Polynomials and the Construction of N-to-1 Graphs

Halley McCormick, Michelle Randolph
University of Washington Mathematics REU

August 2013

Abstract

We show that we can construct an N-to-1 graph for any N using Bernstein polynomials, Legendre polynomials, and complete graph equivalents of graphs composed of multiple 4-stars.

Contents

1 Preliminaries

1.1 Electrical Networks and the Inverse Problem

In our investigation, we consider undirected, connected, finite graphs with no loops and with a set of vertices that can be partitioned into boundary nodes and interior nodes. We distinguish between the two by using closed circles ($\bullet$) to denote boundary nodes and open circles ($\circ$) to denote interior nodes. The set of boundary nodes is always nonempty. In our diagrams, nodes may be drawn more than once; if the same node appears more than once, it is labeled with the same number each time. We will use the terms "nodes" and "vertices" interchangeably.

Definition Let E be the set of edges in a graph G. A conductivity function is a function $\gamma : E \to \mathbb{R}^+$ that assigns each edge, e_l, to a positive real number.

Definition A resistor network, $\Gamma(G, \gamma)$, is a graph G together with a conductivity function γ.

Definition Let $\Gamma = (G, \gamma)$ be a resistor network on a graph G with n nodes $(v_1, v_2, \cdots, v_n)$. The Kirchhoff matrix of Γ is an $n \times n$ matrix K that encodes information about the conductivities on each edge in the following way:

$$K_{i,j} = \begin{cases} \gamma_{i,j} & \text{if } i \neq j \\ -\sum_{i \neq j} \gamma_{i,j} & i = j \end{cases}$$

where

$$\gamma_{i,j} = \sum_{\text{all edges } e_l \text{ joining } v_i \text{ to } v_j} \gamma(e_l)$$

and $\gamma_{i,j} = 0$ if there is no direct edge between vertices i and j.

These are some properties of the Kirchhoff matrix:

- $\gamma_{i,j} \geq 0$ for all $i \neq j$ (i.e., all off-diagonal entries are positive or zero).
- Row sums are zero.
- K is symmetric.

If G has n nodes, let G have m boundary nodes, where $m \leq n$ and all boundary nodes come before interior nodes in the indexing. We can partition the Kirchhoff matrix in the following way:

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B^T} & \mathbf{C} \end{bmatrix}$$

where A is an $m \times m$ matrix and C is an $n - m \times n - m$ matrix.

Definition Let $\Gamma = (G, \gamma)$ be a resistor network on a graph G with n nodes $(v_1, v_2, \cdots, v_n)$, the first $m \leq n$ of which are boundary nodes. The response matrix of Γ is Λ, an $m \times m$ matrix defined as:

$$\Lambda = A - BC^{-1}B^T,$$

where A, B, C, and B^T are submatrices of the Kirchhoff matrix as described above. The entries of Λ are written as $\lambda_{i,j}$.

The response matrix has the same properties as the Kirchhoff matrix:

- $\lambda_{i,j} \geq 0$ for all $i \neq j$ (i.e., all off-diagonal entries are positive or zero).
- Row sums are zero.
- Λ is symmetric.

We will use the notation Λ_γ to refer to the the specific response matrix associated to the network Γ with conductivity function γ.

For more detailed treatment of these definitions, see the work of Curtis and Morrow in [1].

The Inverse Problem The forward problem is to find the response matrix of a graph from its conductivities. The inverse problem, a facet of which we will examine in this paper, is to ask whether we can recover the conductivity γ on each edge given only the graph and the response matrix.

1.2 Graph Structures

Definition An n-star is a graph where one interior node is connected to exactly n boundary nodes, and each boundary node is only connected to that interior node. An example of a 4-star is shown in Figure 1.

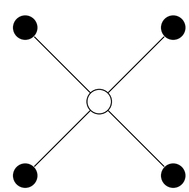

Figure 1: 4-star

The graphs that we will work with are entirely constructed of four-stars patched together in various ways.

Definition A complete graph on n vertices, K_n, is a graph with n boundary nodes where each boundary node is connected by an edge to every other boundary node. An example of a complete graph on four vertices is shown in Figure 2.

Definition A star-K transformation (also known as interiorizing) is a process of removing the interior node in an n-star so that it becomes a complete graph K_n. This process retains information about how each of the boundary nodes in the n-star is connected. See Figure 3 for an example of this transformation on a 4-star.

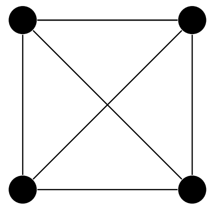

Figure 2: Complete graph on 4 vertices, K_4

For each n-star, there can be more than one way to represent the complete graph. This is primarily determined by the way that the n-stars are connected to one another. In the case of the 4-star, if each boundary node is connected to at least two interior nodes, the complete graph will form a square, and if one of the boundary nodes is only connected to a single interior vertex, it will form a pyramid (see Figures 4 and 5). Notice that the square and the pyramid are identical when they are not part of a larger graph as in Figure 3.

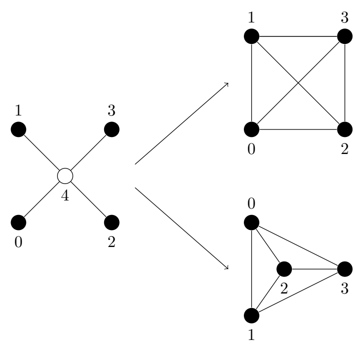

Figure 3: Star-K transformation from a 4-star to K_4. Notice that the square and the pyramid are identical ways of drawing the transformation.

Performing a star-K transformation on a graph composed of multiple n-stars results in the appearance of multiple edges (see Figure 5), which are structures in the graph where there are two or more edges e_α between a pair of boundary nodes i and j. Let the conductivity on e_α be μ_α. All conductivities must be positive, thus, $\mu_\alpha > 0$ for all μ_α and the total conductivity $\lambda_{i,j}$ between two nodes must be the sum of conductivities of the edges between them. For n edges between nodes v_i and v_j:

$$\lambda_{i,j} = \sum_{\alpha=0}^{n} \mu_\alpha \tag{1}$$

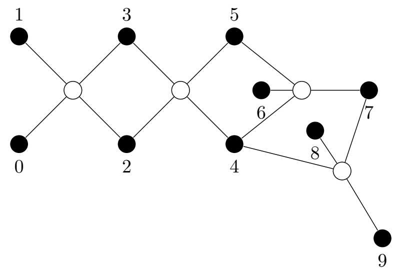

Figure 4: Example graph with multiple 4-stars.

