The Geodesic Integral on Medial Graphs

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We define the geodesic integral defined on paths in the duals of medial graphs on surfaces and use it to study lens elimination and connection properties of circular and annular planar networks.

1 Definitions

1.1 Quasi-medial graph

Let S be an orientable surface and R a region in S with boundary ∂R relative to S. An *undirected quasi-medial graph* in R is a multigraph $\Gamma = (V, E)$ with a set of boundary vertices $\partial V \subseteq V$ together with an embedding into R, subject to the following conditions:

- Every vertex of ∂V has degree 1 and is mapped to a point of ∂R under the embedding;
- Every vertex of $V \setminus \partial V$ degree 4 and is mapped to a point of $R \setminus \partial R$ under the embedding;
- Every face of Γ is simply connected.

A *directed quasi-medial graph* Γ is a directed multigraph with boundary and an embedding into R such that the undirected multigraph with the same vertices, edges (but undirected), boundary, and embedding is an undirected quasi-medial graph, with the additional condition that each interior vertex of Γ have in-degree equal to out-degree.

Quasi-medial graphs appear as medial graphs of multigraphs with boundary embedded in R with all faces simply connected. However, clearly not every quasi-medial graph arises in this fashion, and we sometimes wish to consider the medial graph separately from an underlying primal graph.

A *geodesic* in Γ is a path in Γ which follows the transverse edge at every vertex (i.e., not turning left or right), without regard to orientation. Trivially, the set of geodesics partitions the edges of Γ. Consequently, each vertex lies at the intersection of one or two geodesics; each geodesic either contains no boundary vertices or two distinct boundary vertices. A geodesic has *consistent orientation* if to each vertex on the geodesic there is one inward incident edge of the geodesic and one outward incident edge of the geodesic.

A *loop* in Γ is a segment of a geodesic with the same start and end points with no other self-intersections along this segment. A *simple loop* is one which bounds a simply connected region. A *simple empty loop* is a simple loop which contains no other vertices or edges, or, equivalently, an edge (v, v) which is contractible to a point.

A *lens* in Γ is a pair of segments of geodesics with the same start and end points and having no other intersections except these start and end points. A *simple lens* is one such that these two segments of geodesics bound a simply connected region. A *simple empty lens* is a simple lens which contains no other vertices or edges, or, equivalently, a pair of edges which belong to distinct geodesics and connect the same two vertices.

An undirected quasi-medial graph is called *lensless* if it contains no loops and no lenses. It is called *simply lensless* if it contains no simple loops and no simple lenses.

1.2 Geodesic integral

Let Γ be a quasi-medial graph as above and let Δ be its dual. The conditions on Γ imply that all faces of Δ are simply connected and have four sides.

A path in Δ is a path along the faces of Γ; we call it a *dual path* in Γ.

We define a function associating to each path p in Δ an integer $\int p$, the *geodesic integral along* p, defined as follows. For a path $p = vw$ consisting of two vertices, define

$$\int p = \begin{cases} 1 & \text{if the edge } vw \text{ of } \Delta \text{ crosses its dual edge in } \Gamma \text{ from right to left} \\ -1 & \text{if the edge } vw \text{ of } \Delta \text{ crosses its dual edge in } \Gamma \text{ from left to right} \end{cases}.$$

If $p = v_0 v_1 \dots v_n$, then define $\int p = \sum_{i=1}^{n} \int (v_{i-1} v_i)$, where, by definition, the geodesic integral along a path with no edges is 0. Figure 1 shows an example computation of a geodesic integral.

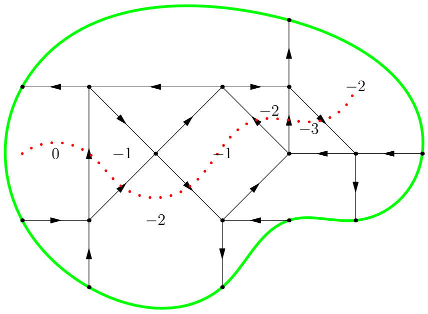

Figure 1: Example of geodesic integral computation along the dotted path.

It is immediate from the definition that the geodesic integral satisfies $\int p = -\int \overline{p}$, where $\overline{p}$ denotes p traversed in the opposite direction, and that $\int pq = \int p + \int q$, where pq is the concatenation of the paths p and q.

