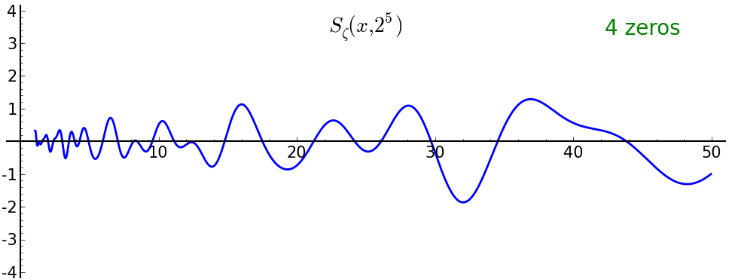

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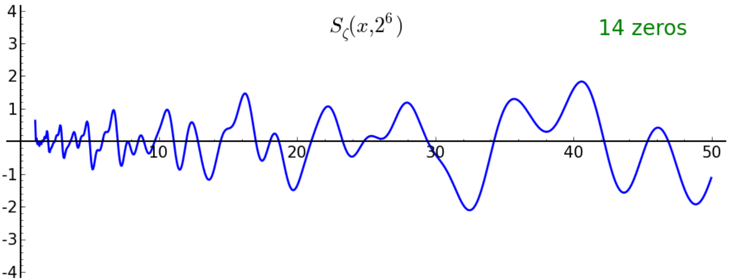

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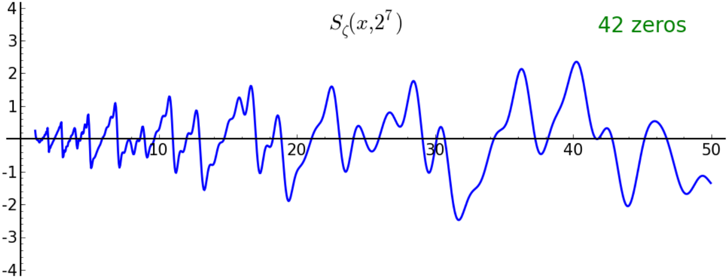

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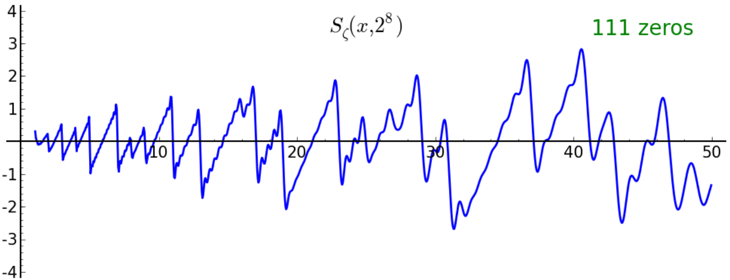

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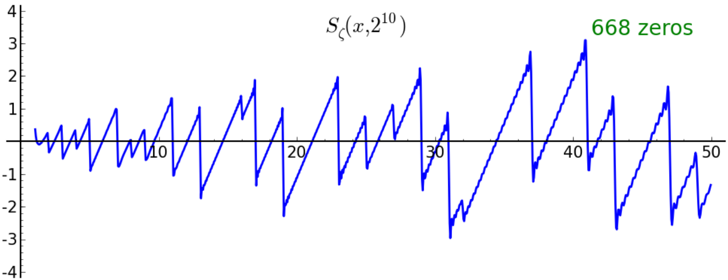

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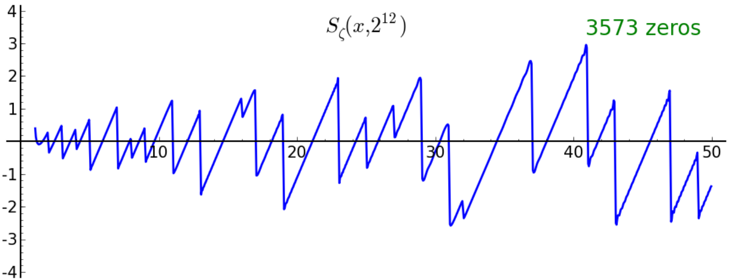

