Parametrizing Response Matrices for n-Wheeled Graphs

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Abstract

We give parametrizations of the response matrix for graphs with k spokes and n wheels. We generalize results found in Zhang's paper for spoked wheel graphs and extend his parametrizations for use in graphs with an arbitrary number of wheels.

1 Introduction

The entries in the response matrix for a given graph are related to the structure of the graph itself. For certain graphs, the number of entries in the response matrix is greater than the number of edges in the graph. This led researchers to investigate ways of parametrizing the response matrix for square lattices [1], $(4n+1)$-spoked wheel graphs [3], $n \times (n+1)$ lattices [3], and general spoked wheel graphs [4].

2 n-Wheeled Graphs

In this section, we discuss properties of n-wheeled graphs and of subdeterminants of their response matrices.

Definition Consider a circular planar graph G. We define G to have k spokes and n wheels if there are k rays in the graph whose endpoints lie on the inner and outer wheels and if each ray intersects each of n wheels exactly once, as shown in BLEHH. The set $V = \{\partial V, V(int)\}$ is the set of vertices on the boundary and in the interior formed by all intersections of rays and wheels. The set $E = \{\partial E, E(int)\}$ is the set of edges formed by all spoke- or wheel segments between two adjacent vertices, where ∂E is the set of boundary-to-boundary edges with two boundary vertices as endpoints. We define the boundary of G to be $\partial G = \{\partial V, \partial E\}$, which is the set containing boundary vertices and boundary-to-boundary edges. We call G an n-wheeled graph.

In such a graph, there are $\binom{k}{2} = \frac{k(k-1)}{2}$ entries in the response matrix but only $k(2n-1)$ edges in the graph (note that we consider only the entries in the response matrix above the main diagonal because of symmetry). This means that we should be able to find a parametrization that specifies $k(2n-1)$ entries in the response matrix and allows $\frac{k(k-1)}{2} - k(2n-1)$ entries to be calculated from the previously established parameters.

Claim The response matrix of a k-spoked and n-wheeled graph should be parametrized with $k(2n-1)$ elements.

Lemma 2.1 *All $2(n-1)$ connections exist in an n-wheeled graph.*

Proof The base case is provided by McCormick in [3] for $n = 2$. We proceed inductively for cases where $n \geq 2$.

Let $(P;Q) = (p_1, p_2, \cdots, p_{2(j-1)}; q_1, q_2, \cdots, q_{2(j-1)}$ be a circular pair of boundary vertices on a j-wheeled graph, where $j = n+1$. Without loss of generality, assume that the vertices are ordered clockwise along the boundary in this way: $(p_1, p_2, \cdots, p_{2(j-1)}, q_{2(j-1)}, q_{2(j-1)-1}, \cdots, q_1)$.

We can always find disjoint paths from p_1 to q_1 and from $p_{2(j-1)}$ to $q_{2(j-1)}$ using only interior vertices adjacent to the boundary. This means we are left with the sequence $(p_2, \cdots, p_{2(j-1)-1}, q_{2(j-1)-1}, \cdots, q_2)$, so that each set has size $2(j-1) - 2 = 2(n-1)$. By the induction hypothesis, we know that all $2(n-1)$ connections exist on an n-wheeled graph. It is straightforward to see that such connections will also exist on an $n+1$-wheeled graph: each path from a boundary vertex uses the interior vertex adjacent to it and then proceeds as in the n-wheeled case. Moreover, the paths formed by the $2(n-1)$ connection are disjoint from the paths formed by the $(p_1, p_{2(j-1)}; q_1, q_{2(j-1)})$ connection described.

Therefore, $2(j-1)$ connections exist on the j-wheeled graph, and by extension, all $2(n-1)$ connections exist on n-wheeled graphs.

Remark Note that each interior wheel can accommodate a two-connection. This provides an explanation for the $2(n-1)$ form of the size of guaranteed connections on an n-wheeled graph.

Theorem 2.2 *Suppose $\Gamma = (G, \gamma)$ is a circular planar resistor network and $(P;Q) = (p_1, p_2, \cdots, p_k; q_1, q_2, \cdots, q_k)$ is a circular pair of sequences of boundary nodes.*

(a) If $(P;Q)$ are not connected through G, then $det(\Lambda_{(P;Q)}) = 0$.

