Eigenvectors and Eigenvalues of Layered Electrical Networks

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Abstract

We will look at the eigenvectors and eigenvalues of the response matrices of different layered electrical networks.

Contents

1 Introduction

Let G=(V,E) be a graph, where V is the set of vertices and E is the set of edges. The set V can be broken down into two sets: *int*V and ∂V, where *int*V is the set of interior vertices and ∂V is the set of boundary vertices. All of the graphs that we will look at are connected circular planar graphs. A graph G is circular planar if G can be embedded in a disk so that the boundary vertices lie on the curve C which bounds the disk.

1.1 Resistor Networks

Given a graph G and a function γ(e), which gives the conductance of an edge e, the resistor network is denoted $\Gamma(G, \gamma(e))$. For a voltage function u defined on every node of Γ, the current c(e) for an edge e with endpoints j and k is given by *Ohm's Law*

$$c(e) = \gamma(e)[u(j) - u(k)].$$

At a node j where the function u is not γ-harmonic, then the current ϕ(j) into the network at j must equal the current from j to its neighboring nodes, which is given by *Kirchhoff's Law*

$$\sum_{k \sim j} \gamma(j,k)[u(j) - u(k)] = \phi(j).$$

If u is a γ-harmonic function then Kirchhoff's Law states that ϕ(j)=0 at all interior nodes. For our purposes, we will only look at γ-harmonic functions.

1.2 Kirchhoff and Response Matrices of a Network

The Kirchhoff matrix of a network, denoted K=$[\kappa_{jk}]$, is the m$\times$m matrix which is denoted by

$$K = \begin{cases} -\gamma_{jk}, & \text{if } j \neq k, \\ \sum_{j \neq k} \gamma_{jk} & \text{if } j = k \end{cases}$$

For K_{jk} where $j \not\sim k$ then the entry in κ_{jk} is zero. This makes the row sums of K to be equal to zero, as $\sigma_{jk} = \sigma_{kj}$. K is a positive semi-definite matrix.

K can be partitioned into four submatrices by ordering and distinguishing between the interior and boundary vertices of the network. This allows the Kirchhoff matrix to be written as

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

where the first n rows and columns are indexed by ∂V and the last p rows and columns are indexed by *int*V and $n + p = m$.

The response matrix, denoted Λ, is the matrix maps boundary voltages to boundary currents. The γ-harmonic function u is defined on all of the nodes of G and defines a current ϕ at the boundary. Given a voltage function, *f*, the response matrix maps f to ϕ. That is,

$$\Lambda f = \phi.$$

The entries of Λ, given by λ_{jk} represent the current at node j due to voltage 1 at node k and voltage 0 everywhere else on the boundary. By repeating this process for every j,k on the graph, we can create the response matrix. But this would be a long and tedious process for all but the most simple networks. There is a much more elegant way of representing Λ in terms of the conductivities on the graph.

Λ is the Schur Complement of the Kirchhoff matrix in terms of the submatrix C. We can write Λ as

$$\Lambda = A - BC^{-1}B^T$$

This formula for Λ can also be gotten by using the Kirchhoff matrix and the vector [*f*,*g*] where f is the vector of voltage imposed at the boundary and g is the vector of resulting voltages at the interior nodes. Because we are working with γ-harmonic functions, Kirchhoff's current law says the net current at the interior nodes of the graph is zero. This gives us

$$Af + Bg = \phi$$
$$B^T f + Cg = 0$$

By solving for g and then ϕ, we get that

$$\phi = (A - BC^{-1}B^T)f$$

Using this notation for Λ, we can clearly see that Λ is an n$\times$n matrix that maps boundary voltages to boundary currents. Like K, Λ is symmetric, positive semi-definite, and has row sums zero. The fact that Λ is a symmetric n$\times$n matrix implies that there exist n linearly independent, orthogonal eigenvectors of Λ that form a basis for $\mathbb{R}^n$.