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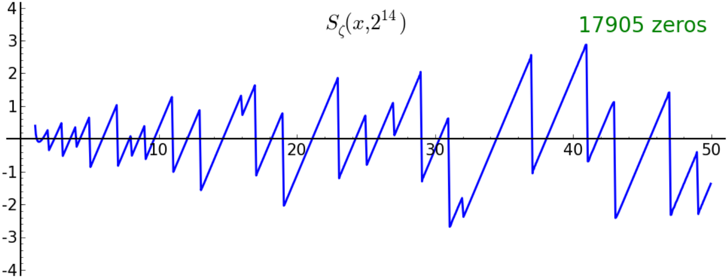

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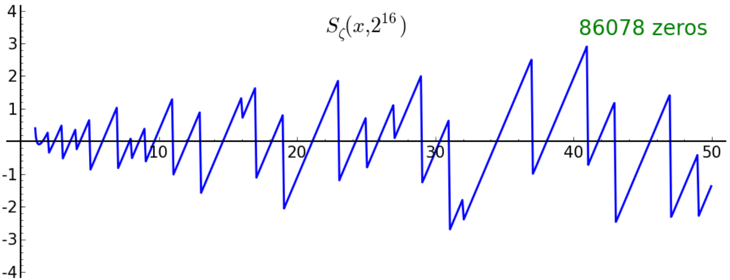

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$$\sum_{\rho} \frac{x^\rho}{\rho} = \lim_{T\to\infty} S_\zeta(x,T) = x - \frac{1}{2}\log\left(1 - 1/x^2\right) - \log(2\pi) - \psi_\zeta(x)$$

where $\psi_\zeta(x) = {\sum_{p^e \leq x}}' \log p$ *is the second Chebyshev function.*

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This is known as (one formulation of) *the explicit formula* for $\zeta(s)$.

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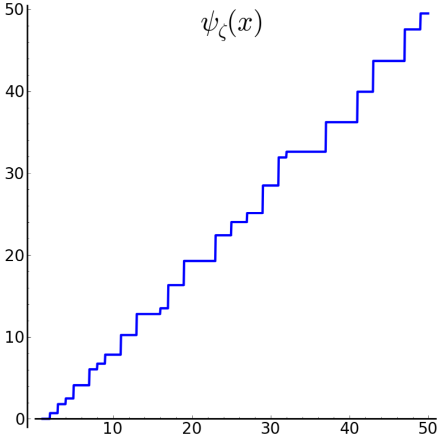

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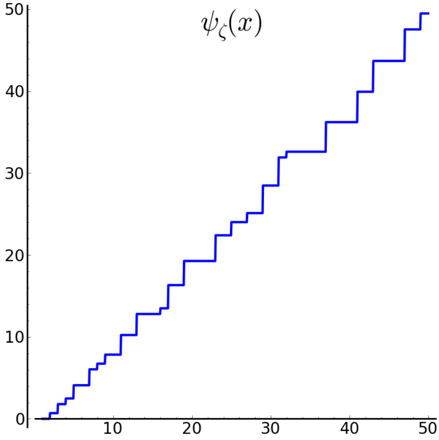

Equivalent Formulation of the Riemann Hypothesis

The above function $\psi_\zeta(x) = x + O(x^{1/2+\epsilon})$ for arbitrarily small $\epsilon > 0$.

L-Functions

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- Can define analogous L-functions attached to other number-theoretic objects:
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  - And many more

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I will show what you can do with elliptic curve L-functions.

Elliptic Curves

Definition

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$$E = 37a : y^2 = x^3 - 16x + 16$$

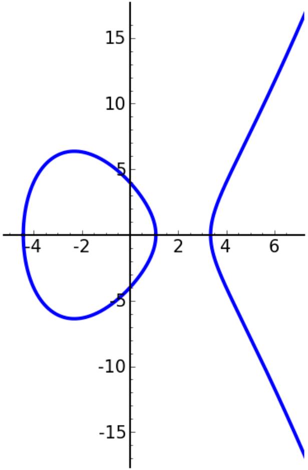

Figure: The Elliptic Curve 37*a*

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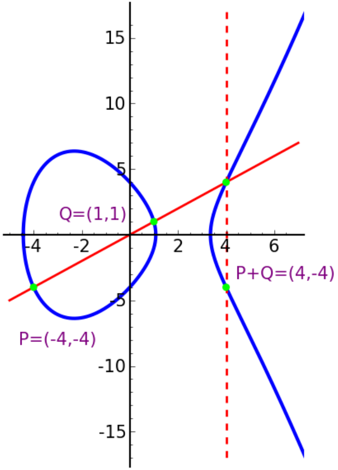