(b) If $(P;Q)$ are connected through G, then $(-1)^k(det\Lambda_{(P;Q)}) > 0$.

Proof This is proven by Curtis and Morrow in [2].

By Lemma 2.1 and Theorem 2.2, any determinant of $\Lambda(P;Q)$, where $P;Q)$ is a circular pair with exactly $2(n-1)$ boundary nodes on the graph, is not equal to zero.

Lemma 2.3 *Let $(P;Q)$ be a circular pair on a spoked wheel graph.*
If $det(\Lambda_{[p_1,p_2,\cdots,p_{2n-1}],[q_1,q_2,\cdots,q_{2n-1}]}) = 0$, then one of the entries in the $(2n-1) \times (2n-1)$ submatrix can be determined in terms of the other $(2n-1)^2 - 1$ entries.

Proof If we compute the determinant using cofactor expansion along the row or column containing our unknown entry, we can clearly see that each term in the determinant is nonzero. This is because every entry in the response matrix is nonzero and because every $2(n-1) \times 2(n-1)$ submatrix has a nonzero determinant (by Lemma 2.1 and Theorem 2.2). We can solve for our unknown entry in terms of these quantities. As noted in [3], a subdeterminant of size $2(n-1) \times 2(n-1)$ will always appear in the denominator, but we have already guaranteed that this subdeterminant is nonzero. Therefore, the solution for the unknown entry is unique and defined.

Definition Let i and j be two boundary vertices on a graph with k spokes and n wheels. Without loss of generality, let $i < j$. We say that i and j are neighbors if $i + 1 = j$ or $i = 1$ and $j = k$.

Lemma 2.4 *Let $(P; Q) = (p_1, p_2, \cdots, p_{2n-1}; q_1, q_2, \cdots, q_{2n-1})$ be a circular pair of sequences of boundary nodes on an n-wheeled graph. None of the nodes in P are neighbors of any nodes in Q if and only if there is no $(2n-1)$-connection between P and Q.*

Proof *If none of the nodes in P are neighbors of any nodes in Q, then there is no $(2n-1)$-connection between P and Q.*

Let $(P; Q) = (p_1, p_2, \cdots, p_{2n-1}; q_1, q_2, \cdots, q_{2n-1}$ be a circular pair of boundary vertices on an n-wheeled graph. Without loss of generality, assume that the vertices are ordered clockwise along the boundary in this way: $(p_1, \cdots, p_{2n-1}, q_{2n-1}, \cdots, q_1)$. We know that the $2(n-1)$ connection through the circular pair $(P \backslash \{p_{2n-1}\}; Q \backslash \{q_{2n-1}\})$ exists and must use the vertices on the first interior circle from $p_{2(n-1)}$ to $q_{2(n-1)}$. Both points in the circular pair $(p_{2n-1}; q_{2n-1})$ lie between $p_{2(n-1)}$ and $q_{2(n-1)}$. Any path from $(p_{2n-1}$ to $q_{2n-1})$ must use the interior vertices adjacent to both of these boundary vertices, but these interior vertices already appear in a path in the $2(n-1)$ connection. Therefore, the paths are not disjoint, and no $2(n-1)$ connection exists.

If there is no $2n-1$ connection between P and Q, then none of the nodes in P are neighbors of any nodes in Q.

It suffices to prove the contrapositive. If one of the nodes p_i in P is the neighbor of one of the nodes q_j in Q, then the path from p_i to q_j need not use any interior vertices; it can use the direct boundary-to-boundary edge between the two. This guarantees that none of the vertices used by the $2(n-1)$ connection will be used by the $(p_i; q_j)$ connection. Therefore, the paths of each connection are disjoint and the $2n-1$ connection exists. By the contrapositive, if the $2n-1$ connection does not exist, then none of the nodes in P are neighbors of any nodes in Q.

3 Parametrizing n-Wheeled Graphs

3.1 Even Number of Spokes ($k = 2m$)

Here, we consider specifically the n-wheeled graphs with an even number of spokes, where k can be written as $2m$. This means that the response matrix will consist of $2m$ columns and $2m$ rows. We partition the response matrix in the following way:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{D} \end{pmatrix}$$

Here, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$ are $m \times m$ blocks. Again, we need only determine the entries above the main diagonal because of the symmetry of the response matrix.

