3-1 Graph Construction

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These are notes on the construction of n-to-1 graphs and the verification of sign conditions on conductivities in the response matrix, where each $\lambda_{i,j}$ is an entry in the response matrix. We choose to use the convention that all off-diagonal entries in the response matrix are positive; this is different from the standard convention.

This paper contains a 3-to-1 graph construction along with the calculations required to verify the sign conditions. Unfortunately, the construction method used for the 3-to-1 graph does not hold up well for larger n as the relationships between the λ's became far too complicated—and, in some attempted constructions, impossible—for all of them to be positive. The primary tool for propagation of information through our graph is the quadrilateral rule.

1 3-to-1 Graph

1.1 Main Arm

Here, we propagate the unknown function, $f_1 = x$, through the "main arm" of the graph.

$$f_1 = x \tag{1}$$

$$f_2 = f_1 = x, \text{ if } \lambda_{1,2} = \lambda_{2,3} \tag{2}$$

$$f_3 = \lambda_{0,3} - f_2 = \lambda_{0,3} - x \tag{3}$$

$$f_4 = f_3 = \lambda_{0,3} - x, \text{ if } \lambda_{0,4} = \lambda_{4,6} \tag{4}$$

$$f_5 = \lambda_{3,6} - f_4 = \lambda_{3,6} - (\lambda_{0,3} - x) = x, \text{ if } \lambda_{3,6} = \lambda_{0,3} \tag{5}$$

$$f_6 = f_5 = x, \text{ if } \lambda_{3,5} = \lambda_{5,9} \tag{6}$$

$$f_7 = \lambda_{6,9} - f_6 = \lambda_{6,9} - x \tag{7}$$

$$f_8 = f_7 = \lambda_{6,9} - x, \text{ if } \lambda_{6,7} = \lambda_{7,8} \tag{8}$$

$$f_9 = \lambda_{8,9} - f_8 = \lambda_{8,9} - (\lambda_{6,9} - x) = x, \text{ if } \lambda_{6,9} = \lambda_{8,9} \tag{9}$$

$$f_{10} = f_9 = x, \text{ if } \lambda_{9,10} = \lambda_{10,11} \tag{10}$$

$$f_{11} = \lambda_{8,11} - f_{10} = \lambda_{8,11} - x \tag{11}$$

$$f_{12} = f_{11} = \lambda_{8,11} - x, \text{ if } \lambda_{8,12} = \lambda_{12,14} \tag{12}$$

$$f_{13} = \lambda_{11,14} - f_{12} = \lambda_{11,14} - (\lambda_{8,11} - x) = x, \text{ if } \lambda_{11,14} = \lambda_{8,11} \tag{13}$$

$$f_{14} = f_{13} = x, \text{ if } \lambda_{11,13} = \lambda_{13,15} \tag{14}$$

$$f_{15} = \lambda_{14,15} - f_{14} = \lambda_{14,15} - x \tag{15}$$

$$f_{16} = f_{15} = \lambda_{14,15} - x, \text{ if } \lambda_{14,16} = \lambda_{16,17} \tag{16}$$

$$f_{17} = \lambda_{15,17} - f_{16} = \lambda_{15,17} - (\lambda_{14,15} - x) = x, \text{ if } \lambda_{15,17} = \lambda_{14,15} \tag{17}$$

$$f_{18} = f_{17} = x, \text{ if } \lambda_{0,18} = \lambda_{15,18} \tag{18}$$

$$f_{19} = \lambda_{0,17} - f_{18} = \lambda_{0,17} - x \tag{19}$$

1.2 Secondary Arm

Here, we propagate the unknown function, $f_1 = x$, through the "secondary arm" of the graph. Where the secondary arm branches off from the main arm, we switch to $\tilde{f}$ notation. Note that we have replaced Vertex $\tilde{8}$ with Vertex 1.

$$f_1 = x \tag{20}$$

$$\tilde{f}_2 = f_1 = x, \text{ if } \lambda_{0,2} = \lambda_{2,3} \tag{21}$$

$$\tilde{f}_3 = \lambda_{1,3} - \tilde{f}_2 = \lambda_{1,3} - x \tag{22}$$

$$\tilde{f}_4 = \tilde{f}_3 = \lambda_{1,3} - x, \text{ if } \lambda_{1,\tilde{4}} = \lambda_{\tilde{4},\tilde{6}} \tag{23}$$

$$\tilde{f}_5 = \lambda_{3,\tilde{6}} - \tilde{f}_4 = \lambda_{3,\tilde{6}} - (\lambda_{1,3} - x) = x, \text{ if } \lambda_{3,\tilde{6}} = \lambda_{1,3} \tag{24}$$

$$\tilde{f}_6 = \tilde{f}_5 = x, \text{ if } \lambda_{3,\tilde{5}} = \lambda_{\tilde{5},\tilde{9}} \tag{25}$$