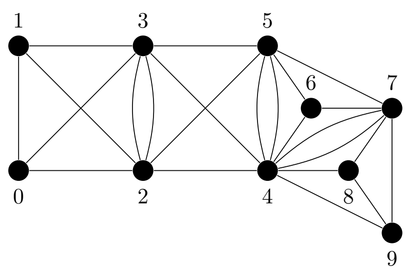

Figure 5: Star-K equivalent of Figure 4.

We will perform star-K transformations on our original graphs and focus on their complete graph equivalents.

Previous documents on this topic (namely, Kempton's work in [4] and Wu's work in [5]) use the term "R-multigraph" to refer to the graph constructed by performing a star-K transformation on a graph composed of multiple n-stars. We will not use this term; when we need to refer to this graph by name, we will simply call it the "complete graph equivalent" of the original graph.

Definition The quadrilateral rule describes the relationship among the conductivities of each edge in any quadrilateral in a complete graph. Given the complete graph in Figure 6, with conductivity μ_α on each edge e_α, the quadrilateral rule states that $\mu_1\mu_2 = \mu_3\mu_4 = \mu_5\mu_6$. The same quadrilateral rule applies to the pyramid shown in Figure 7. More detailed definitions of the quadrilateral rule can be found in [4] and [5].

Theorem 1.1. *Let $\Gamma = (G, \gamma)$ be a resistor network where G is a graph composed of n-stars. Let K_n be the complete graph obtained by performing a star-K transformation on G. The network on G is response-equivalent to the network on K_n if and only if the conductivities on K_n satisfy the quadrilateral rule.*

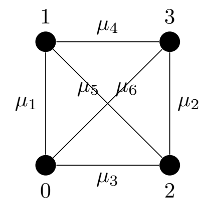

Figure 6: Quadrilateral rule for a square K_4 graph. $\mu_1\mu_2 = \mu_3\mu_4 = \mu_5\mu_6$

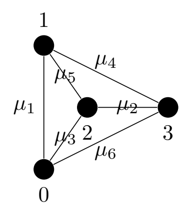

Figure 7: Quadrilateral rule for a pyramid K_4 graph. $\mu_1\mu_2 = \mu_3\mu_4 = \mu_5\mu_6$

Proof. This is proven in [3]. □

The quadrilateral rule and the appearance of multiple edges from the transformation of connected n-stars into a complete graph allows us to consider the propagation of unknown functions through the graph, which contributes to our understanding of N-to-1 graphs, which we define below.

1.3 N-to-1 Graphs

As we saw above in Figure 4 and Figure 5, a star-K transformation of a graph composed of multiple n-stars can result in the creation of multiple edges. However, the response matrix only provides information about the total conductivity between two boundary nodes, not about the individual conductivities of each of the edges that constitute that connection. Therefore, the response matrix does not give us full information about the conductivities on the edges of the complete graph equivalent. This leads us to investigate the ways in which we can use the double edges introduced by the star-K transformation to propagate unknown functions through the graph.

Definition Let G be a graph such that if we fix G, we can associate n distinct conductivity functions to it to form n different networks:

$$\Gamma_1 = (G, \gamma_1), \Gamma_2 = (G, \gamma_2), \cdots, \Gamma_n = (G, \gamma_n).$$

If each of these networks has the same response matrix, i.e:

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} = \cdots = \Lambda_{\gamma_n},$$

then we call G an N-to-1 graph.

If we set the conductivity of one of the multiple edges in the complete graph equivalent equal to an unknown function—for example, let e_α be one of the multiple edges between nodes v_1 and v_2 and let $e_\alpha = x$ —we can leave the response matrix unchanged but introduce a variable into the set of conductivities on the complete graph equivalent. Later in this paper, we will show that we can use different graph structures to propagate this unknown function $f(x) = x$ through the graph to yield polynomials in x with N distinct positive real roots in $(0, 1)$. Because these polynomials have N distinct positive real roots, we will show that such graphs can also have N distinct sets of conductivities corresponding to the same response matrix and are therefore N-to-1.

Let us begin with $\mu_\alpha = f(x) = x$ as the conductivity of one of the multiple edges making up $\lambda_{0,1}$, where $\lambda_{0,1}$ is the entry in the response matrix corresponding to nodes v_0 and v_1. We will construct a polynomial, $p(x)$, by propagating x through the graph and looping back around to $\lambda_{0,1}$. By Equation 1, we will have:

$$\lambda_{0,1} = p(x) + x \tag{2}$$

And we will define:

$$\sigma(x) = p(x) + x \tag{3}$$

This is the polynomial for which we will want to find N distinct positive real roots. In order for the graph to have valid conductivites, $\sigma(x)$ must have N distinct positive real solutions to $\sigma(x) = \lambda_{0,1}$ for some $\lambda_{0,1} > 0$. All of the solutions must also produce positive conductivities on each edge in the graph. For our construction of N-to-1 graphs, we will construct $p(x)$ to be a linear combination of $x^k(C_j - x)^n$. We will let $C_j = 1$ for all j because this choice makes $p(x)$ a very common type of polynomial, and we may use Legendre and Bernstein polynomials. Notice that this will restrict our roots to $x < 1$ otherwise it would produce zero or negative conductivities. We will show how to construct $x^k(C_j - x)^n$ in Section 3. We will then show how to explicitly construct $p(x)$ and address these conditions more specifically in Sections 5 and 6.

2 Bernstein Polynomials

Definition The Bernstein basis polynomials, $b_{r,n}$, are of the form

$$b_{r,n}(x) = \binom{n}{r} x^r (1-x)^{n-r} \tag{4}$$

where $0 \leq r \leq n$.

The Bernstein basis polynomials of degree n form a basis for the vector space of polynomials of degree less than or equal to n.