Figure: The Elliptic Curve 37a

Theorem (Mordell 1922, Weil 1928)

$$E(\mathbb{Q}) \approx E(\mathbb{Q})_{TOR} \times \mathbb{Z}^r$$

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Example

For $E = 37a$, we have $E(\mathbb{Q}) \approx \mathbb{Z}^1$, generated by $P = (0,4)$:

n	0	1	2	3	4	5	6
nP	$\mathcal{O}$	$(0,4)$	$(4,4)$	$(-4,-4)$	$(8,-20)$	$(1,-1)$	$(24,116)$

n	7	8	9
nP	$\left(-\frac{20}{9}, \frac{172}{27}\right)$	$\left(\frac{84}{25}, -\frac{52}{125}\right)$	$\left(-\frac{80}{49}, -\frac{2108}{343}\right)$

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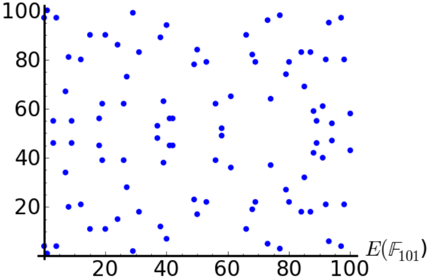

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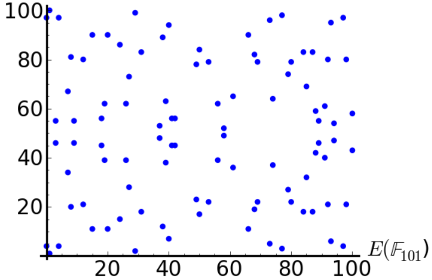

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Elliptic Curves over finite fields

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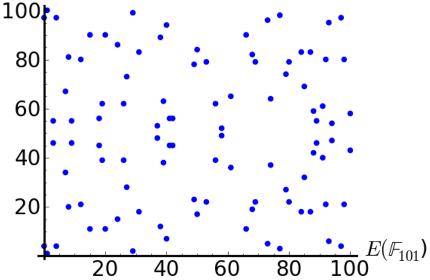

Let $\#E(\mathbb{F}_p)$ be the number of points on E modulo the prime p.

Theorem (Hasse, 1936)

$$p + 1 - 2\sqrt{p} \leq \#E(\mathbb{F}_p) \leq p + 1 + 2\sqrt{p} \quad \textit{for all } p.$$

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For prime p , let $a_p(E) = p + 1 - \#E(\mathbb{F}_p)$.

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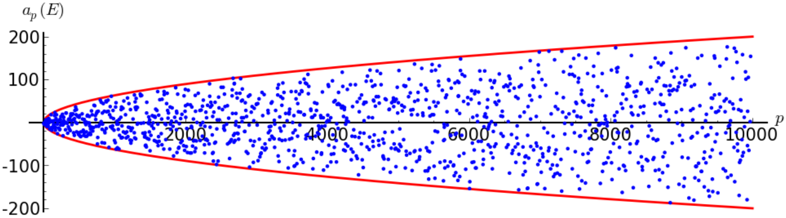
Example

$E = 37a$

p	2	3	5	7	11	13	17	19	23	29	31	37
a_p	-2	-3	-2	-1	-5	-2	0	0	2	6	-4	-1



The Conductor of a Curve

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The conductor N of an elliptic curve E is a positive integer that encapsulates primes of bad reduction for E, i.e. primes for which when we look at the set of points on E modulo p, *bad stuff** happens.

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Example

The conductor of $37a$ is $N = 37$, hence its name. That is, bad stuff only happens for this elliptic curve at $p = 37$.