1.3 Some Useful Definitions

It might be helpful to define some terms that will be used frequently, as well as the definitions for the types of graphs that will be looked at in this paper. The exact specifications for each graph will be given in more detail later.

Definition 1.1. The **depth** of the graph is the least number of edges needed to connect the boundary to the center of the graph.

Definition 1.2. An electrical network with **layered conductances** is a network in which all edges that are a given distance from the root have the same conductance. We will denote the conductance of an edge jk by σ_{jk}.

Definition 1.3. A **boundary antenna** consists of two boundary nodes which are neighbors of the same interior vertex. For layered networks, the two edges that make up the boundary antenna will have the same conductance.

Definition 1.4. The **degree of the root** is the number, r, of edges which have the root as an endpoint. We will also refer to r as the number of root branches, or edges which extend from the root.

Definition 1.5. For any vertex v, the **degree of a vertex**, given by deg(v), is the number vertices which are connected by an edge to v.

1.4 Eigenvalues and Eigenvectors of Λ

Because Λ is a map from boundary voltages to boundary currents, it is often interesting to look at cases where the boundary currents are a multiple of the boundary voltages. Eigenvectors and eigenvalues of Λ allow us to do that. If λ_k is an eigenvalue of Λ, then for some vector, v, of voltages on the boundary,

$$\Lambda v = \lambda v.$$

The vector, v, is the eigenvector that corresponds to λ.

If Λ is an n×n matrix, then there exists n linearly independent orthogonal eigenvectors and at most n eigenvalues. There may be eigenvalues with multiplicity greater than one.

Lemma 1.1. *λ=0 is an eigenvalue for all response matrices with multiplicity 1.*

Proof. Let v be a constant vector, that is, v assigns the same voltage to all boundary nodes on the network. Then because row sums of Λ are zero, Λv = 0. So λv = 0, and λ = 0.

■

Lemma 1.2. *All eigenvalues of the response matrix are greater than or equal to zero.*

Proof. As shown above, if v is a constant eigenvector, then λ=0. If v is not a constant eigenvector we can normalize v so that $||v||_\infty$=1. Let j be a boundary point where v(j)=1. If j is connected to k other nodes, then

$$\sum_{j\sim k}(1 - v_k)\gamma_{jk} = \lambda * v_j = \lambda$$

where v_k is the voltage at node k and γ_{jk} is the conductivity on the edge between j and k. Because v is normalized, -1 $\leq$ v $\leq$ 1, so (1-v_k) $\geq$ 0. So $\lambda \geq 0$.

$\blacksquare$

Corrolary 1.3. *$\lambda = \sigma_1$ is the largest eigenvalue for networks with at least one boundary antenna.*

A proof of the general case is given in [*reference*]. Because we are only looking at networks with layered conductances, a_m=σ_1 and $\lambda \leq \sigma_1$.

2 Layered Square Lattice Networks

A layered square lattice network Γ with depth n has a grid-like structure with n boundary nodes on each edge. A layered square lattice network has no boundary to boundary edges. The conductances are layered such that the edges, ij, which connect a boundary node, i, to an interior node, j, have conductance σ_1, the edges connecting the interior nodes j have conductance σ_2 and so forth as shown below.

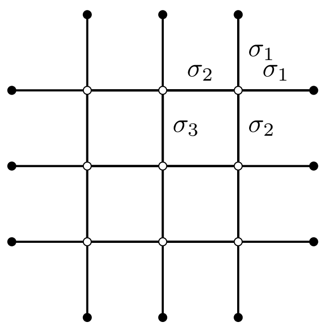


Figure 1.

2.1 The Smallest Non-Trivial Case: n=2

The n=2 case consists of two lines laid over two lines such that they form a 2×2 grid, similar to a tic-tac-toe board. If we layer the conductivities such that σ_1 is on the outermost layer and σ_2 is on the innermost layer, and number the boundary vertices as shown in figure 2, we get a graph that looks like this:

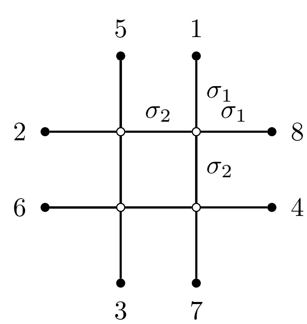


Figure 2.