Definition Given a function f on $[0, 1]$, the Bernstein polynomial, B_n, is a linear combination of the $b_{r,n}$:

$$\begin{aligned} B_n(f, x) &= \sum_{r=0}^{n} f\left(\frac{r}{n}\right) b_{r,n}(x) \\ &= \sum_{r=0}^{n} f\left(\frac{r}{n}\right) \binom{n}{r} x^r (1-x)^{n-r}. \end{aligned} \tag{5}$$

2.1 Properties of Bernstein Polynomials

Property 2.1. *Given a function $f \in C[0, 1]$ and any $\epsilon > 0$, there exists an integer N such that*

$$|f(x) - B_n(f; x)| < \epsilon,\ 0 \le x \le 1$$

for all $n \ge N$; i.e., the Bernstein polynomials for a given continuous function f on $[0, 1]$ will converge uniformly to f on $[0, 1]$.

Proof. The proof is provided in [2]. □

Property 2.2. *If $f \in C^k[0, 1]$, for some integer $k \ge 0$, then $B_n^{(k)}(f; x)$ converges uniformly to $f^{(k)}(x)$ on $[0, 1]$.*

Proof. The proof is provided in [2]. □

As a result of Property 2.1 and since the Bernstein polynomials form a basis, x can be written in terms of Bernstein polynomials. Given precisely by Equation 5, x is:

$$x = \sum_{r=0}^{n} \frac{r}{n} \binom{n}{r} x^r (1-x)^{n-r}, \text{ for } n \ge 1. \tag{6}$$

Property 2.3. *The Bernstein basis polynomials form a partition of unity. Thus,*

$$\sum_{r=0}^{n} b_{r,n} = \sum_{r=0}^{n} \binom{n}{r} x^r (1-x)^{n-r} = 1.$$

Proof. Consider that $(1 - x + x)^n = 1$. Applying the binomial theorem, we get that

$$1 = (1 - x + x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r (1-x)^{n-r}.$$

□

Property 2.4. *For $C \in \mathbb{R}$ and $p_n(x)$, a Bernstein polynomial of degree n such that $p_n(x) = \sum_{r=0}^{n} a_r \binom{n}{r} x^r (1-x)^{n-r}$, the polynomial can be shifted up by a constant C by adding C to every coefficient a_r of $p_n(x)$:*

$$p_n(x) + C = \sum_{r=0}^{n} (a_r + C) \binom{n}{r} x^r (1-x)^{n-r}$$

Proof.

$$\sum_{r=0}^{n} (a_r + C) \binom{n}{r} x^r (1-x)^{n-r} = \sum_{r=0}^{n} a_r \binom{n}{r} x^r (1-x)^{n-r} + C \sum_{r=0}^{n} \binom{n}{r} x^r (1-x)^{n-r}.$$

Using Property 2.3,

$$\begin{aligned} \sum_{r=0}^{n} a_r \binom{n}{r} x^r (1-x)^{n-r} + C \sum_{r=0}^{n} \binom{n}{r} x^r (1-x)^{n-r} &= \sum_{r=0}^{n} a_r \binom{n}{r} x^r (1-x)^{n-r} + C \\ &= p_n(x) + C. \end{aligned}$$

□

3 Construction of Bernstein Polynomials Through a Network of Four-Stars

We will show that we can construct a Bernstein polynomial with positive coefficients.

Lemma 3.1. *We can construct a graph such that if we propagate an unknown edge conductivity x through it, we can obtain a conductivity whose value is defined by x or $\lambda - x$, where λ is a positive constant.*

Proof. Let us take the edge corresponding to f_1 to have an unknown conductivity, x, and propagate this through the graph in Figure 8. Notice that Figure 8 is only one of many pyramids that would be part of the graph. Using the quadrilateral rule, we get:

$$\begin{aligned} f_2 = f_1 = x \quad &\text{if } \lambda_{1,2} = \lambda_{2,3} \\ &\text{and} \\ f_3 = \lambda_{0,3} - f_2 &= \lambda_{0,3} - x. \end{aligned}$$

Thus, through this pyramid, we have transformed an x into a $\lambda - x$.

Once again, if we take the edge corresponding to f_1 to have an unknown conductivity x and propagate this through the graph in Figure 9, we get:

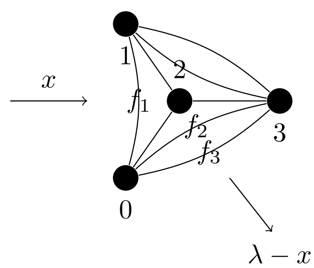

Figure 8: Here we propagate an x through the graph to output a $\lambda - x$.

$$\begin{aligned} f_2 = f_1 = x \quad &\text{if } \lambda_{1,2} = \lambda_{2,3} \\ &\text{and} \\ f_3 = \lambda_{0,3} - f_2 &= \lambda_{0,3} - x. \end{aligned}$$

We propagate this through the second pyramid to get:

$$\begin{aligned} f_4 = f_3 = \lambda_{0,3} - x \quad &\text{if } \lambda_{0,4} = \lambda_{4,5}, \\ f_5 = \lambda_{3,5} - f_4 &= \lambda_{3,5} - (\lambda_{0,3} - x), \\ &\text{and} \\ f_5 = x \quad &\text{if } \lambda_{0,3} = \lambda_{3,5} \end{aligned}$$

Thus, through this pyramid, we have propagated an x through the graph without transforming it. It is important to note that these pyramids would be connected to others in the complete graph and thus the x could be propagated through the graph in more than one direction to produce several edges in the graph with a conductivity defined by x. We can repeat this processes and extend the graph by simply adding cycles of pyramids.