Elliptic Curve *L*-Functions

Definition

The L-function attached to E is

$$L_E(s) := \prod_{p|N} \frac{1}{1 - a_p p^{-s}} \prod_{p\nmid N} \frac{1}{1 - a_p p^{-s} + p^{1-2s}} = \sum_{n=1}^{\infty} a_n n^{-s}$$

for $\Re(s) > \frac{3}{2}$.

The a_n are defined by multiplying out the Euler product.

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Definition

The *completed* L-function attached to E is

$$\Lambda_E(s) := N^{s/2}(2\pi)^{-s}\Gamma(s)L_E(s)$$

Analytic Continuation of $L_E(s)$

Theorem (Breuille, Conrad, Diamond, Taylor, Wiles et al, 1999,2001)

$L_E(s)$ extends to an entire function on $\mathbb{C}$. Specifically,

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where $w = 1$ or -1 depending on the elliptic curve.

Notably, unlike $\zeta(s)$, $L_E(s)$ has no poles on $\mathbb{C}$ for any given elliptic curve E.

The Zeros of $L_E(s)$

Three flavors:

- A simple zero at $0, -1, -2, -3, \ldots$
- A zero of order r_{an} at $s = 1$; r_{an} is called the *analytic rank* of E
- Countably infinite zeros in the strip $0 < \Re(s) < 2$, symmetric about $\Re(s) = 1$ and x-axis.

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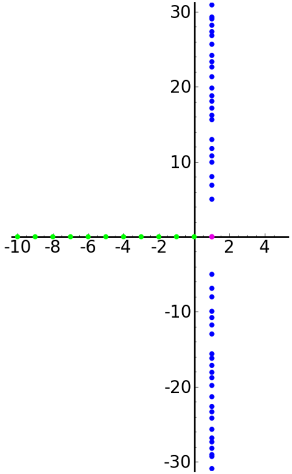

Figure: The zeros of $L_E(s)$ for $E = 37a$

The BSD Conjecture

Conjecture (Birch, Swinnerton-Dyer 1960s)

- $r_{an} = r$, *i.e. the order of vanishing of* $L_E(s)$ *at* $s = 1$ *equals the rank of the free part of* $E(\mathbb{Q})$

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- *The leading coefficient of* $L_E(s)$ *at* $s = 1$ *is*

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where

- *Ω_E is the real period of (an optimal model of) E,*
- *Reg_E is the regulator of E,*
- *$\#Ш(E/\mathbb{Q})$ is the order of the Shafarevich-Tate group attached to $E/\mathbb{Q}$,*
- *$\prod_p c_p$ is the product of the Tamagawa numbers of E, and*
- *$\#E_{Tor}(\mathbb{Q})$ is the number of rational torsion points on E.*

Things You Can Do With EC L-functions

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Proposition

If $E/\mathbb{Q}$ has conductor N and analytic rank r then

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Better results (S.), although nowhere close to effective yet:

r	$N \geq$	Smallest Known Conductor
0	3	11
1	6	37
2	16	389
3	55	5077
4	232	234446
5	1192	19047851
6	6696	5187563742

Things You Can Do With EC L-functions

Contingent on GRH and BSD we have a complete description of the Taylor series of L_E about $s = 1$. Specifically:

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Proposition

Let $L_E(s+1) = s^r \left(a + b \cdot s + c \cdot s^2 + O(s^3)\right)$,
where a is the leading coefficient described by BSD. Then

$$\frac{b}{a} = \eta + \log\left(\frac{2\pi}{\sqrt{N}}\right)$$
$$\frac{c}{a} = \frac{1}{2}\left[\eta + \log\left(\frac{2\pi}{\sqrt{N}}\right)\right]^2 - \frac{\pi^2}{12} + \sum_{\gamma>0} \gamma^{-2}$$

where γ runs over the imaginary parts of the nontrivial zeros of $L_E(s)$ (excluding $s = 1$), and $\eta = 0.57721566\ldots$ is the Euler-Mascheroni constant.

Recursive formulae exist for higher coefficients as well.