$$\tilde{f}_7 = \lambda_{\tilde{6},\tilde{9}} - \tilde{f}_6 = \lambda_{\tilde{6},\tilde{9}} - x \tag{26}$$

$$\tilde{f}_8 = \tilde{f}_7 = \lambda_{\tilde{6},\tilde{9}} - x, \text{ if } \lambda_{\tilde{6},\tilde{7}} = \lambda_{\tilde{7},\tilde{8}} \tag{27}$$

$$\tilde{f}_9 = \lambda_{1,\tilde{9}} - \tilde{f}_8 = \lambda_{1,\tilde{9}} - (\lambda_{\tilde{6},\tilde{9}} - x) = x, \text{ if } \lambda_{1,\tilde{9}} = \lambda_{\tilde{6},\tilde{9}} \tag{28}$$

1.3 Tertiary Arm

Here, we propagate the unknown function, $f_9 = x$ (obtained from propagating $f_1 = x$ through the main arm), through the "tertiary arm" of the graph. Where the tertiary arm branches off from the main arm, we switch to $f\prime$ notation.

$$f_9 = x \tag{29}$$

$$f_{10}\prime = f_9 = x, \text{ if } \lambda_{8,10} = \lambda_{10,11} \tag{30}$$

$$f_{11}\prime = \lambda_{9,11} - f_{10}\prime = \lambda_{9,11} - x \tag{31}$$

$$f_{12}\prime = f_{11}\prime = \lambda_{9,11} - x, \text{ if } \lambda_{9,12\prime} = \lambda_{12\prime,14\prime} \tag{32}$$

$$f_{13}\prime = \lambda_{11,14\prime} - f_{12}\prime = \lambda_{11,14\prime} - (\lambda_{9,11} - x) = x, \text{ if } \lambda_{11,14\prime} = \lambda_{9,11} \tag{33}$$

$$f_{14}\prime = f_{13}\prime = x, \text{ if } \lambda_{11,13\prime} = \lambda_{13\prime,15\prime} \tag{34}$$

$$f_{15}\prime = \lambda_{14\prime,15\prime} - f_{14}\prime = \lambda_{14\prime,15\prime} - x \tag{35}$$

$$f_{16}\prime = f_{15}\prime = \lambda_{14\prime,15\prime} - x, \text{ if } \lambda_{14\prime,16\prime} = \lambda_{16\prime,17\prime} \tag{36}$$

$$f_{17}\prime = \lambda_{15\prime,17\prime} - f_{16}\prime = \lambda_{15\prime,17\prime} - (\lambda_{14\prime,15\prime} - x) = x \text{ if } \lambda_{15\prime,17\prime} = \lambda_{14\prime,15\prime} \tag{37}$$

1.4 Multipliers

In this section, we describe the propagation of information when the "arms" of the graph come together at the square multipliers.

Secondary and Tertiary Arm Input Here, we consider the square multiplier with edges $(1, \tilde{9})$ and $(15\prime, 17\prime)$.

$$\tilde{f_{10}} = \frac{\tilde{f_9} \cdot f_{16}\prime}{\lambda_{\tilde{9},17\prime}} = \frac{x^2}{\lambda_{\tilde{9},17\prime}} \tag{38}$$

$$\tilde{f_{11}} = \lambda_{1,15\prime} - \tilde{f_{10}} = \lambda_{1,15\prime} - \frac{x^2}{\lambda_{\tilde{9},17\prime}} \tag{39}$$

Main Arm Input and Polynomial Output Here, we consider the square multiplier with edges $(0, 17)$ and $(1, 15\prime)$. We will carry these computations through the square multiplier to obtain the polynomial which will ultimately allow us to define our roots.

$$f_{20} = \frac{\tilde{f_{11}} \cdot f_{19}}{\lambda_{15\prime,17}} = (\lambda_{1,15\prime} - \frac{x^2}{\lambda_{\tilde{9},17\prime}}) \cdot (\lambda_{0,17} - x) \cdot (\frac{1}{\lambda_{15\prime,17}}) \tag{40}$$

We then add the double edge between $\lambda_{0,1}$ and plug in values to produce our λ-dependent polynomial.

$$f_{20} + f_1 = \lambda_{0,1} \tag{41}$$

$$(\lambda_{1,15\prime} - \frac{x^2}{\lambda_{\tilde{9},17\prime}}) \cdot (\lambda_{0,17} - x) \cdot (\frac{1}{\lambda_{15\prime,17}}) + x = \lambda_{0,1} \tag{42}$$

$$(\lambda_{1,15\prime} - \frac{x^2}{\lambda_{\tilde{9},17\prime}}) \cdot (\lambda_{0,17} - x) + x(\lambda_{15\prime,17}) - \lambda_{0,1}\lambda_{15\prime,17} = 0 \tag{43}$$

$$(\lambda_{1,15\prime}\lambda_{\tilde{9},17\prime} - x^2) \cdot (\lambda_{0,17} - x) + x(\lambda_{15\prime,17}\lambda_{\tilde{9},17\prime}) - \lambda_{0,1}\lambda_{15\prime,17}\lambda_{\tilde{9},17\prime} = 0 \tag{44}$$

$$x^3 - \lambda_{0,17}x^2 - \lambda_{1,15\prime}\lambda_{\tilde{9},17\prime}x + \lambda_{1,15\prime}\lambda_{\tilde{9},17\prime}\lambda_{0,17} + x(\lambda_{15\prime,17}\lambda_{\tilde{9},17\prime}) - \lambda_{0,1}\lambda_{15\prime,17}\lambda_{\tilde{9},17\prime} = 0 \tag{45}$$