It will be useful to notice that we could add a positive coefficient to the output of x by changing the relationship between $\lambda_{4,5}$ and $\lambda_{0,4}$. So that,

$$
\begin{aligned}
f_4 &= \frac{f_3\lambda_{4,5}}{\lambda_{0,4}}\\
&= \frac{(\lambda_{0,3}-x)\lambda_{4,5}}{\lambda_{0,4}}\\
&= \frac{\lambda_{0,3}\lambda_{4,5}-\lambda_{4,5}x}{\lambda_{0,4}}
\end{aligned}
$$

and

$$
\begin{aligned}
f_5 &= \lambda_{3,5} - f_4\\
&= \lambda_{3,5} - \frac{\lambda_{0,3}\lambda_{4,5}}{\lambda_{0,4}} + \frac{\lambda_{4,5}x}{\lambda_{0,4}}
\end{aligned}
$$

If we then required that $\lambda_{3,5} = \frac{\lambda_{0,3}\lambda_{4,5}}{\lambda_{0,4}}$, we would obtain an output of $\frac{\lambda_{4,5}}{\lambda_{0,4}}x$. $\square$

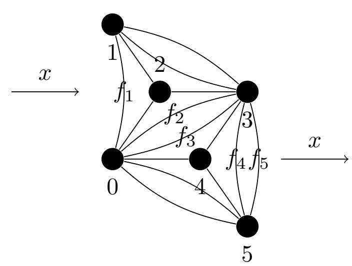

Figure 9: Here we propagate an x through the graph to output an x.

Lemma 3.2. *We can construct a graph such that if we propagate an unknown function x through it, we can obtain a conductivity whose value is defined by x^k.*

Proof. From Lemma 3.1, we can obtain an x or a $\lambda - x$ from the graph. We can then extend the graph and wrap it around itself to input the x or $\lambda - x$ into the square multipliers in the graph. Using the graph shown in Figure 10, which would be part of a larger graph (such as the one in Figure 11), we can input x into the edges e_1 and e_2 and propagate through the network.

Notice the input from the top and bottom edges is x. Since these will have come from propagation through another part of the graph, $\lambda_{1,3}$ and $\lambda_{2,4}$ will both have double edges which we have not shown in Figure 10. Thus by the

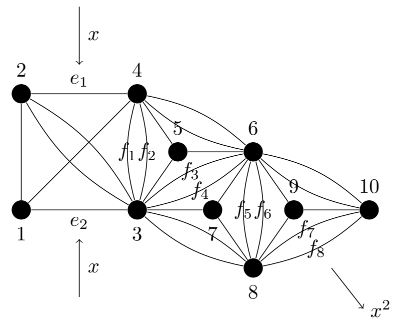

Figure 10: Here, we propagate two x's through the graph to output an x^2.

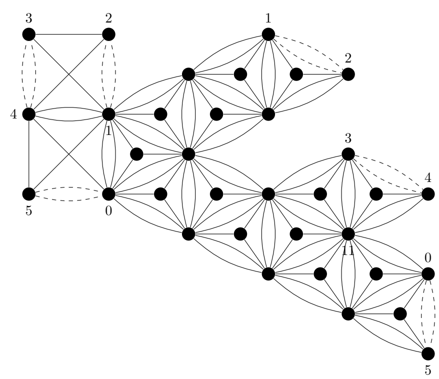

Figure 11: Example of a simple 3-to-1 graph construction. The arms wrap around to input back into the graph. The nodes labeled with the same number are the same nodes drawn twice and each of the dotted edges is drawn twice.

quadrilateral rule:

$$f_1 = \frac{x^2}{\lambda_{1,2}}$$

and

$$f_2 = \lambda_{3,4} - f_1 = \lambda_{3,4} - \frac{x^2}{\lambda_{1,2}}.$$

We use the first pyramid remove $\lambda_{3,4}$:

$$
\begin{aligned}
f_3 = f_2 &= \lambda_{3,4} - \frac{x^2}{\lambda_{1,2}} \quad \text{if } \lambda_{5,6} = \lambda_{4,5} \\
f_4 &= \lambda_{3,6} - f_3 = \lambda_{3,6} - (\lambda_{3,4} - \frac{x^2}{\lambda_{1,2}}) \\
&= \frac{x^2}{\lambda_{1,2}} \quad \text{if } \lambda_{3,6} = \lambda_{3,4}.
\end{aligned}
$$

We use the second pyramid to remove $\lambda_{1,2}$:

$$
\begin{aligned}
f_5 &= \frac{f_4 \lambda_{7,8}}{\lambda_{3,7}} = \frac{x^2 \lambda_{7,8}}{\lambda_{3,7}\lambda_{1,2}} \\
&= x^2 \quad \text{if } \frac{\lambda_{7,8}}{\lambda_{3,7}} = \lambda_{1,2}
\end{aligned}
$$

Notice that though $f_5 = x^2$, it is not an edge that can be input into the rest of the graph, thus we must continue propagation.

$$f_6 = \lambda_{6,8} - f_5 = \lambda_{6,8} - x^2.$$

We use the third pyramid to output x^2:

$$
\begin{aligned}
f_7 = f_6 = \lambda_{6,8} - f_5 &= \lambda_{6,8} - x^2 \quad \text{if } \lambda_{6,9} = \lambda_{9,10} \\
f_8 = \lambda_{8,10} - f_7 &= \lambda_{8,10} - (\lambda_{6,8} - x^2) \\
&= x^2 \quad \text{if } \lambda_{8,10} = \lambda_{6,8}
\end{aligned}
$$

At this point, the x^2 is on the second part of the double edge which means that it would be input into the next peice of the graph simply as x^2. We could easily repeat this process using the generated x^2 and another x to produce x^3, then do the same for successively higher powers and generate x^k for any integer k. □

Lemma 3.3. *We can construct a graph such that if we propagate an unknown function x through it, we can obtain a conductivity whose value is defined by $(1-x)^k$.*

Proof. From Lemma 3.1, we have shown that we can produce $\lambda - x$ from the graph. We can simply set $\lambda = 1$ to produce $(1-x)$. This will mean that we must restrict the values of x to $x < 1$ to maintain our sign conditions. Then, as in the proof of Lemma 3.2, we can substitute $(1-x)$ everywhere that x appears; the same process applies and the result will be of the form $(1-x)^k$ instead of x^k. □

Lemma 3.4. *We can construct a graph such that if we propagate an unknown function x through it, we can obtain a conductivity whose value is defined by $Cx^r(1-x)^{n-r}$, where $C > 0$.*