Lemma 1. *A closed path in Δ which is contractible to a point in R has geodesic integral 0.*

Proof. Let p be a closed path. We may suppose p has no self-intersections; if it does, then the integral along each simple closed component is 0, as the following argument will show. Without loss of generality, suppose that Δ traces counterclockwise the boundary of the simply connected region F which it encloses.

Write I_f for the geodesic integral counterclockwise around a face f of Δ. Then $\int p = \sum_f I_f$, where the sum is taken over all faces f inside F. Indeed, each edge in the interior of F appears twice on the right side of the equality, but in opposite directions, and the integrals along these edges cancel out. This leaves the counterclockwise edges along the boundary of F.

But the condition that the in-degree is equal to the out-degree in Γ translates to the statement that the geodesic integral around any face of Δ is 0. Therefore, $\int p = 0$. □

Note 2. *A path in R passes through certain faces of Γ, defining a path in Δ. The preceding lemma shows that the path's behavior near boundaries of faces is irrelevant. Thus, the geodesic integral an be used to define a group morphism from the fundamental group of R to $\mathbb{Z}$.*

Questions of algebraic topology aside, we have the following.

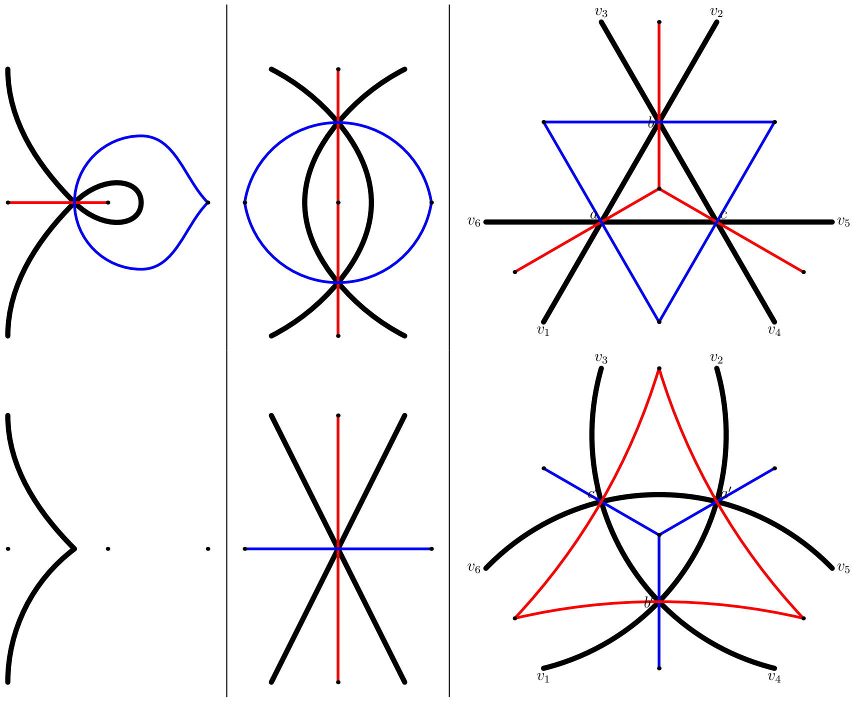

Figure 2: Lens elimination operations: (1) loop deletion, (2) lens contraction, (3) triangle inversion. Medial graph shown in black, primal graph in red, dual graph in blue.

Corollary 3. *If two paths p and q in Δ are homotopic relative to their start and end points, then $\int p = \int q$.*

Proof. pq is contractible to a point. $\square$

Corollary 4. *If R is simply connected, then the geodesic integral along any loop is 0 and the integral between two points is independent of path.*

Proof. Trivial from the previous. $\square$

2 Lens elimination

In the present section we will use the geodesic integral to obtain results about lens elimination in certain kinds of quasi-medial graphs.

Suppose that we are given an undirected quasi-medial graph and are permitted to perform the following three *lens elimination* operations and their inverses.

(1) Given a simple empty loop, we may delete it and contract the resulting two series edges into one. (This corresponds to elimination of pendants or self-loops in a primal graph.) We must note the special case that a vertex has two self-loops. In this case, after elimination of one of the loops, contraction along the second loop is just deletion of the remaining vertex and edge.

(2) Given a simple empty lens, we may contract its two constituent edges. (This corresponds to contraction of one of a pair of series edges or deletion of one of a pair of parallel edges in a primal graph.)