We are able to use the symmetries and the fact that the conductances are constant on layers to find the eigenvectors for this square lattice network.

We exploited the fact that the square lattice network has rotational symmetry to find our eigenvalues. By using the permutation matrix, P, for a rotation by 90°, we can get eight orthogonal eigenvectors for the 2×2 case.

The vertices are numbered such that 1 maps to 2, 2 to 3, and so on after each rotation of 90°. We can then use P=$C_4 \oplus C_4$. The vectors

$$V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, V_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, V_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, V_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

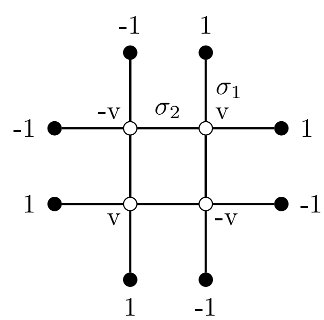
can be used to form the eigenvectors for Λ. V_1 and V_4 are eigenvectors of C_4, while V_2 and V_3 are the real and imaginary parts, respectively, of eigenvectors of C_4 ($V_3 + iV_2$ and $V_3 - iV_2$ are eigenvectors of C_4).

Using these four orthogonal vectors, we can build the orthogonal eigenvectors for Λ.

In order for these vectors to be eigenvectors, they must satisfy Λx = λx, where λ is the corresponding eigenvalue. Λ maps voltages to currents, so the ∂ - current will be a scalar multiple of the ∂ - voltage.

To verify that these vectors are in fact eigenvectors, we will use Kirchhoff's Law and Ohm's Law. As an example, we will use $V_4 \oplus -V_4$, but this process can be repeated to prove the validity of the seven other eigenvectors.

Given the voltages on the boundary correspond to $V_4 \oplus -V_4$, it is clear that if we let the voltage at nodes 9 and 11 will be v, then the voltage at nodes 10 and 12 will be -v.



Using Kirchhoff's Law at an interior vertex with voltage v, we get

$$\sum_{k \sim j} I_{jk} = 0 = 2(v-1)\sigma_1 + 2(v+v)\sigma_2$$

Solving for v yields v = $\frac{\sigma_1}{\sigma_1+2\sigma_2}$. To determine the current at the boundary, we use Ohm's Law which gives the current at node 1,

$$I_1 = (1-v)\sigma_1 = (1 - \frac{\sigma_1}{\sigma_1 + 2\sigma_2})\sigma_1 = \frac{2\sigma_1\sigma_2}{\sigma_1 + 2\sigma_2}$$

This value I_1 is a multiple of the voltage at node 1, specifically $1 * \frac{2\sigma_1\sigma_2}{\sigma_1+2\sigma_2}$. So $V_4 \oplus V_4$ is an eigenvector with eigenvalue $\lambda = \frac{2\sigma_1\sigma_2}{\sigma_1+2\sigma_2}$.

This process can be repeated for the seven other ∂-vertices. This table gives the eigenvectors for the 2×2 square lattice network first in terms of V_1, V_2, V_3, V_4 using the rotational ordering, and second in the traditional circular ordering, with the first entry corresponding to same vertex 1 used in the rotational ordering with the rest following in a counterclockwise orientation.