Proof. In Lemmas 3.2 and 3.3, we have shown that we can construct a graph to produce conductivities defined by $(1-x)^k$ and x^k. It follows that simply by inputting them into either side of our square multiplier, we can produce a function:

$$f = \frac{1}{\lambda_{i,j}} x^{k_1}(1-x)^{k_2} \tag{7}$$

for some conductivity $\lambda_{i,j}$ and integers k_1 and k_2. Notice that it would not be difficult to choose k_1 and k_2 such that $k_1 = r$ and $k_2 = n - r$ for given n and r. Also, since $\lambda_{i,j}$ does not have a double edge, the only requirement is that $\lambda > 0$ and it must satisfy the quadrilateral rule in the square. Thus it would not be difficult to choose $\frac{1}{\lambda_{i,j}}$ such that $\frac{1}{\lambda_{i,j}} = C$. Alternatively, this constant, C can be produced by adding a coefficient to the x as it is propagated through the graph as mentioned briefly in Lemma 3.1. Notice that $\binom{n}{r}$ is a positive constant for given n and r, thus by this process and choosing the appropriate C, we can produce any Bernstein basis polynomial. □

Lemma 3.5. *Given a function f that is strictly positive on $[0,1]$, we can construct $B_n(f;x)$ by creating multiple edges between a set of nodes.*

Proof. We have shown that it is possible to produce any Bernstein basis polynomial through squares and pyramids in the graph. Notice that each element of the linear combination, B_n, is simply a Bernstein basis polynomial with a coefficient. Since the function f is positive on $[0,1]$, these coefficients will all be positive. We have already shown that we can alter the conductivities on the graph to produce any positive coefficient. By adding additional arms to the graph, we can produce multiple Bernstein basis polynomials. If each of these edges with conductivity defined by the produced Bernstein basis polynomial corresponds to the same two nodes, the total conductivity between the nodes, $\lambda_{0,1}$, is simply equal to the sum of the conductivities of all of the edges as in Equation 1. An example of this is shown in Figure 12. Each of the shaded nodes is connected to other nodes in the graph that are not shown.

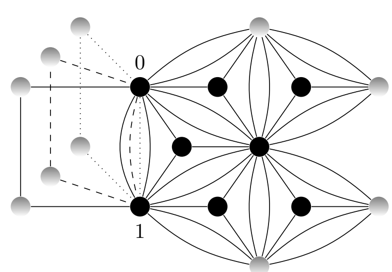

Figure 12: Here we add edges that have been propagated through the graph.

□

Corollary 3.6. *The above construction of a graph producing a Bernstein polynomial preserves all sign conditions for each edge's conductivity.*

Proof. All choices for $\lambda_{i,j}$ are independent of each other, except for the equalities specified when propagating the x. Therefore, choosing one set of conductivities will not affect choices for the other conductivities. This allows us to choose any conductivities that will maintain correct sign conditions, namely that each $\lambda_{i,j}$ that needs to have x subtracted from it is greater than x or, when we create larger polynomials, $\lambda_{i,j} > x^k$. For the $\lambda_{i,j}$ that do not need to fulfill this criterion, there is no restriction on how large their value has to be, so they can be exactly the value necessary to yield the desired coefficients on each Bernstein basis polynomial. $\square$

4 Bernstein Polynomial Approximations of Functions

By Lemma 3.5, we can propagate an x through the graph to yield a positive linear combination of the Bernstein basis polynomials associated to a function $f(x)$ that is strictly positive on $[0, 1]$. This propagation will yield the polynomial $B_n(f; x)$. Because this propagated polynomial loops around the graph and is made up of the functions representing the conductivities on all but one of the multiple edges in $\lambda_{0,1}$, with the other edge having conductivity x, $\lambda_{0,1}$ will have a value given by

$$B_n + x = \sigma(x),$$

where $\sigma(x)$ is the same polynomial that we referenced in Equation 3. If we can show that the equation $\sigma(x) = \lambda_{0,1}$ has exactly N distinct positive real solutions on $(0, 1)$, then we will know that we can construct an N-to-1 graph by producing the appropriate B_n.

We will determine the "appropriate" B_n by designating a function, $h(x)$, that is approximated by the Bernstein polynomial $B_n(h; x)$ which can be constructed through a graph. Before we can define $h(x)$, however, we must define a function $g(x)$ that has properties similar to those of $\sigma(x)$.

Let $g(x)$ (shown in Figure 13) be a function that satisfies the following conditions on $[0, 1]$:

- $g(x)$ is continuous and differentiable
- $g(x) > 1$ at all points on $(0, 1)$
- $g(x) = \lambda$ has N solutions in the interval $(0, 1)$, where $\lambda > 1$

The number of solutions that $g(x) = \lambda$ has on $(0, 1)$ is equal to the number of distinct positive conductivities that we can place on the graph, thus giving rise to an N-to-1 graph. Note that we choose roots on the open interval $(0, 1)$ because no edge can have conductivity zero and because we want to be able to choose $\lambda_{i,j} = 1$.

Let $h(x) = g(x) - x$. We know that $h(x)$ is positive because $g(x)$ is greater than 1, and we only wish to approximate the function on the interval $[0, 1]$.

Therefore, by Lemma 3.5, we can construct $B_n(h;x)$ through the graph. By Theorem 2.1, $B_n(h;x)$ approximates $h(x)$, and by Theorem 2.2, $B_n(h;x)+x=\lambda$ will have the same number of solutions as $g(x)=\lambda$ on $[0,1]$. We obtain the relationship

$$B_n(h;x) \approx g(x) - x,$$

so

$$B_n(h;x) + x \approx g(x),$$

and

$$\sigma(x) \approx g(x).$$

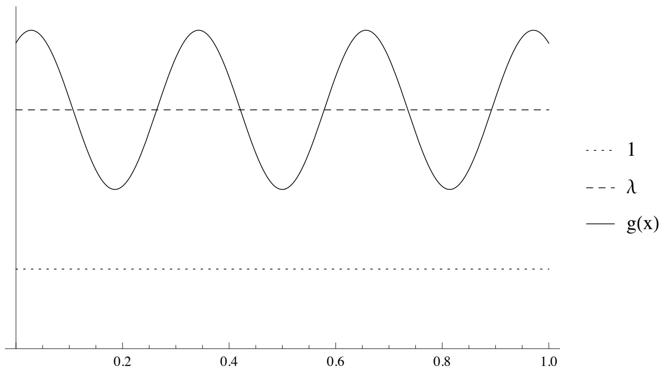

Figure 13: $g(x)$

Because the sign conditions on each $\lambda_{i,j}$ have been satisfied by $B_n(h;x)$ (by Corollary 3.6), we have created a polynomial $\sigma(x)$ with N distinct positive real roots on $(0,1)$ such that each root will produce distinct conductivities on the graph but will produce the same response matrix. We only consider those roots on $(0,1)$ because, as mentioned before, any roots that lie outside that interval will result in invalid (i.e. negative or complex) conductivities. Also, we have guaranteed that each coefficient in the linear combination of the Bernstein basis polynomials is positive by the conditions we placed on $g(x)$. Therefore, we have shown that we can construct an N-to-1 graph for any N.