(3) Given a triangle of three edges enclosing a simply connected region, we may invert it. That is: suppose we are given a triangle abc, where a, b, and c are interior vertices. Let the geodesic segments surrounding the edges of this triangle be v_1abv_2, v_3bcv_4, v_5cav_6, where the v_i are not necessarily distinct. We may delete the vertices a, b, and c and insert new vertices a', b', and c' and edges along the paths $v_1b'a'v_2$, $v_3c'b'v_4$, and $v_5a'c'v_6$. (This corresponds to a Y-Δ transformation in a primal graph.)

It is easy to see the graph resulting after any of these transformations remains quasi-medial: the degree conditions are trivial to check, and the condition on simply connected faces follows because all three operations are local. Figure 2 shows these three operations and the corresponding operations on a primal graph.

If one undirected quasi-medial graph can be obtained from another by a finite sequence of these operations, then the two graphs are called *equivalent*. If these operations are considered to be induced by pendant and self-loop eliminations, parallel and series edge simplifications, and Y-Δ modifications in an electrical network, then equivalent-medial graphs imply electrically (response-)equivalent networks.

Theorem 5. *Every undirected quasi-medial graph is equivalent to some simply lensless undirected quasi-medial graph.*

Proof. We shall show that every quasi-medial graph which is not simply lensless is equivalent to a quasi-medial graph with strictly fewer edges. Because the number of edges is bounded below by 0, the result will follow.

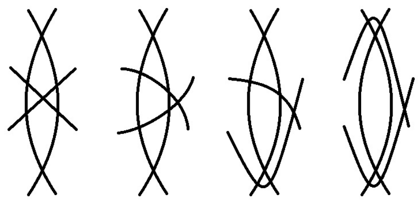

Figure 3: Emptying a minimal simple lens.

Suppose that Γ is an undirected quasi-medial graph which is not simply lensless. Choose a simple loop or simple lens ℓ which is minimal (contains no other simple loops or simple lenses).

If ℓ is empty, then we may eliminate it by operation (1) or (2), creating an equivalent graph with fewer edges.

Otherwise, suppose ℓ is a loop. Then any geodesic γ entering the interior of ℓ must cross the loop where it exists, forming a simple lens inside ℓ because ℓ is simple and γ has no loops inside ℓ. This contradicts minimality of ℓ.

So, ℓ is a lens. Let it be composed of the geodesics γ and δ. If a geodesic enters ℓ crossing γ, then it must exit crossing δ; else, a simple lens would be formed inside ℓ. The symmetric statement is also true; thus, every geodesic entering ℓ exits through the opposite side and no two geodesic segments cross more than once inside ℓ. Performing a series of operations (3), which do not change the number of edges, results in an equivalent graph with an empty lens. The details of verifying this are left to the reader to prove or find in [1]; see Figure 3 for an example. Finally, this empty lens may be removed by operation (2), which reduces the number of edges.

This completes the proof. $\square$

We will now show how to perform these three operations on directed quasi-medial graphs, changing the orientations of only the edges affected by the operations.

(1) When we eliminate an empty loop, the orientations of the remaining two edges incident to the point next to the vertex of the loop must be opposite with respect to this vertex. So these two edges may be contracted and

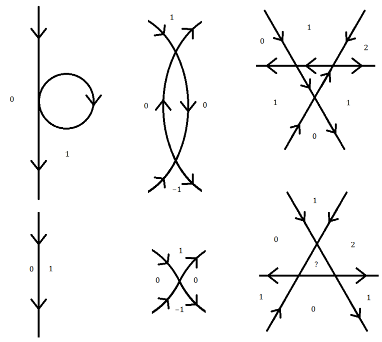

Figure 4: Lens elimination operations on a directed graph: (a) loop deletion, (b) lens contraction, (c) triangle inversion.

given the same orientation. This preserves the property that the in-degree is equal to the out-degree at all vertices. (See Figure 4(a).)

(2) When we eliminate an empty loop by contracting its two edges, we do not change the orientations of any other edges. The geodesic integral around the resulting new vertex is still 0 because it is equal to the integral around the entire loop in the original graph. (See Figure 4(b).)