The parametrization for an n-wheeled graph with $2m$ spokes is given as follows:

- All entries $a_{j,j+1}$ (these are the entries directly above the main diagonal),
- top $2(n-1)$ rows of $\mathbf{B}$,
- rightmost $2(n-1)$ columns of $\mathbf{B}$,
- $a_{2n-1,k-2(n-1)}$,
- upper right triangle in $\mathbf{A}$ with height and width $2n-3$,
- upper right triangle in $\mathbf{D}$ with height and width $2n-3$,
- entries $a_{j,m-2n+3+j}$ for $a_{1,m-2n+4}$ through $a_{n-1,m-n+4}$,
- and BLAHHHH.

Essentially, this parametrization allows us to calculate unknown entries using the fact that $(2n-1) \times (2n-1)$ subdeterminants are zero if no nodes in the circular pair are neighbors. We will be able to use these calculated entries to propagate information through the rest of the matrix. Note that the number of parameters is equal to the number of edges in the graph (the check is left to the reader).

Theorem 3.1 *The parametrization given above completely determines every entry of a response matrix for an n-wheeled graph with an even number of spokes.*

Proof We can compute the remaining entries in $\mathbf{B}$ by using the $(2n-1)\times(2n-1)$ subdeterminant that places the unknown entry in the bottom left corner. Because none of the entries in this subdeterminant correspond to nodes that are neighbors, we can solve for the unknown entry (by Theorem 2.2, Lemma 2.3, and Lemma 2.4).

We can compute the remaining entries in $\mathbf{A}$ and $\mathbf{D}$ in a similar way, again by using the $(2n-1) \times (2n-1)$ subdeterminant that places the unknown entry in the bottom left corner. However, the entries used in these subdeterminants are not necessarily contiguous in the matrix. In $\mathbf{A}$, they may "wrap around" the top and bottom edges of the matrix to form a complete $(2n-1) \times (2n-1)$ subdeterminant. In $\mathbf{D}$, they may "wrap around" the sides.

Consider an entry $\lambda_{i,j}$ in $\mathbf{A}$, where $i < 2n-1$, $j > m-(2n-1)$, the entries directly above and to the right of $\lambda_{i,j}$ are parameters, and $\lambda_{i,j}$ is not a parameter.

We claim that this entry can be calculated by placing it in the bottom left corner of a $(2n-1) \times (2n-1)$ subdeterminant. We have already noted that rows of this subdeterminant will "wrap around" the top and bottom of the matrix. In order to calculate $\lambda_{i,j}$, each of the other entries in the subdeterminant must be known. It is easy to see that all the needed entries in $\mathbf{A}$ and $\mathbf{B}$ are known because all entries above and to the right of $\lambda_{i,j}$ are parameters. We now have to examine the entries in the "wrap-around" region of the subdeterminant. We know that all needed entries in $\mathbf{B}^T$ are present because we know all entries in $\mathbf{B}$. If we want to calculate $\lambda_{i,j}$, we must know the entries in a $(2n-1-i) \times (2n-1) - (m-j)$ block in the bottom left corner of $\mathbf{D}$. We know that these entries are parameters, however, because by symmetry, these entries appear as the parametrized entries mentioned above.

In this way, we can calculate all entries in $\mathbf{A}$ that have parameters directly above them and to their right. Subsequently, we can propagate this information down and to the left through $\mathbf{A}$ until we reach the above-diagonal parameters. Note that for this submatrix, we also avoided using subdeterminants that included neighboring nodes.

The same argument applies to $\mathbf{D}$. Simply apply the subdeterminants that "wrap around" the sides of the matrix rather than the top and bottom.

By symmetry, we can reflect these known entries over the main diagonal to fill in all other entries, except those on the main diagonal. However, we can use the fact that row sums of the response matrix are zero to solve for entries on the diagonal.

Thus, all entries are able to be determined using the parametrization outlined above.

Example Consider the specific parametrization of the response matrix for a graph with 12 spokes and 3 wheels.

1-40. Parameters.
41. Calculated from $det\Lambda(2,3,4,5,6;8,9,10,11,12) = 0$.
42-45. Parameters.
46. Calculated from $det\Lambda(1,2,3,4,5;7,8,9,10,11) = 0$.
47-53. Parameters.
54. Calculated from $det\Lambda(12,1,2,3,4;6,7,8,9,10) = 0$.
55-59. Parameters.
60. Calculated from $det\Lambda(3,4,5,6,7;9,10,11,12,1) = 0$.
61-63. Parameters.
64. Calculated from $det\Lambda(11,12,12,1,2,3;5,6,7,8,9) = 0$.
65. Parameter.
66. Calculated from $det\Lambda(4,5,6,7,8;10,11,12,1,2) = 0$.