This shows us that producing an N-to-1 graph is possible. However, it does not give us a specific way to construct such a graph. For any $B_n(f;x)$, the role of the function f is to define the coefficients needed on the Bernstein basis polynomials $b_{r,n}$ in order to correctly approximate the function. The method we have just outlined hinges upon finding an appropriate function (in our case, $g(x)$) that satisfies the conditions we have stated. Because this method does not allow us to state explicitly what an N-to-1 graph construction would be, we use Legendre polynomials in the next section to give us explicit coefficients needed for each Bernstein basis polynomial.

5 Legendre Polynomials

Definition Bonnet's recursion formula, $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$, where $P_0(x) = 1$ and $P_1(x) = x$ gives us an explicit representation of the Legendre polynomials:

$$P_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \left(\frac{1+x}{2}\right)^{n-k} \left(\frac{1-x}{2}\right)^k \tag{8}$$

The standard Legendre polynomials have roots in the interval $(-1, 1)$. Because we are trying to construct polynomials with n distinct positive real roots and we have restricted $x < 1$, we want the roots of the Legendre polynomials to instead be in the interval $(0, 1)$. We can shift the Legendre polynomials to the desired interval.

Definition The shifted Legendre polynomials are constructed by setting $x = 2x - 1$ so that Equation 8 becomes:

$$P_n(2x-1) = \tilde{P}_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 x^{n-k} (1-x)^k . \tag{9}$$

The first few shifted Legendre polynomials are:

$$\begin{aligned}
n = 0 &: \quad \tilde{P}_0(x) = 1 \\
n = 1 &: \quad \tilde{P}_1(x) = 2x - 1 \\
n = 2 &: \quad \tilde{P}_2(x) = (1-x)^2 - 4x(1-x) + x^2 \\
n = 3 &: \quad \tilde{P}_3(x) = -(1-x)^3 + 9x(1-x)^2 - 9x^2(1-x) + x^3 \\
n = 4 &: \quad \tilde{P}_4(x) = (1-x)^4 - 16x(1-x)^3 + 36x^2(1-x)^2 - 16x^3(1-x) + x^4
\end{aligned}$$

For our purposes, we will want the Legendre polynomials to be structured similarly to Bernstein polynomials. Therefore, we will take $r = n - k$ which makes $k = n - r$ and plug this into the shifted Legendre polynomial. With this substitution and the fact that $\binom{n}{n-r} = \binom{n}{r}$, Equation 9 becomes:

$$\tilde{\sigma}(x) = \tilde{P}_n(x) = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r}^2 x^r (1-x)^{n-r} . \tag{10}$$

We will call this $\tilde{\sigma}(x)$ because it is almost the $\sigma(x)$ that we would like to construct however, it does not satisfy all the requirements for $\sigma(x)$.

6 Explicit Polynomial Construction for an N-to-1 Graph

We have shown that we can construct a polynomial $p(x)$ of linear combinations of $x^k(1-x)^{n-k}$ where $\sigma(x) = p(x) - x$. Notice that all coefficients of $p(x)$ must be positive and that all the roots of $\sigma(x) = \lambda_{0,1}$ must be less than 1 in order to ensure that the conductivity of each edge is positive. Our final polynomial $\sigma(x)$ will be a shifted version of a Legendre polynomial and our constructed polynomial $p(x)$ will be a Bernstein polynomial.

We will begin constructing our polynomial with Legendre polynomials. In the discussion of Legendre polynomials, we shifted them to have roots in the interval $(0, 1)$ so that all of the roots of the polynomial we construct will be positive and since we have set $C_j = 1$, we cannot have any roots larger than 1. For now we will be working with $\tilde{\sigma}(x)$. From Equation 6, we have an explicit representation of x in terms of Bernstein polynomials and from Equation 10, we have $\tilde{\sigma}(x)$. Thus we can write an "almost" $p(x)$ function,

$$\begin{aligned}
\tilde{p}(x) &= \tilde{\sigma}(x) - x \\
&= \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r}^2 x^r (1-x)^{n-r} + \sum_{r=0}^{n} \frac{r}{n} \binom{n}{r} x^r (1-x)^{n-r} \\
&= \sum_{r=0}^{n} \left((-1)^{n-r} \binom{n}{r} - \frac{r}{n} \right) \binom{n}{r} x^r (1-x)^{n-r} . \qquad (11)
\end{aligned}$$

Notice that $\tilde{p}(x)$ does not maintain the sign conditions on all of our conductivities for our $\lambda_{i,j}$'s. We fix this by shifting the entire polynomial up. We do this by adding a sufficiently large constant to ensure positive coefficients using Property 2.4. The largest possible magnitude for a negative coefficient of $\tilde{p}(x)$ will occur when $\binom{n}{r}$ has the largest value; this occurs at $\binom{n}{\lceil \frac{n}{2} \rceil}$. Since $\frac{r}{n}$ will never be greater than 1, adding $\binom{n}{\lceil \frac{n}{2} \rceil} + 1$ to each coefficient is sufficient to produce all positive coefficients of $\tilde{p}(x)$.

$$\begin{aligned}
p(x) &= \tilde{p}(x) + \binom{n}{\lceil \frac{n}{2} \rceil} + 1 \\
&= \sum_{r=0}^{n} \left((-1)^{n-r} \binom{n}{r} - \frac{r}{n} + \binom{n}{\lceil \frac{n}{2} \rceil} + 1 \right) \binom{n}{r} x^r (1-x)^{n-r} . \qquad (12)
\end{aligned}$$