Eigenvector	Eigenvector	Eigenvalue
$V_1 \oplus V_1$	(1,1,1,1,1,1,1,1)	$\lambda_0 = 0$
$V_1 \oplus -V_1$	(1,-1,1,-1,1,-1,1,-1)	$\lambda_1 = \sigma_1$
$V_2 \oplus V_3$	(1,0,0,1,-1,0,0,-1)	
$V_3 \oplus V_2$	(0,1,1,0,0,-1,-1,0)	
$V_4 \oplus V_4$	(1,1,-1,-1,1,1,-1,-1)	
$V_2 \oplus -V_3$	(1,0,0,-1,-1,0,0,1)	$\lambda_2 = \frac{\sigma_1\sigma_2}{\sigma_1+\sigma_2}$
$V_3 \oplus V_2$	(0,1,1,0,0,-1,-1,0)	
$V_4 \oplus -V_4$	(1,-1,-1,1,1,-1,-1,1)	$\lambda_3 = \frac{2\sigma_1\sigma_2}{\sigma_1+2\sigma_2}$

2.2 Example: n=3 Case

We can use a similar method to find the eigenvectors for the n=3 case. Using P = $C_4 \oplus C_4 \oplus C_4$, we can get twelve orthogonal eigenvectors. These can all be shown to be eigenvectors of Λ through the same process given for the n=2 case.

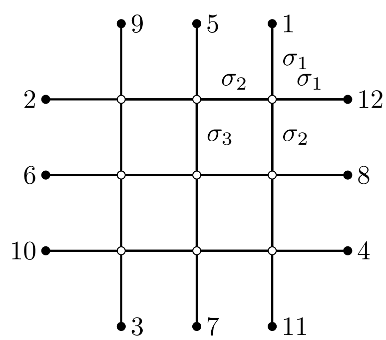


Figure 3.

For this case, we can get eight farily simple eigenvectors, some of which resemble the eigenvectors for the n=2 case. Because all twelve eigenvectors have to be orthogonal, the last four are a little harder to find.

Eigenvector	Eigenvector	Eigenvalue
$V_1 \oplus V_1 \oplus V_1$	(1,1,1,1,1,1,1,1,1,1,1,1)	$\lambda_0 = 0$
$V_2 \oplus 0 \oplus V_3$	(1,0,0,0,0,1,-1,0,0,0,0,-1)	$\lambda_1 = \sigma_1$
$V_3 \oplus 0 \oplus -V_2$	(0,0,-1,1,0,0,0,0,1,-1,0,0)	
$V_4 \oplus 0 \oplus V_4$	(1,0,1,-1,0,-1,1,0,1,-1,0,-1)	
$V_1 \oplus 0 \oplus -V_1$	(1,0,-1,1,0,-1,1,0,-1,1,0,-1)	
$V_4 \oplus 0 \oplus -V_4$	(1,0,-1,-1,0,1,1,0,-1,-1,0,1)	$\lambda_2 = \frac{\sigma_1\sigma_2}{\sigma_1+\sigma_2}$
$V_1 \oplus -2V_1 \oplus V_1$	(1,-2,1,1,-2,1,1,-2,1,1,-2,1)	$\lambda_3 = \frac{2\sigma_1\sigma_2}{\sigma_1+2\sigma_2}$
$0 \oplus V_4 \oplus V_4$	(0,1,0,0,-1,0,0,1,0,0,-1,0)	$\lambda_4 = \frac{2\sigma_1\sigma_2+\sigma_1\sigma_3}{\sigma_1+2\sigma_2+\sigma_3}$
$V_2 + V_3 \oplus a_+V_2 \oplus V_2 - V_3$	(1,a_+,1,1,0,-1,-1,$-a_+$,-1,-1,0,1)	$\lambda_5 = \frac{\sigma_3+3\sigma_2 \mp \sqrt{\sigma_3^2+5\sigma_2^2+2\sigma_2\sigma_3}}{\sigma_3+3\sigma_2+2\sigma_1 \pm \sqrt{\sigma_3^2+5\sigma_2^2+2\sigma_2\sigma_3}}$
$V_2 + V_3 \oplus a_-V_2 \oplus V_2 - V_3$	(1,a_-,1,1,0,-1,-1,$-a_-$,-1,-1,0,1)	
$V_3 - V_2 \oplus a_+V_2 \oplus V_2 + V_3$	(-1,0,1,1,a_+,1,1,0,-1,-1,$-a_+$,-1)	
$V_3 - V_2 \oplus a_-V_3 \oplus V_2 + V_3$	(-1,0,1,1,a_-,1,1,0,-1,-1,$-a_-$,-1)	

For the eigenvectors of λ_5, $a_\pm = -\frac{\sigma_2+\sigma_3 \pm \sqrt{\sigma_3^2+5\sigma_2^2+2\sigma_2\sigma_3}}{\sigma_2}$. As the network grows, the eigenvectors begin to depend more heavily on the values of the conductivities.

