GENERALIZATIONS OF THE CUT-POINT LEMMA

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Abstract. The cut-point lemma of [1] is generalized to a multiple-source situation, and to the annulus; and a framework is provided that may prove useful in constructing a cut-point lemma for medial graphs on annulus-alikes with more than one center hole.

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Introduction

The cornerstone of the use of medial graphs to analyze the recoverability of circular-planar graphs is the Cut-Point Lemma (from [1]; also below as corollary 2.3). The statement of the lemma considers a graph embedded in a disc, and relates the *maximal connection* of that graph, which can be thought of as a maximal flow on the graph given unit node capacities, from the nodes contained in a boundary interval designated as the source, to the nodes contained in the complementary boundary interval, which is designated as a sink.

In this note we wish to investigate generalizations to the Cut-Point Lemma: specifically, we wish to consider cases in which source and sink are discontiguous, consisting of more than one boundary interval each.

The proof of the Cut-Point Lemma given in [1] uses only arc uncrossing at the boundary to convert the medial graph into one whose underlying graph has the same maximal connection size, but which contains no arc crossings.

One difficulty with naïvely applying the cut-point lemma to our divided-source case is that we might run out of empty boundary triangles while uncrossing the chords, since we are only guaranteed to have three disjoint empty boundary triangles, but we have four cut points. We must be careful not to disturb our counting by making an inappropriate uncrossing at one of the cut points.

By allowing $Y - \Delta$ transformations, however, and certain interior uncrossings in addition to uncrossings at the boundary, we can reach a crossing-free arrangement of the geodesic arcs that has the same maximal connection.

Another difficulty is that, once in reduced form, the black polygons bounded by our chords might have edges that intersect more than one source interval; we must

Date: July 17, 2013.

impose conditions on the original sets of chords to prevent this from happening if we hope to retain as simply expressible a conslusion as $m + r = n$.

In this note, the cut-point lemma of [1] is generalized to a multiple-source situation, and to the annulus; and a framework is provided that may prove useful in constructing a cut-point lemma for medial graphs on higher annulus-alikes with more than one center hole.

1. Reduction to Equivalent Crossing-Free Form

The following notation will be implicit in the propositions below. We are discussing finite collections of curves in the disc, all of which have both endpoints on the boundary circle, no three of which intersect at any single point of the disc, and no two of which intersect at more than one point, or at any point of the boundary circle. Such collections of these curves are denoted in calligraphic letters ($\mathcal{M}$, $\mathcal{N}$). The boundary circle is partitioned into a set of nonempty open intervals S_i and T_i, $i = 1, \ldots, k$, together with their endpoints, arranged so that T_i lies immediately counterclockwise from S_i and immediately clockwise from $S_{(i+1) \bmod k}$. The intervals S_i are designated as *source intervals*, and the T_i are designated *sink intervals*. Let $S = \bigcup_{i=1}^k S_i$ and $T = \bigcup_{i=1}^k T_i$. In addition, a two-coloring of the regions of the disc under $\mathcal{M}$ (respectively $\mathcal{N}$) is assumed, with an associated graph $G_\mathcal{M}$ divided into boundary nodes (black cells that lie on the boundary circle) and interior nodes (black cells not incident on the boundary circle). If a black cell does not contain an endpoint of any boundary source interval, but it has at least one edge incident on a boundary source interval, the corresponding node of $G_\mathcal{M}$ is designated as a source node; similarly, if a black cell does not contain an endpoint of any boundary sink interval, but it has at least one edge incident on a boundary sink interval, the corresponding node of $G_\mathcal{M}$ is designated as a sink node. $m_\mathcal{M}$, or simply m if there is no confusion, denotes the maximum flow in $G_\mathcal{M}$ between source and sink nodes, if unit node capacities are imposed. $n_\mathcal{M}$, or simply n, denotes the number of black cell edges contained in source intervals; and $r_\mathcal{M}$, or simply r, denotes the number of chords of $\mathcal{M}$ both of whose endpoints lie in the same source interval S_i.

The purpose of this section will be the reduction of any finite chord collection $\mathcal{M}$ to an equivalent collection $\mathcal{N}$ in which no chords cross. By analyzing the steps necessary to reduce $\mathcal{M}$ to $\mathcal{N}$, and by looking at the connection size in $\mathcal{N}$, we can develop criteria to predict the connection size in $\mathcal{M}$.