(3) For this operation, we do not change the orientations of the affected edges incident to $v_1, \ldots, v_6$, but assign orientations to the new edges $a'b'$, $b'c'$, and $c'a'$. Choose a face f of the original graph adjacent to the triangle and label the six faces surrounding this triangle with the geodesic integrals from f around the triangle as shown in Figure 4(c). If the orientations of the old edges are not to be changed, the geodesic integrals from the chosen face to all the faces surrounding the triangle must remain the same. If we can choose a value of the geodesic integral for the interior of the new triangle that differs by 1 from each of the faces adjacent to it, this determines the orientations of the new edges $a'b'$, $b'c'$, and $c'a'$. But these faces are the three faces of the old graph which had been labeled with odd numbers. Notice that no two of them can differ

by more than 2 – either they are all equal or two are the same and the third differs from them by 2. In both cases it is clearly possible to choose an admissible value for the triangle's interior.

2.1 Lens elimination: circular case

In the case that R is a disc in the plane, a quasi-medial graph Γ in R is called *circular planar*. Suppose that the boundary of R is partitioned into two arcs, the *upper* and *lower* arcs, and that each boundary vertex of Γ lies on one of the arcs. The two points on the boundary where the arcs meet are called the *left* and *right cut points* such that the lower arc is directly clockwise of the right cut point.

Following the terminology of [3], we classify the geodesics of Γ according to their endpoints:

- A geodesic with both endpoints on the lower arc is called a *rock*;
- A geodesic with both endpoints on the upper arc is called a *cloud*;
- A geodesic with one endpoint on each arc is called a *tree*;
- A geodesic with no endpoints is called an *island*.

Let $t(\Gamma)$ denote the number of tree geodesics in Γ.

We would like to study how lens elimination affects the number geodesics of each type.

Theorem 6. *Suppose that a circular planar undirected quasi-medial graph Γ° is equivalent to a lensless circular planar undirected quasi-medial graph Γ'°. Then $t(\Gamma'^\circ) \geq t(\Gamma^\circ)$.*

Proof. Suppose that Γ° is a circular planar undirected quasi-medial graph and consider the circular planar directed quasi-medial graph Γ, defined to be Γ with every edge oriented in the direction leading to the lower arc along the geodesic containing it. In this way, at every vertex the degree conditions are satisfied.

Consider a path p in the dual of Γ from the face containing the left cut point to the face containing the right cut point. $\int p$ does not depend on the choice of p, so choose it to be the path following the lower arc counterclockwise. p crosses each tree along the lower arc once in a direction with positive geodesic integral and each rock along the lower arc twice (once in a direction with positive geodesic integral and once in the opposite direction), and no other geodesics. So, $\int p = t(\Gamma^\circ)$.

Perform on Γ the same lens elimination operations that are used to obtain $\Gamma' \circ$ from Γ° to result in a simply lensless graph Γ'. The orientations of the edges incident to boundary vertices have not changed as a result of these operations. So, if p' is a dual path in Γ' from the left cut point to the right cut point, then $\int p = \int p'$.

Γ' is simply lensless, so it could not contain clouds which intersect rocks (else, there would be a simple lens). Choose a dual path q' in Γ' from the left cut point to the right cut point which does not cross any rocks or clouds and crosses each tree exactly once, so $\int q' \leq t(\Gamma'^\circ)$. Because R is simply connected, $\int q' = \int p'$. So $t(\Gamma'^\circ) = \int q' = \int p' = \int p = t(\Gamma^\circ)$. □

Corollary 7. *Suppose that a circular planar undirected quasi-medial graph Γ° in which all geodesics are trees is equivalent to a lensless circular planar undirected quasi-medial graph Γ'°. Then all geodesics in Γ'° are trees.*

Proof. It is trivial from the preceding theorem that there are no rocks or clouds in Γ'°. Observe that the operations of lens elimination do not change the number of connected components of the graph, so there are no islands in Γ'. (If an island did not form a connected component, there would be a simple lens.) □

2.2 Lens elimination: annular case

In the case that R is an annulus in the plane, a quasi-medial graph Γ in R is called *annular*. R has two boundary segments, the *interior* and *exterior circles*.

Similarly to the circular case, we classify the geodesics of Γ according to their endpoints:

- A geodesic with both endpoints on the exterior circle is called a *rock*;
- A geodesic with both endpoints on the interior circle is called a *cloud*;
- A geodesic with one endpoint on each circle is called a *tree*;
- A geodesic with no endpoints is called an *island* if it encloses a simply connected region and a *river* otherwise.