Although there are many symmetries of the layered square lattice network, as n grows, the number of symmetries stays the same. The eigenvalue $\lambda_1 = \sigma_1$ will always have multiplicity 4, no matter how large n gets. That means that the eigenvectors corresponding to $\lambda_1 = \sigma_1$ cover less of the eigenspace as n grows, which causes eigenvalues and eigenvectors to become increasingly more complicated as n gets larger.

In networks with more symmetries, we find that the eigenvalues and eigenvectors grow in multiplicity with the size of the graph, which prevents the complicated eigenvectors seen in the layered square lattice network.

3 Layered Tree Networks

A rooted layered tree network Γ with depth n and root degree r has r subtrees that are binomial trees. The boundary nodes are all at a depth n from the root, and there are no boundary to boundary edges. For our examples we will use r=3 for simplicity. The layered tree network has a much more symmetric structure than the layered square lattice network because the number of symmetries grows with the size of the graph. We will number the vertices in a counter-clockwise manner, with nodes 1 and 2 on the same boundary antenna, and the conductances will be layered as shown below.

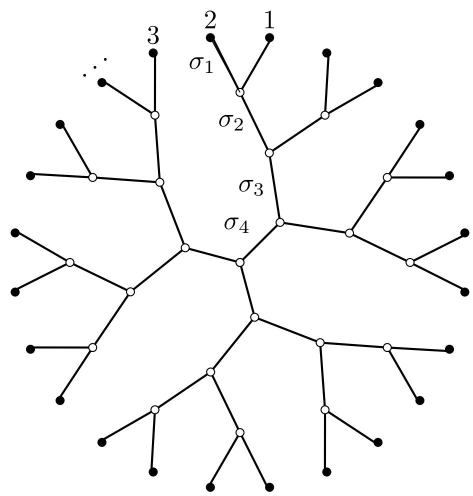


Figure 4.

3.1 The Smallest Non-Trivial Case: n=2, r=3

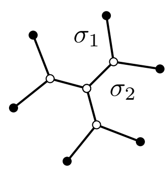


Figure 5.

Eigenvector	Eigenvalue
(1,1,1,1,1,1)	$\lambda_0 = 0$
(1,-1,0,0,0,0)	$\lambda_1 = \sigma_1$
(0,0,1,-1,0,0)	
(0,0,0,0,1,-1)	
(1,1,-1,-1,0,0)	$\lambda_2 = \frac{\sigma_1\sigma_2}{2\sigma_1+\sigma_2}$
(0,0,1,1,-1,-1)	

3.2 Example: n=3, r=3

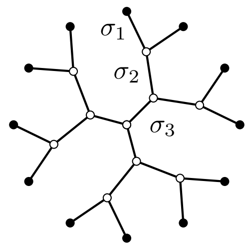


Figure 6.

Eigenvector	Eigenvalue
(1,1,1,1,1,1,1,1,1,1,1,1)	$\lambda_0 = 0$
(1,-1,0,0,0,0,0,0,0,0,0,0)	$\lambda_1 = \sigma_1$
(0,0,1,-1,0,0,0,0,0,0,0,0)	
(0,0,0,0,1,-1,0,0,0,0,0,0)	
(0,0,0,0,0,0,1,-1,0,0,0,0)	
(0,0,0,0,0,0,0,0,1,-1,0,0)	
(0,0,0,0,0,0,0,0,0,0,1,-1)	
(1,1,-1,-1,0,0,0,0,0,0,0,0)	$\lambda_2 = \frac{\sigma_1\sigma_2}{2\sigma_1+\sigma_1}$
(0,0,0,0,1,1,-1,-1,0,0,0,0)	
(0,0,0,0,0,0,0,0,1,1,-1,-1)	
(1,1,1,1,-1,-1,-1,-1,0,0,0,0)	$\lambda_3 = \frac{\sigma_1\sigma_2\sigma_3}{4\sigma_1\sigma_2+2\sigma_1\sigma_3+\sigma_2\sigma_3}$
(0,0,0,0,1,1,1,1,-1,-1,-1,-1)	