The Explicit Formula for Elliptic Curves

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Definition

Let

- $$S_E(x, T) := \sum_{|\gamma|<T} \frac{x^{i\gamma}}{i\gamma} = \sum_{0<\gamma<T} \frac{2\sin(\gamma \log x)}{\gamma}$$

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- $$\psi_E(x) := {\sum_{n\leq x}}' c_n(E)$$

  where $c_n(E) = -\left(p^e + 1 - \#\widetilde{E}(\mathbb{F}_{p^e})\right) \cdot \frac{\log(p)}{p^e}$ for $n = p^e$ a perfect prime power, and 0 otherwise.

The Explicit Formula for Elliptic Curves

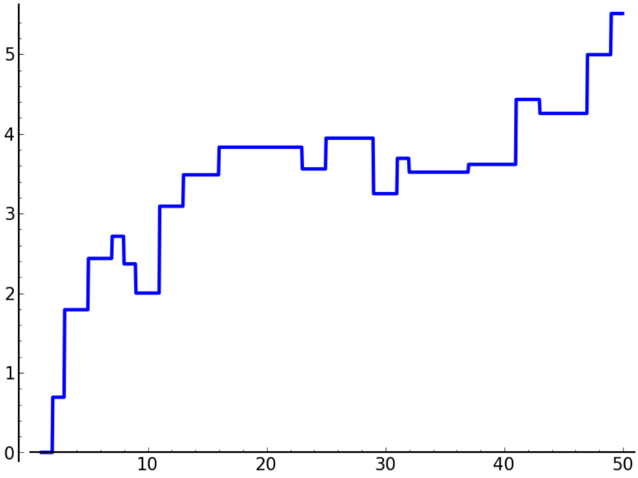

Figure: $\psi_E(x)$ for $E = 37a$

The Explicit Formula for Elliptic Curves

Theorem

For any any $E/\mathbb{Q}$ with conductor N and for any $x > 1$ the partial sum function $S_E(x, T)$ converges as $T \to \infty$. Specifically,

$$\lim_{T\to\infty} S_E(x, T) = \sum_{\gamma>0} \frac{2\sin(\gamma \log x)}{\gamma}$$

$$= -\eta - \log\left(\frac{2\pi}{\sqrt{N}}\right) - r_{an}\log x - \log(1 - x^{-1}) + \psi_E(x)$$

where η is the Euler-Mascheroni constant $= 0.5772156649\ldots$

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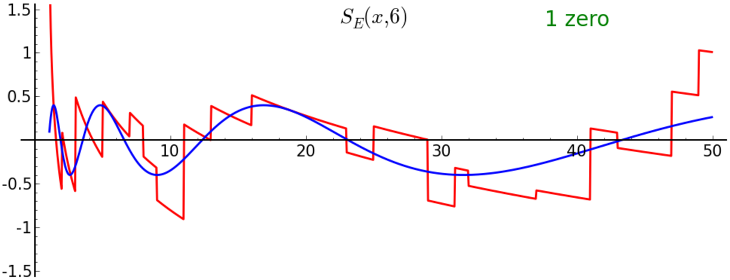

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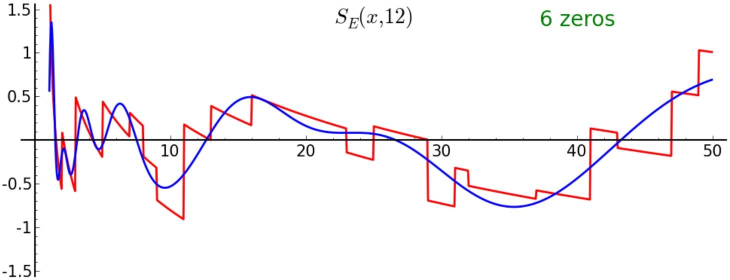

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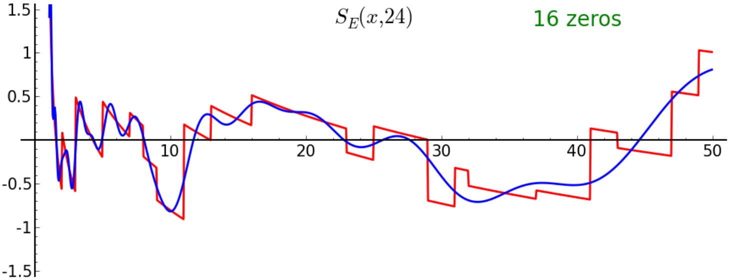