To begin with, the following two lemmas from [1] will be necessary:

Lemma 1.1. *Let $\mathcal{M}$ be as above. Denote by p the crossing of xy closest to x, and let the chord that intersects xy be called uv. Then there is an empty boundary triangle contained in $\triangle xpu$.*

Proof. If $\triangle xpu$ is empty, the conclusion is valid. If not, there must be another chord that intersects chord uv closer to u than p. Let q be the intersection nearest u, and denote by st the chord that intersects uv at q. st cannot intersect the segment xp, it cannot intersect uv twice, and $q \neq p$, so one endpoint, say s, must lie within $\triangle xpu$. Thus the triangle $\triangle uqs \subset \triangle xpu$, and $\triangle uqs$ has the same property that one leg is the boundary and a second leg has no crossings. Since $\mathcal{M}$ is finite, continuing in this way we must reach a boundary triangle that is contained within $\triangle xpu$ and which is empty. □

Lemma 1.2 (Three Triangles Lemma)**.** *Let $\mathcal{M}$ be as above. Then the chords from $\mathcal{M}$ form at least three disjoint empty boundary triangles.*

Proof. Let a be the endpoint of one of the intersecting chords, and let b be the intersection on that chord closest to a. By lemma 1.1, there exists an empty triangle $\triangle cde$ somewhere on the boundary. Let c and d be the points on the boundary, and let f and g be the opposite endpoints of their arcs, respectively. Apply lemma 1.1 to $\triangle ceg$ and $\triangle def$, respectively. Since $\triangle cde$, $\triangle ceg$ and $\triangle def$ are all disjoint, the three empty boundary triangles we have found must all be disjoint as well. □

Definition 1.3 (Connection-Equivalent)**.** Let $\mathcal{M}$, $\mathcal{N}$, S and T be as above, and suppose that $\mathcal{M}$ and $\mathcal{N}$ color the boundary circle identically. $\mathcal{M}$ and $\mathcal{N}$ are said to be *connection-equivalent* if $m_{\mathcal{M}} = m_{\mathcal{N}}$.

Definition 1.4 (nr-Equivalent)**.** Let $\{S_i\}$ be a set of disjoint open intervals on the unit circle, and let $\mathcal{M}$ and $\mathcal{N}$ be collections of chords in the disc, each with a respective two-coloring. Define $n_{\mathcal{M}}$ as the number of black boundary intervals of $\mathcal{M}$ contained in any of the intervals S_i, define as $r_{\mathcal{M}}$ the number of chords of $\mathcal{M}$ with both endpoints in any of the S_i; and similarly define $n_{\mathcal{N}}$ and $r_{\mathcal{N}}$.

$\mathcal{M}$ and $\mathcal{N}$ are said to be *nr-equivalent* if $n_{\mathcal{M}} - r_{\mathcal{M}} = n_{\mathcal{N}} - r_{\mathcal{N}}$.

Fact 1.5. Let $\mathcal{M}$ and $S_1, \ldots, S_k$ be as above. Let two chords ab and cd in $\mathcal{M}$ have endpoints a and c within the same source interval S_i, and let ab and cd intersect at a point e. Suppose $\triangle aec$ is empty. Then the chord collection $\mathcal{N}$ produced from $\mathcal{M}$ by uncrossing ab and cd at e is both connection-equivalent and nr-equivalent to $\mathcal{M}$.

Definition 1.6 (Gridwork Form)**.** Let $\mathcal{M}$ be as above, and let the boundary circle be partitioned into n boundary segments $B_1, \ldots, B_n$, ordered counterclockwise around the circle, such that every endpoint of every curve of $\mathcal{M}$ is contained in one of the B_i. Denote by $\mathcal{M}_i$ all the chords of $\mathcal{M}$ that have at least one endpoint in B_i. $\mathcal{M}$ is said to be in *gridwork form* as respects $\{B_i\}$ if no chords of $B_i \cap B_j$ intersect one another for any i, j, and if addition $\mathcal{M}$ contains no chords that are reentrant in any of the boundary segments B_i.

Proposition 1.7. *Let $\mathcal{M}$ and S be as above, with $k = 1$. Let every chord of $\mathcal{M}$ have at least one endpoint in S. Suppose that there are two chords in $\mathcal{M}$ that intersect. Then there is an empty boundary triangle one of whose sides lies completely in S.*

Proof. Let $a \in S$ be the endpoint of any chord $ab \in \mathcal{M}$ that intersects at least one other chord of $\mathcal{M}$. Let cd intersect ab at e such that there are no other chord intersections on the segment ae. Either c or d must lie in S, by hypothesis; suppose c. Then by lemma 1.1, there exists an empty boundary triangle somewhere within $\triangle aec$. $ac \subset S$, so our conclusion is valid. □