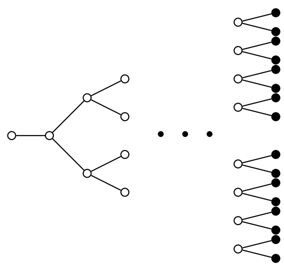
3.3 Expanding to Larger Cases of n and r

In the larger cases, as is evident in the smaller cases, the number of boundary vertices grows with the values of n and r. The number of boundary vertices is given by $2^{n-1}r$. The number of boundary vertices is also the number of linearly independent orthogonal eigenvectors of the network.

For layered tree networks, there are a small number of eigenvalues compared to the number of eigenvectors. This results in eigenvalues with large multiplicities.

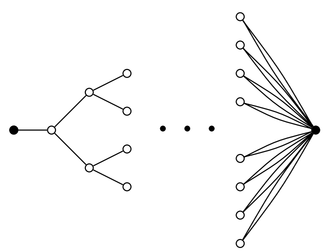
After looking at cases with larger n and r, a pattern for the eigenvalues and their multiplicities begins to emerge. Given that the graph is a tree graph with layered conductances as described above, all eigenvalues, as well as their multiplicities, can be found.

Given a rooted tree graph, let p$\sim$r, where r is the root vertex. Every ∂-vertex has a unique path to r. An **arm** through p is the set of all edges and vertices for which this unique path includes p. We will denote an arm of a rooted tree graph by G.



An Arm, G, of a Rooted Tree Graph with Depth n

If we take an arm, G, of a rooted tree graph, we can construct a new graph, $\widetilde{G}_n$, where we identify all ∂-vertices. If we let the root vertex, r, be viewed as a ∂-vertex, then we have a graph which looks like this



Graph $\widetilde{G}_n$

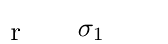
These two graphs, G and $\widetilde{G}_n$, are electrically equivalent, when r has voltage 0 and all the other ∂-vertices have voltage 1. The **effective conductance** of $\widetilde{G}_n$, denoted by μ_n, is the inverse of the sum of the resistances of each edge in the series.

Theorem 3.1. *The effective conductance of an arm of a rooted tree graph is*

$$\mu_n = \frac{1}{\frac{1}{\sigma_n} + \frac{1}{2\sigma_{n-1}} + \cdots + \frac{1}{2^{n-1}\sigma_1}}$$

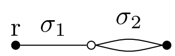
where n is the depth of $\widetilde{G}_n$.

Proof. We can prove this by induction.Given an arm $\widetilde{G}_1$ with depth n=1, the effective conductance is trivial.



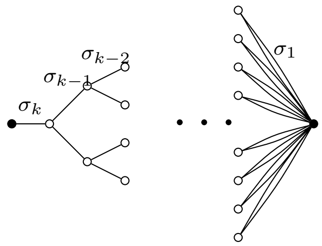
$\mu_1 = \frac{1}{\frac{1}{\sigma_1}} = \sigma_1$.

A better example is an arm $\widetilde{G_2}$ with depth n=2.

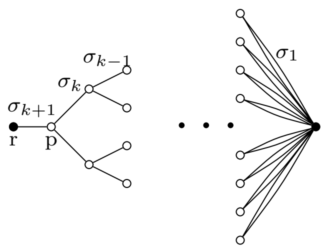


$\mu_2 = \frac{1}{\frac{1}{\sigma_2}+\frac{1}{2\sigma_1}}$.