Let $t(\Gamma)$ denote the number of tree geodesics in Γ.

We have the following analogues of the circular planar case. Note that "lensless" has been weakened to "simply lensless".

Theorem 8. *Suppose that an annular planar undirected quasi-medial graph Γ° is equivalent to a simply lensless annular planar undirected quasi-medial graph Γ'°. Then $t(\Gamma'^\circ) \geq t(\Gamma^\circ)$.*

Proof. Let Γ be the the directed graph which is Γ° with all trees oriented from the interior circle to the exterior circle. The geodesic integral around a path encircling the center of the annulus once counterclockwise is equal to the number of trees. The rest of the proof proceeds identically to its circular planar analogue. □

Corollary 9. *Suppose that an annular planar undirected quasi-medial graph Γ° in which all geodesics are trees is equivalent to a simply lensless annular planar undirected quasi-medial graph Γ'°. Then all geodesics in Γ'° are trees.*

Proof. Trivial. □

2.3 Lens elimination: further questions

In the circular planar case, the following is true.

Theorem 10. *Suppose Γ_1 and Γ_2 are equivalent lensless undirected circular planar quasi-medial graphs. Then Γ_1 can be obtained from Γ_2 by applying operation (3) repeatedly.*

Proof. [1] □

In particular, any two graphs we obtain from a circular planar quasi-medial graph after eliminating lenses are "the same": the endpoints of the geodesics occur in the same order along the boundary.

Question 11. *Under what conditions on an undirected annular planar quasi-medial graph Γ can any two simply lensless graphs equivalent to Γ be obtained from one another by operation (3)?*

If G is an electrical network embedded in an annulus with simply lensless medial graph Γ, nonexistence of simply lensless graphs equivalent to Γ which

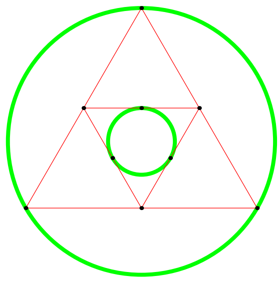

Figure 5: Embedding of the 2-to-1 triangle-in-triangle graph in the annulus.

cannot be obtains from Γ by operation (3) is a necessary condition for recoverability of G from its electrical response. It is not a sufficient condition, as illustrated by the graph shown in Figure 5. This graph's medial graph (shown in red) is simply lensless, but the graph is not recoverable ([2]).

Question 12. *What is the behavior of the number of geodesics of different types in annular planar quasi-medial graphs if only rivers and trees, or rivers and no trees, are present?*

Question 13. *What geodesics result after elimination of lenses in graphs embedded in annulus-like regions with $n > 1$ holes?*

The fundamental group π of such a region is the (non-abelian) free group on n generators, but the morphism from π to $\mathbb{Z}$ defined by $\int$ factors through its abelianization $\mathbb{Z}^n$. That is, the geodesic integral around any path depends only on how many times the path encircles each hole.

Perhaps the simplest case to consider would be a region where all geodesics join two distinct boundary circles.

A generalization of the geodesic integral may be useful to study directed quasi-medial graphs in these regions. We may assign to each edge of a quasi-medial graph a basis element of some $\mathbb{Z}$-algebra ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, etc.) and require that the in-degree be equal to the out-degree at every vertex when one counts only edges of a given type. Now, when a dual path crosses an edge, the geodesic integral increases or decreases not by 1, but by the basis element assigned to the edge.

In this case, it can be shown with some casework that the operations of lens elimination can still be done in a way which changes only locally the orientations of the edges.

3 Maximal connections

In this section we study the connection properties of planar graphs embedded in circles and annuli using the geodesic integral on their medial graphs.

If Γ is a graph with boundary $\partial\Gamma$ and A and B are two subsets of $\partial\Gamma$, a *connection* from A to B is a set of disjoint paths in Γ, each joining a vertex of A to a vertex of B. The *maximal connection* of Γ with respect to A and B is the maximal number of paths in a connection from A to B among all such connections.[1]

Note that A and B need not be disjoint. A single vertex $v \in A \cap B$ is considered to be a path from A to B.

A graph with boundary embedded in a surface such that each face is simply connected is *critical* if its medial graph is simply lensless.