Notice that $p(x)$ maintains the sign conditions and has n distinct roots in the interval $(0, 1)$. Thus, we can write $\sigma(x)$ in this way:

$$\begin{aligned}
\sigma(x) &= \tilde{\sigma}(x) + \binom{n}{\lceil \frac{n}{2} \rceil} + 1 \\
&= \tilde{p}(x) + \binom{n}{\lceil \frac{n}{2} \rceil} + 1 + x \\
&= p(x) + x \\
&= \sum_{r=0}^{n} \left((-1)^{n-r} \binom{n}{r} - \frac{r}{n} + \binom{n}{\lceil \frac{n}{2} \rceil} + 1 \right) \binom{n}{r} x^r (1-x)^{n-r} + x \qquad (13)
\end{aligned}$$

We now have an explicit formula for the nth degree polynomial that we can use to construct an N-to-1 graph. Let $\lambda_{0,1} = \binom{n}{\lceil \frac{n}{2} \rceil} + 1$, because we shifted the polynomial up by $\binom{n}{\lceil \frac{n}{2} \rceil} + 1$ and thus the solutions to $\sigma(x) = \lambda_{0,1}$ would occur at the zeros of the original shifted Legendre polynomial. Notice that $\sigma(x)$ satisfies all the necessary properties, $\sigma(x) = \lambda_{0,1}$ has n solutions in the interval $(0, 1)$ and since $C_j = 1$ for all j, $C_j > x$ for all solutions. $\sigma(x)$ and $\lambda_{0,1}$ for $n = 2$ through $n = 5$ are plotted in Figures 14-17.The first few $\sigma(x)$ and $p(x)$ are:

$$
\begin{aligned}
n = 1: \quad & p(x) = (1-x) + 2x \\
& \sigma(x) = (1-x) + 2x + x \\
n = 2: \quad & p(x) = 4(1-x)^2 + x(1-x) + 3x^2 \\
& \sigma(x) = 4(1-x)^2 + x(1-x) + 3x^2 + x \\
n = 3: \quad & p(x) = 3(1-x)^3 + 20x(1-x)^2 + x^2(1-x) + 4x^3 \\
& \sigma(x) = 3(1-x)^3 + 20x(1-x)^2 + x^2(1-x) + 4x^3 + x \\
n = 4: \quad & p(x) = 8(1-x)^4 + 11x(1-x)^3 + 75x^2(1-x)^2 + 9x^3(1-x) + 7x^4 \\
& \sigma(x) = 8(1-x)^4 + 11x(1-x)^3 + 75x^2(1-x)^2 + 9x^3(1-x) + 7x^4 + x \\
n = 5: \quad & p(x) = 10(1-x)^5 + 79x(1-x)^4 + 6x^2(1-x)^3 + 204x^3(1-x)^2 + 26x^4(1-x) + 11x^5 \\
& \sigma(x) = 10(1-x)^5 + 79x(1-x)^4 + 6x^2(1-x)^3 + 204x^3(1-x)^2 + 26x^4(1-x) + 11x^5 + x.
\end{aligned}
$$

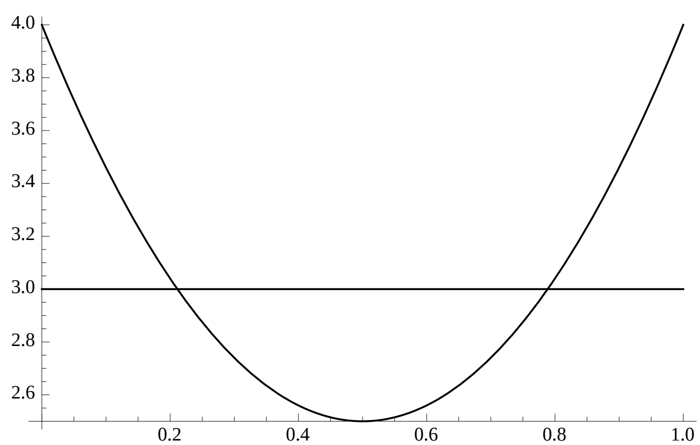

Figure 14: When $n = 2$ there are two solutions to $\sigma(x) = \lambda_{0,1} = 3$

To summarize the previous steps for constructing $\sigma(x) = p(x) + x$ with the necessary properties:

• Take the shifted Legendre polynomial with roots in the interval $(0, 1)$. This is $\tilde{\sigma}(x)$.

• Write x in terms of Bernstein polynomials.

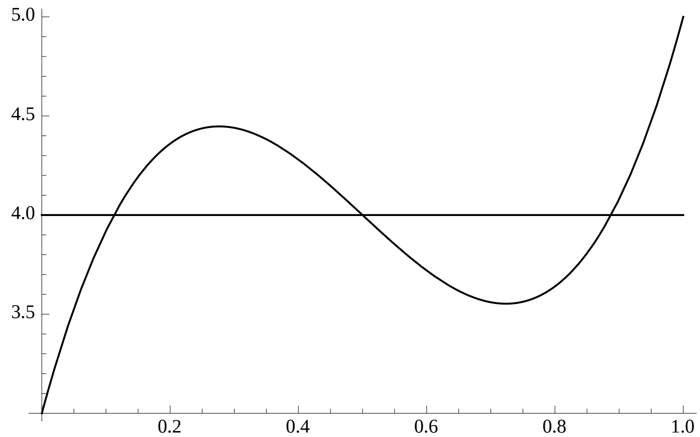

Figure 15: When $n = 3$ there are three solutions to $\sigma(x) = \lambda_{0,1} = 4$

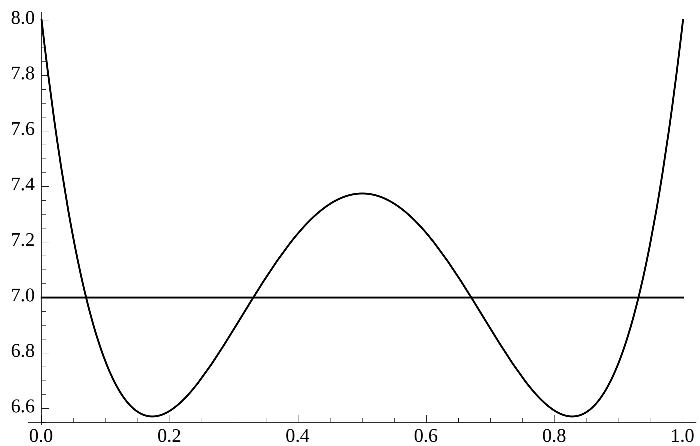

Figure 16: When $n = 4$ there are four solutions to $\sigma(x) = \lambda_{0,1} = 7$

• Define $\tilde{p}(x) = \tilde{\sigma}(x) - x$ and write it in Bernstein polynomials.
• Define $p(x) = \tilde{p}(x) + \binom{n}{\lceil \frac{n}{2} \rceil} + 1$ to shift $\tilde{p}(x)$ up so that all of the coeficents are positive.