This is our λ-dependent polynomial:

$$x^3 - \lambda_{0,17}x^2 + \lambda_{\tilde{9},17\prime}(\lambda_{15\prime,17} - \lambda_{1,15\prime})x + \lambda_{\tilde{9},17\prime}(\lambda_{1,15\prime}\lambda_{0,17} - \lambda_{0,1}\lambda_{15\prime,17}) = 0 \tag{46}$$

1.5 Sign Conditions and Necessary Assumptions

From the construction of the graph, in order to make all of the conductivities positive, the following sign conditions must be met.

$$\lambda_{3,6} = \lambda_{0,3} > x \tag{47}$$

$$\lambda_{8,9} = \lambda_{6,9} > x \tag{48}$$

$$\lambda_{11,17} = \lambda_{8,11} > x \tag{49}$$

$$\lambda_{15,17} = \lambda_{14,15} > x \tag{50}$$

$$\lambda_{3,\tilde{6}} = \lambda_{1,3} > x \tag{51}$$

$$\lambda_{1,\tilde{9}} = \lambda_{\tilde{6},\tilde{9}} > x \tag{52}$$

$$\lambda_{11,14'} = \lambda_{9,11} > x \tag{53}$$

$$\lambda_{15',17'} = \lambda_{14',15'} > x \tag{54}$$

$$\lambda_{1,15'} > \frac{x^2}{\lambda_{\tilde{9},17'}} \tag{55}$$

$$\lambda_{0,17} > x \tag{56}$$

1.6 Choosing Roots and Appropriate Conductivities

Now that we have a λ-dependent polynomial and sign conditions in place, we can designate our desired roots and choose appropriate values for each $\lambda_{i,j}$. Let our roots be a, b, and c, given by the polynomial:

$$(x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc = 0$$

We can use the coefficients of each polynomial to choose appropriate values for each $\lambda_{i,j}$.

$$a + b + c = \lambda_{0,17} \tag{57}$$

$$ab + ac + bc = \lambda_{\tilde{9},17'}(\lambda_{15',17} - \lambda_{1,15'}) \tag{58}$$

$$-abc = \lambda_{\tilde{9},17'}(\lambda_{1,15'}\lambda_{0,17} - \lambda_{0,1}\lambda_{15',17}) \tag{59}$$

We then use sign conditions to choose conductivities that fulfill all restrictions noted above.

1.7 Example: Roots at $\mathbf{x = 1, 2, 3}$

We can show that this polynomial can have three distinct positive roots and all sign conditions are met. If we choose $x = 1, 2, 3$ as the roots of the polynomial, this is not difficult to verify. Thus, we need to choose the λ's such that our polynomial becomes:

$$x^3 - 6x^2 + 11x - 6 = 0. \tag{60}$$

Thus, from Equations 57, 58, 59, 60, we get that:

$$\lambda_{0,17} = 6, \tag{61}$$

$$\lambda_{\tilde{9},17'}(\lambda_{15',17} - \lambda_{1,15'}) = 11, \tag{62}$$

$$\lambda_{\tilde{9},17'}(\lambda_{1,15'}\lambda_{0,17} - \lambda_{0,1}\lambda_{15',17}) = -6. \tag{63}$$

If we choose

$$\lambda_{\tilde{9},17'} = 11, \tag{64}$$

Equation 62 becomes:

$$\lambda_{15',17} - \lambda_{1,15'} = 1, \tag{65}$$

and Equation 63 becomes:

$$6\lambda_{1,15'} - \lambda_{0,1}\lambda_{15',17} = \frac{-6}{11}. \tag{66}$$

Thus, if we choose

$$\lambda_{0,1} = 5, \tag{67}$$

we can pick

$$\lambda_{15',17} = \frac{60}{11} \tag{68}$$

and

$$\lambda_{1,15'} = \frac{49}{11}. \tag{69}$$

Notice that the final choices for $\lambda_{15',17}$ and $\lambda_{1,15'}$ were not arbitrary, and not all λ's that satisfy Equation 66 will satisfy the sign conditions.

It is not difficult to verify based on our choices for the λ's that the previously mentioned sign conditions are met and that we have constructed a 3-to-1 Graph with all positive conductivities.

2 3-to-1 Graphs with Nodes and Edges Labled