3.1 Maximal connections: circular case

Let Γ be a critical graph with boundary embedded in a circle and let M be its medial graph. M is a quasi-medial graph. Take two cut points as defined above, both placed in segments of the boundary corresponding to faces of Γ. Thus, for any boundary vertex of Γ, the entire segment between the boundary vertices of M containing it is entirely within the upper arc or the lower arc, and we may speak of a boundary vertex of Γ lying in the upper arc or in the lower arc.

Lemma 14. *The maximal connection between the upper and lower boundaries of* Γ *is not greater than* $\frac{1}{2}t(M)$.

Proof. Take a path from the left cut point to the right cut point which crosses each tree exactly once and crosses no other geodesics of Γ. The path crosses $t(M)$ geodesics, passing through $\frac{1}{2}t(m)$ faces of Γ corresponding to vertices of M. Every path in a connection in M between the arcs must cross this path, i.e., use one of these $\frac{1}{2}t(M)$ vertices. But the paths in a connection are disjoint, so there could be no more than $\frac{1}{2}t(m)$ paths in a connection. $\square$

In fact, the following is true:

[1] It does not seem correct to call a connection maximal if it has the greatest number of paths among all connections. If "maximal connection" denoted a connection, then it should refer to a connection maximal with respect to inclusion: there is no larger connection containing all the paths of this connection.

Theorem 15. *The maximal connection between the upper and lower arcs of Γ is equal to $\frac{1}{2}t(M)$.*

Proof. Let N be the directed quasi-medial graph with the same vertices and embedding as M and edge orientations defined as follows. Each tree geodesic has all edges oriented from the upper face to the lower face. Each rock is oriented counterclockwise along the lower arc (i.e., the left endpoint of each rock has an outward edge and the right endpoint has an inward edge). Each tree is oriented counterclockwise along the upper arc.

Label each face of M with the geodesic integral along a path from the face containing the left cut point, as shown in Figure 7.

We will now perform a series of edge deletions and contractions in Γ which do not increase the maximal connection.

Choose an interior vertex of N. Because it has two geodesics with consistent orientations intersecting, there are two faces around this vertex with the same label and lying opposite each other with respect to this vertex. We remove this vertex by uncrossing the two geodesics in such a way as to join the two regions with the same label (see Figure 6). This operation is done locally and preserves the properties of a directed quasi-medial graph. Note, too, that each geodesic remains consistently oriented after this operation. This represents the deletion or contraction of an edge of Γ and clearly does not increase the maximal connection. Continue these operations until a graph N' with no interior vertices is obtained, the medial graph of a graph Γ'.

N' is critical because it has no interior vertices and could not possibly contain loops or lenses. Take a path p from the left cut point to the right cut point in N which crosses each tree exactly once and crosses no rocks or clouds. The resulting graph N' has the same labels as N along the faces along the boundary circle, so $\int p = t(N) = t(M)$. Thus, p passes through at least $\frac{1}{2}t(M)$ distinct faces of N', each of which touches both boundary arcs and corresponds to a vertex in Γ' which has been contracted from at least one vertex in the upper arc and at least one vertex in the lower arc. The vertices and edges in Γ from which a face was contracted contain a path between the lower arc and the upper arc in Γ. Thus we have found $\frac{1}{2}t(M)$ disjoint paths between the two arcs. $\square$

This is equivalent to the cut-point lemma of [1], which states that the number of vertices on the lower boundary arc is the sum of the maximal

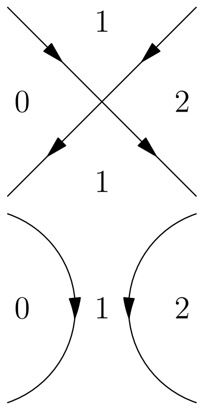

Figure 6: Uncrossing of geodesics.

connection between the two arcs and the number of geodesics having both endpoints on the lower arc.

Corollary 16. *There is a connection between the upper arc and the lower arc in Γ with $t(M)$ paths $p_1, \ldots, p_{\frac{1}{2}t(M)}$, where p_i uses only vertices whose corresponding faces in M have label $2i-1$.*

Proof. This is obvious from the previous theorem. One must only make the observation that all integers from 0 to $t(M)$ appear as labels along each of the arcs, which follows directly from the way in which the arcs were oriented, and that no other labels appear along both of the arcs. Among these integers there are $\frac{1}{2}t(M)$ odd ones, corresponding to vertices. □

This method of finding the maximal connection and the connection which achieves it can be extended to segments of the boundary whose union is not the entire boundary. Indeed, let Γ and M be as above and take four cut points, P, Q, R, S, in counterclockwise order. As above, we have assumed that the cut points are in segments of the boundary corresponding to faces of Γ. Let the segments of the boundary be A, B, C, D, in counterclockwise order, where A is between P and Q. Suppose that there are no geodesics from B to D.