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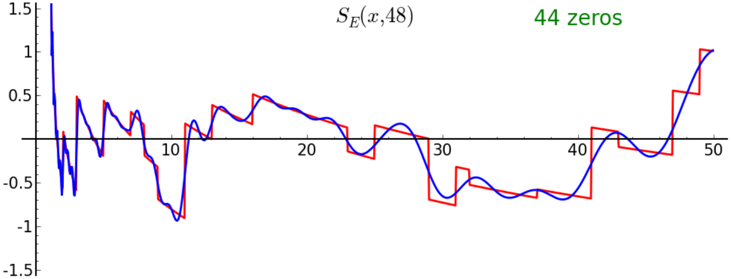

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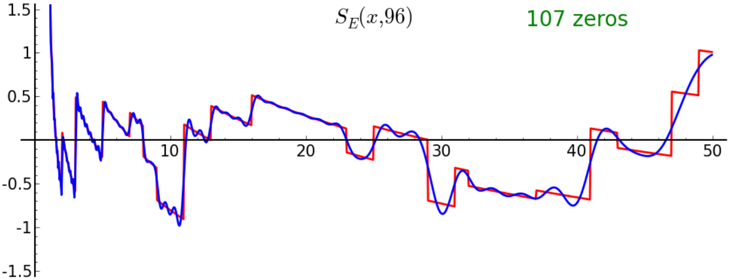

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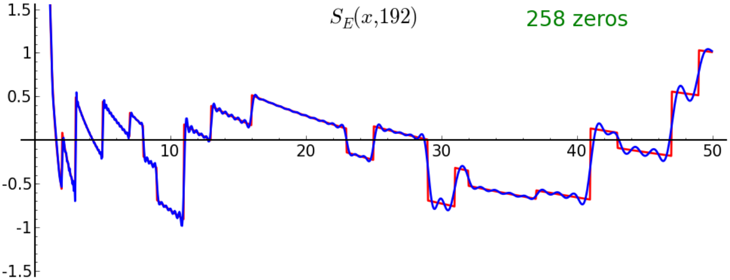

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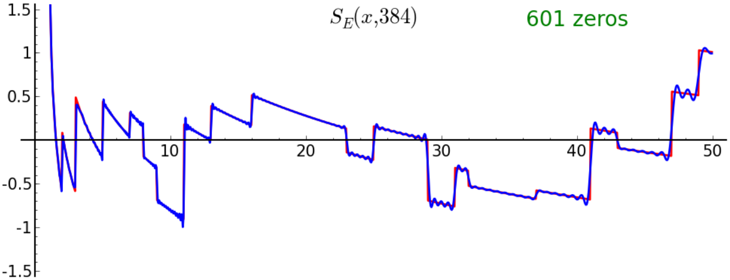

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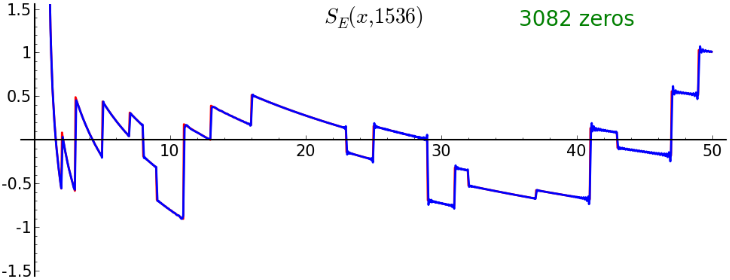

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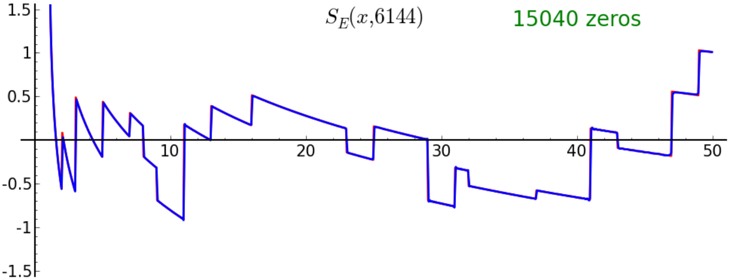