Proposition 1.8. *Let $\mathcal{M}$ and $S_1, \ldots, S_k$ be as above. Then $\mathcal{M}$ is both connection-equivalent and nr-equivalent to a set $\mathcal{N}$ of chords, where $\mathcal{N}$ is in gridwork form.*

Proof. Using Ringel's Theorem, we can "comb" the crossings of all chords incident on S_i all into the region D_i reachable from S_i without crossing any chord not incident on S_i, using only Reidemeister type 3 movements [2][3]. Since Reidemeister type 3 movements in the medial graph are equivalent to $Y - \Delta$ transformations in

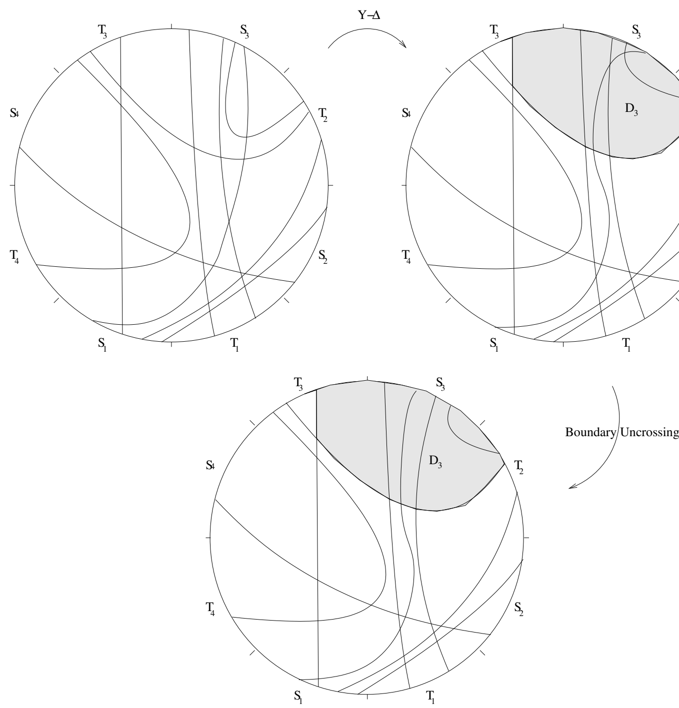

FIGURE 1. An illustration of the "comb and unravel" operation described in proposition 1.8.

the underlying graph, this "combing" does not change the connection properties of $\mathcal{M}$.

Now consider the chords incident on S_i restricted to D_i; call this family of chords, so restricted, $\mathcal{M}_i$. By proposition 1.7, if any chords of $\mathcal{M}_i$ intersect within D_i, there exists an empty boundary triangle on S_i. We can then uncross the chords on the boundary segment S_i without changing the size of the maximal connection using fact 1.5. Since $\mathcal{M}$ contains a finite number of chords, so does $\mathcal{M}_i$, so a finite number of boundary uncrossings will yield no crossings in $\mathcal{M}_i$.

Performing these two operations for S_i and T_i, for each $i = 1, \ldots, k$, will result in a chord collection for which no two chords with endpoints in the same source or sink interval intersect, while preserving m and $n - r$.

Finally, removal of any arc reentrant in any of the S_i or any of the T_i does not change m or $n - r$. □

Proposition 1.9. *Let $\mathcal{M}$ and $S_1, \ldots, S_k$ be as above, with $\mathcal{M}$ in gridwork form, and suppose that $\mathcal{M}$ contains two intersecting chords. Then there exists a pair of chords that may be uncrossed to produce a collection $\mathcal{N}$ of chords, also in gridwork form, that is both connection-equivalent and nr-equivalent to $\mathcal{M}$.*

Proof. Let $\mathcal{M}'$ be the subset of $\mathcal{M}$ consisting of chords that intersect other chords. By hypothesis, $\mathcal{M}'$ is non-empty; so by lemma 1.2 $\mathcal{M}'$ has at least three empty boundary triangles. Choose one of these: let chords ab and cd intersect at a point

e, so that $\triangle aec$ is an empty boundary triangle in $\mathcal{M}'$. Note that, since $\mathcal{M}$ is in gridwork form, $\triangle aec$ must contain one or more cut points.

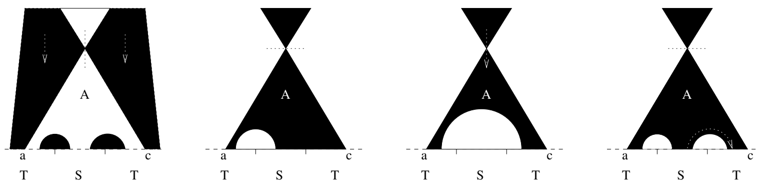

FIGURE 2. The four cases described in the proof of proposition 1.9. White dotted lines indicate a connection that must be preserved; black dotted lines indicate a lack of connection, and hence an acceptable breaking point.