Each calculated entry is found using the fact that 5×5 subdeterminants are zero if no nodes in the circular pair are neighbors. To avoid using nonzero subdeterminants, we did not use any subdeterminants containing the entries directly above the main diagonal (entries 30-40) or the entry in the upper right corner (entry 5). At times, this forced us to use determinants that "wrapped around" the sides of the matrix (this occurred for entries 54, 60, 64, and 66).

3.2 Odd Number of Spokes ($k = 2m + 1$)

Here, we consider specifically the n-wheeled graphs with an odd number of spokes, where k can be written as $2m+1$. This means that the response matrix will consist of $2m+1$ columns and $2m+1$ rows. For this reason, let us partition the response matrix in the following way:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{D} \\ \mathbf{B}^T & \mathbf{C} & \mathbf{E} \\ \mathbf{D}^T & \mathbf{E}^T & \mathbf{F} \end{pmatrix}$$

Here, $\mathbf{A}$, $\mathbf{D}$, and $\mathbf{F}$ are $m \times m$ blocks. $\mathbf{B}$ is $m \times 1$, $\mathbf{E}$ is $1 \times m$, and $\mathbf{C}$ is a single entry. As noted previously, we need only consider the entries above the main diagonal and use symmetry to determine the remaining entries.

The parametrization for an n-wheeled graph with $2m + 1$ spokes is given as follows:

- All entries $a_{j,j+1}$ (these are the entries directly above the main diagonal),
- top $2(n-1)$ rows of $\mathbf{D}$,
- rightmost $2(n-1)$ columns of $\mathbf{D}$,
- $a_{2n-1,k-2(n-1)}$,
- top $2(n-1)$ entries of $\mathbf{B}$,
- rightmost $2(n-1)$ entries of $\mathbf{E}$,
- upper right triangle in $\mathbf{A}$ with height and width $2n-3$,
- and upper right triangle in $\mathbf{F}$ with height and width $2n-3$.

We use similar techniques as in the even case. Again, note that the number of parameters is equal to the number of edges (the check is still left to the reader).

Theorem 3.2 *The parametrization given above completely determines every entry of a response matrix for an n-wheeled graph with an odd number of spokes.*

Proof The proof for computing entries in $\mathbf{D}$ is identical to that for computing entries in $\mathbf{B}$ in the even case.

Computing an unparametrized entry in $\mathbf{B}$ and $\mathbf{E}$ is similarly simple because it can always be placed in the lower left corner of a $(2n-1)\times(2n-1)$ submatrix whose determinant is zero.

The same argument for calculating entries in $\mathbf{A}$ and $\mathbf{E}$ applies as in the even case.

By symmetry, we can reflect these known entries over the main diagonal to fill in all other entries, except those on the main diagonal. However, we can use the fact that row sums of the response matrix are zero to solve for entries on the diagonal.

Thus, all entries are able to be determined using the parametrization outlined above.

Remark

3.3 Relationship to Zhang's Work

3.4 Generalized n-Wheeled Graphs

Corollary 3.3 *Let G be an n-wheeled graph. Consider vertices i and $j \in \partial V$, where i and j are neighbors and without loss of generality, $i < j$. If we delete the edge between them, we can remove entry $\lambda_{i,j}$ in the response matrix from our list of parameters.*

Proof If there is no edge between i and j, then any path between them uses an interior vertex. By the proof of Lemma 2.4, we can see that the $(2n-1)\times(2n-1)$ connection containing i and j (but no other neighboring vertices) does not exist, and therefore, the determinant of the $(2n-1)\times(2n-1)$ submatrix containing $\lambda_{i,j}$ is equal to zero (by Theorem 2.2). Therefore, we can calculate $\lambda_{i,j}$ (by Lemma 2.3) and do not need to include it as a parameter.

Remark In a purely numerical sense, this result is logical because deleting an edge removes one of the unknowns from the system and thus removes the need for one independent parameter.

4 Further Research

References

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