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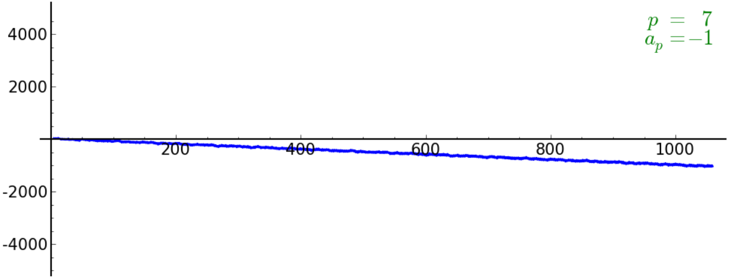

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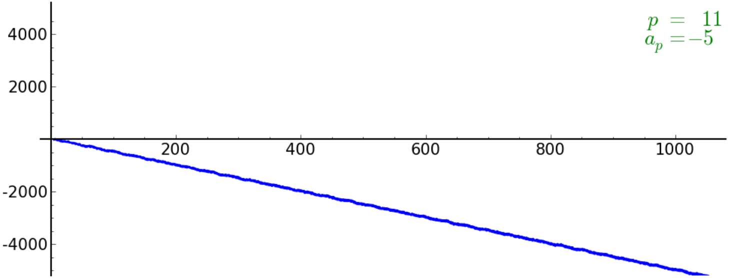

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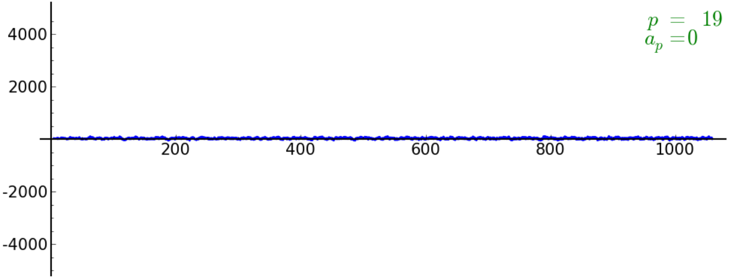

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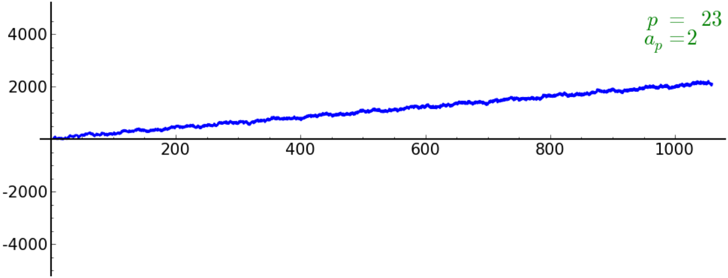

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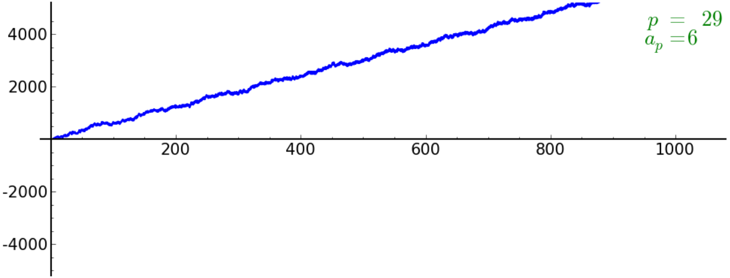

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Conjecture - Alternate BSD Part 1 (Sarnak, Mazur)

For any given $E/\mathbb{Q}$,

$$\lim_{x\to\infty} \frac{1}{\log(x)} \sum_{p\leq x} \frac{-a_p \log(p)}{p} = r$$

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Where does this comes from?

Take explicit formula:

$$\sum_{\gamma} \frac{\sin(\gamma \log x)}{\gamma} = -\eta - \log\left(\frac{2\pi}{\sqrt{N}}\right) - r\log x - \log(1-1/x) + \psi_E(x)$$

Divide both sides by $\log(x)$ and take limits*.

Ngiyabonga Kakhulu

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Hamba Kahle!