Let A be the single-colored polygon contained within $\triangle aec$ and having as two of its edges ae and ce. Choose one of the two possible uncrossings of these two edges, depending on the color of A and whether it contains any cut points or source or sink intervals, according to the following cases. (Note that for some chord collections, both directions of uncrossing may be valid.)

Case 1. A is white. There might be connections to the black regions to its left or right, but since both are boundary sink nodes, they cannot participate in the same connection. Uncross to create chords ad and bc.

Case 2. Case 1 is not satisfied, and A contains a cut point. Although A is black, it cannot participate in a connection. Uncross to create chords ac and bd.

Case 3. Cases 1 and 2 are not satisfied, and the boundary intervals contained in A are either all source intervals or all sink intervals. A could participate in a connection as an endpoint, but within itself it cannot form a connection. Uncross to create chords ad and cb.

Case 4. Cases 1–3 are not satisfied. Then A is black and does not contain any cut points, and the boundary intervals contained in A include at least one source interval and at least one sink interval. There might or might not be a connection using A as an endpoint; there is definitely a connection with A as both source and sink. Since both connections cannot exist simultaneously, we choose in favor of the definite connection and uncross to create chords ac and bd.

Uncrossing chords ab and cd in either direction does not change n or r, and in all cases we have preserved m.

It may be that by uncrossing ab and cd, we have created a collection that is no longer in gridwork form; if that is the case, we can apply proposition 1.8 to return us to gridwork form without affecting the number of crossings in the gridwork. □

Lemma 1.10. *Let $\mathcal{M}$ and $S_1, \ldots, S_k$ be as above. Then $\mathcal{M}$ is both connection-equivalent and nr-equivalent to a collection $\mathcal{N}$ of chords in the disc containing no intersections and for which $r = 0$.*

Proof. Use proposition 1.8 to convert $\mathcal{M}$ to gridwork form, and then repeatedly apply proposition 1.9 to remove the remaining intersections until none remain. □

Fact 1.11. Let $\{S_i\}$, $\{T_i\}$, and $\mathcal{M}$ be as above, and suppose that no two chords of $\mathcal{M}$ intersect, and that $\mathcal{M}$ contains no reentrant chords. Then $m_{\mathcal{M}}$ is given by the number of black polygons one of whose edges lies in some source interval S_i, one of whose edges lies in some sink interval T_j, and which does not contain any of the endpoints of any source interval S_k.

2. Cut Point Lemmas

At this point, we have a means to reduce an arbitrary $\mathcal{M}$ to an equivalent intersection-free form $\mathcal{N}$, where connections are easy to count. It is not correct to conclude that $m+r=n$ in all cases, however, since $\mathcal{N}$ could contain black polygons that have edges contained in multiple different source intervals, or polygons one of whose edges is completely contained within a source interval S_i but another of whose edges contains the endpoint of a different source interval S_j.

If after reducing to intersection-free form, however, every adjacent pair of sink intervals T_i were connected by a chord, then the cases that present us difficulty could be avoided.

Definition 2.1 (Encapsulation Condition)**.** Let $\mathcal{M}$, $\{S_i\}$, and $\{T_i\}$ be as above. Let s_i be the number of chords with one endpoint in S_i and the other endpoint not in S_i, T_i, or $T_{(i+k-1) \mod k}$, and let t_i be the number of chords with one endpoint in T_i and the other in $T_{(i+k-1) \mod k}$. If $t_i > s_i$ for all i, then $\mathcal{M}$ is said to satisfy the *encapsulation condition.*

Lemma 2.2 (k-source Cut Point Lemma)**.** *Let $\mathcal{M}$, $\{S_i\}$, and $\{T_i\}$ be as above, and suppose that if $k > 1$, $\mathcal{M}$ satisfies the encapsulation condition. Then $m + r = n$.*

Proof. Using lemma 1.10, we can transform $\mathcal{M}$ to an equivalent $\mathcal{N}$ containing no intersections and no reentrant chords in S or T. If $\mathcal{M}$ has $t_i > s_i$ for all i, then $\mathcal{N}$ has $t_i > 0$ for all i; since $\mathcal{N}$ also has no intersections and no reentrant chords, then no black polygon of $\mathcal{N}$ has edges that intersect more than one source interval, and any black polygon of $\mathcal{N}$ has at most one edge that inersects any source interval. That is also (trivially) true if $k = 1$. Thus $\mathcal{N}$ has $m = n$ with $r = 0$; so $\mathcal{M}$ has $m + r = n$. □