This process will produce $p(x)$ from Equation 12 and $\sigma(x)$ from Equation 13. By setting $\sigma(x) = \binom{n}{\lceil \frac{n}{2} \rceil} + 1$ this will produce n positive real solutions with positive edge conductivities for all n solutions.

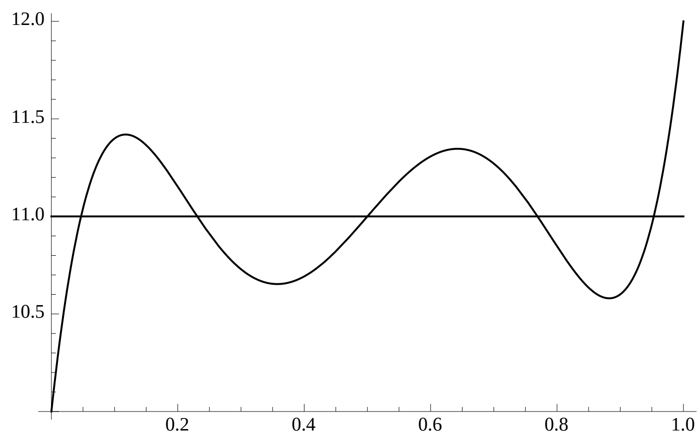

Figure 17: When $n = 5$ there are five solutions to $\sigma(x) = \lambda_{0,1} = 11$

7 Discussion

7.1 Comparing Constructions Using Bernstein Polynomials and Legendre Polynomials

We have seen that it is possible to propagate a function $f(x) = x$ through a variety of graph structures to yield Bernstein polynomials with positive coefficients. The limitation of using only Bernstein polynomials is that in order to construct an N-to-1 graph, we must find a function $g(x)$ such that it has N solutions to the equation $g(x) = \lambda$ on the open interval $(0, 1)$. Such a function may be difficult to specify, so the method outlined in Section 4 only tells us that it is possible to construct such a graph; it does not provide us with specific tools to do so.

The advantage that Legendre polynomials afford us is the ability to write down exact graph constructions explicitly and obtain known (though complicated) roots. Legendre polynomials of degree n are known and calculable, and when shifted, they are also able to be written as Bernstein basis polynomials with known coefficients. We saw above that if $\sigma(x) = p(x) + x$, where $\sigma(x)$ is a shifted positive Legendre polynomial, we can write $\sigma(x)$ in the form of a Bernstein polynomial. Because x can also be written as a Bernstein polynomial (see Section 2.1), we can write $p(x) = \sigma(x) - x$ as a Bernstein polynomial and therefore construct a graph that produces it. In this case, we have specific steps for this construction: because the Legendre polynomial of degree n is known, we can specify exact relationships between the conductivities of some edges on the graph. We can construct the exact graph that we need.

One thing to note here is that although we have presented an explicit way of constructing these graphs, this method is far from pleasant to implement. Because Bernstein basis polynomials depend on the multiplication of large numbers of x and $(1-x)$ terms, the graphs needed to construct them are very large and complicated. It seems unlikely to us that actually drawing such a graph would be useful or illustrative.

7.2 Use of Other Polynomials

In this document, we focused on the use of two specific polynomials—Bernstein and Legendre—to construct N-to-1 graphs. We did this because Bernstein polynomials and Legendre polynomials have known and convenient properties that make construction easier and more straightforward or universally explicit. Instead of using Legendre polynomials, however, we can choose to use any polynomial we want as long as its roots lie on $(0,1)$. This is because the Bernstein basis polynomials $b_{r,n}$ form a basis for all polynomials of degree at most n, so we can always write any polynomial as a linear combination of the $b_{r,n}$. As shown in Section 3, we can propagate an x through our graph to yield this linear combination.

7.3 Further Research

Very little is known about N-to-1 graphs. There is much work to be done in studying them and how they fit into our understanding of electrical networks. What follows is a list of possible topics to explore regarding N-to-1 graphs:

- <u>Finding the genus of an N-to-1 graph.</u> Is it dependent on N only, or on other factors as well?
- <u>Parametrizing the response matrix.</u> Is there a more explicit way of doing this? We have identified some relationships between edge conductivities based on the quadrilateral rule, but is there a way to formalize this? Can we read off some or all these relationships directly from the response matrix?
- <u>Computing conductivities for an example graph.</u> We noted in Section 7.1 that actually writing down an example of an N-to-1 graph constructed using these methods is tedious. However, it may be possible and useful to write a program that carries the computations through the graph without having to draw the graph itself.
- <u>Negative and/or complex roots.</u> In this paper, we focused solely on roots that were positive and real. However, it would be conceivable to construct polynomials that yielded negative or even complex roots. In the context of electrical networks, what does it mean for a graph to have non-positive, non-real conductivities on its edges? How does this change our assumptions about restrictions on the $\lambda_{i,j}$?

References

[1] Edward B. Curtis and James A. Morrow. *Inverse Problems for Electrical Networks.* World Scientific: New York, 2000.

[2] George McArtney Phillips. *Interpolation and Approximation by Polynomials.* Springer: New York, 2003.

[3] Jeff Russell. *Star and K Solve the Inverse Problem.* University of Washington Math REU (2003).

[4] Courtney Kempton. *N-1 Graphs.* University of Washington Math REU (2011).

[5] Cynthia Wu. *n to 1 Graphs.* University of Washington Math REU (2012).