In the directed quasi-medial graph N, which has the same vertices and embedding as M, we give each geodesic a consistent orientation as follows (shown in Figure 8):

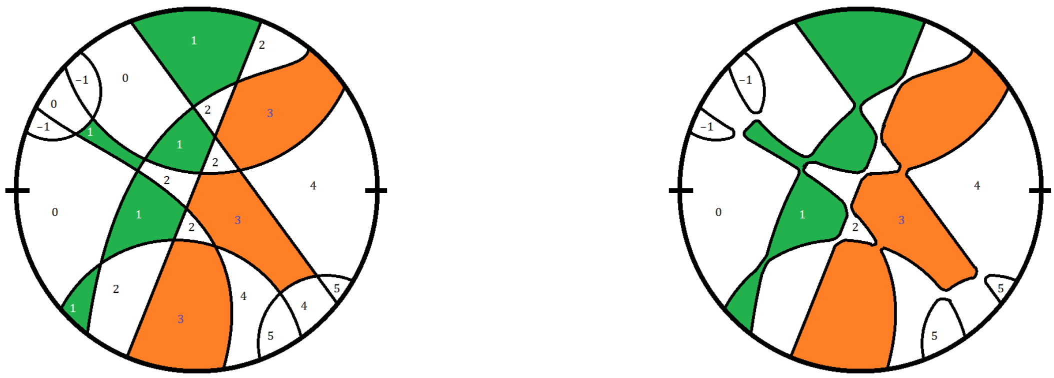

Figure 7: (a) Geodesic integral from the left cut point labeled on each face of a medial graph. (b) The same graph after contractions. Connection of size 2 shown in color.

- Geodesics with both endpoints in C, D, or A are oriented counterclockwise. Geodesics with both endpoints in B are oriented clockwise.
- Geodesics with an endpoint in C are oriented out of C. Geodesics with an endpoint in A are oriented into A.

Corollary 17. *Label each face of N with the geodesic integral along a path from P. Then the maximal connection from A to C is equal to the number of labels that appear along A and along C and do not appear along B or along D.*

3.2 Maximal connections: annular case

As with the results about elimination of lenses, the above can be generalized to graphs embdeed in an annulus.

Let Γ be a graph with boundary embedded in an annulus such that each face is simply connected.

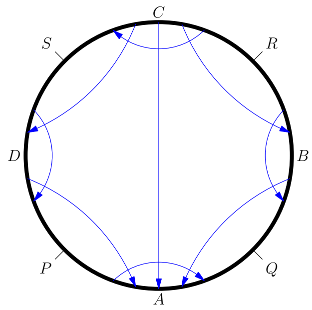

Figure 8: Orientations of geodesics in a graph with four cut points.

Theorem 18. *Suppose that* Γ *is a graph with boundary embedded in an annulus such that each face is simply connected and that its medial graph* M *is simply lensless. The maximal connection between the vertices on one boundary circle and the vertices on the other boundary circle is* $\frac{1}{2}t(M)$*.*

Proof. The trees are oriented from outer to inner boundary circle, the rocks are oriented so the inner circle is to the left when one travels along a rock, and the clouds are oriented such that the outer circle is to the left when one travels along a cloud. Rivers are oriented counterclockwise.

We label each face with the geodesic integral to this face along a path from some fixed face. These labels are unique up to equivalence modulo $t(M)$. The rest is as above. □

3.3 Maximal connections: further questions

Planar graphs with boundary nodes on annuli generalize planar graphs with boundary nodes on circles. Another generalization is as follows.

Question 19. *Suppose a graph is embedded in a orientable surface of some genus other than 2 (i.e., a torus with* n *holes) such that each face is simply connected and the boundary nodes lie on some circle (closed path) on this surface which is contractible to a point. What is the relation of the geodesic integral computed from a fixed point to the maximal connection between two arcs of the boundary circle?*

Question 20. *Which of the results of [4] can be restated or generalized using this method?*

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