Corollary 2.3 (Cut Point Lemma)**.** *Let $\mathcal{M}$, S, and T be as above, with $k = 1$. Then $m + r = n$.*

Corollary 2.4 (Double-Source Cut Point Lemma)**.** *Let $\mathcal{M}$, $\{S_i\}$, $\{T_i\}$, and m, n, and r be defined as above, with $k = 2$. Suppose that there are strictly more chords in $T_1 \times T_2$ than there are in $S_1 \times S_2$. Then $m + r = n$.*

The double-source cut-point lemma allows us to form a cut-point lemma for the annulus, subject to some restrictions. There are two natural ways the cut-point lemma might be expressed on the annulus: first, we might want two contiguous source regions, one on the inner and one on the outer boundary; second, we might want a single source region, either on the outer boundary or on the inner boundary. The following two propositions address these cases.

Proposition 2.5. *Let $\mathcal{M}$ be a collection of curves in the annulus. Let $S_1 = \widehat{X_1, Y_1}$ be a segment on the inside boundary of the annulus, and let $S_2 = \widehat{X_2, Y_2}$ be a segment on the outside boundary. Let $B_1 = \overline{Y_1, X_2}$ and $B_2 = \overline{Y_2, X_1}$ be curves drawn on the annulus such that every curve of $\mathcal{M}$ intersects B_1 or B_2 at most once. (This is always possible.) Denote by h_1 the number of chords that span between B_1 and B_2 within the region $X_1Y_1X_2Y_2$, and by v_2 the number of chords that span between B_1 and B_2 within the region $X_2Y_1X_1Y_2$. Denote by v_1 the number of chords that span*

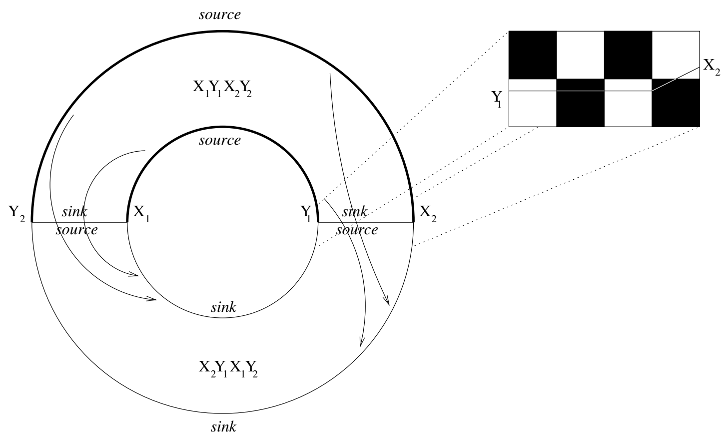

FIGURE 3. An illustration of chaining the two-source cut-point lemma to provide a form of cut-point lemma on the annulus.

between S_1 and S_2 within the region $X_1Y_1X_2Y_2$ and by h_2 the number of chords that span between T_1 and T_2 within the region $X_2Y_1X_1Y_2$. Suppose that $h_1 > v_1$ and $h_2 > v_2$. Then $m + r = n$.

Proof. Chain the two semi-annular regions together as shown in figure 3. Because the curves B_1 and B_2 cannot cross any medial line twice, and cannot have an endpoint on a medial endpoint, the nodes on either side of curves B_1 and B_2 are identically matched, and no medial line is reentrant in B_1 or B_2. If the conditions for corollary 2.4 are met on $X_2Y_1X_1Y_2$, then, any connection in $X_1Y_1X_2Y_2$ has a matching path in $X_2Y_1X_1Y_2$ that will take it to the boundary. □

Proposition 2.6. *Let $\mathcal{M}$ be a collection of curves in the annulus. Let $S_1 = \widehat{X_1, Y_1}$ be a segment on the inside boundary of the annulus, and let $\widehat{X_2, Y_2}$ be a segment on the outside boundary. Let $B_1 = \overline{Y_1, X_2}$ and $B_2 = \overline{Y_2, X_1}$ be curves drawn on the annulus such that every curve of $\mathcal{M}$ intersects B_1 or B_2 at most once. Denote by v_2 the number of chords that span between B_1 and B_2 within the region $X_2Y_1X_1Y_2$, and by h_2 the number of chords that span between T_1 and T_2 within the region $X_2Y_1X_1Y_2$. Suppose that $h_2 > v_2$. Then $m + r = n$.*

Proof. The proof is the same as for the previous proposition; but using corollary 2.3 to count the connections in $X_1Y_1X_2Y_2$ instead of corollary 2.4. □

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