Assume $\mu_k = \frac{1}{\frac{1}{\sigma_k}+\frac{1}{2\sigma_{k-1}}+\cdots+\frac{1}{2^{k-1}\sigma_1}}$, where k≥n. The graph $\widetilde{G}_k$ with depth k looks like



We need to show that $\mu_{k+1} = \frac{1}{\frac{1}{\sigma_{k+1}}+\frac{1}{2\sigma_k}+\cdots+\frac{1}{2^k\sigma_1}}$. The graph $\widetilde{G}_{k+1}$ with depth n=k+1 looks like



The graph $\widetilde{G}_{k+1}$ consists of two arms, $\widetilde{G}_k$, whose roots are glued together at a point p and an edge connecting the new root, r, to p. From this, we can see that the effective conductance of $\widetilde{G}_{k+1}$ is

$$\begin{aligned}
\mu &= \frac{1}{\frac{1}{\sigma_{k+1}}+\frac{1}{2\mu_k}} \\
&= \frac{1}{\frac{1}{\sigma_{k+1}}+\frac{1}{2}[\frac{1}{\sigma_k}+\frac{1}{2\sigma_{k-1}}+\cdots+\frac{1}{2^{k-1}\sigma_1}]} \\
&= \frac{1}{\frac{1}{\sigma_{k+1}}+\frac{1}{2\sigma_k}+\cdots+\frac{1}{2^k\sigma_1}} \\
&= \mu_{k+1}
\end{aligned}$$

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Theorem 3.2. *Given an eigenvector with voltage 1 on one arm of a graph of depth n, voltage -1 on an adjacent arm and voltage 0 elsewhere, the eigenvalue will be* $\lambda_k = \frac{1}{\sum_{j=1}^{n} \frac{2^{j-1}}{\sigma_j}}$*, where* $\lambda_0 = 0$*.*

Proof. A graph of depth n has $2^{n-1}r$ ∂-vertices, where r is the degree of the root vertex, so there are 2^{n-1} ∂-vertices on each arm. We know the effective conductance of an arm $\widetilde{G}_n$, with voltage 1 on the boundary is $\mu_n = \frac{1}{\frac{1}{\sigma_n}+\frac{1}{2\sigma_{n-1}}+\cdots+\frac{1}{2^{n-1}\sigma_1}}$. So the current on the boundary of $\widetilde{G}_n$ will be $(1-0)\mu_n = \mu_n$. The original graph G has 2^{n-1} ∂-vertices, so we have to divide the current, μ_n by 2^{n-1} to get the current at each ∂-vertex in G.

$$\begin{aligned}
\frac{\mu_n}{2^{n-1}} &= \frac{1}{2^{n-1}} \frac{1}{\frac{1}{\sigma_n} + \frac{1}{2\sigma_{n-1}} + \cdots + \frac{1}{2^{n-1}\sigma_1}} \\
&= \frac{1}{2^{n-1}[\frac{1}{\sigma_n} + \frac{1}{2\sigma_{n-1}} + \cdots + \frac{1}{2^{n-1}\sigma_1}}] \\
&= \frac{1}{\frac{2^{n-1}}{\sigma_n} + \frac{2^{n-2}}{\sigma_{n-1}} + \cdots + \frac{2}{\sigma_2} + \frac{1}{\sigma_1}} \\
&= \frac{1}{\sum_{j=1}^{n} \frac{2^{j-1}}{\sigma_j}} \qquad\qquad = \lambda_n
\end{aligned}$$

■

Corrolary 3.3. *The multiplicity, m, of any eigenvalue* λ_k *of a rooted tree graph is*

$$m = \begin{cases} r \cdot 2^{n-1-k} & \text{when } 1 \le k \le n-1 \\ r-1 & \text{when } k = n, \\ 1 & \text{when } k = 0 \end{cases}$$

where n *is the depth of the graph,* r *is the degree of the root, and* k *is the subscript of* λ*.*

Proof. The number of linearly independent eigenvectors of a tree graph with depth n and root degree r is equal to the number of ∂-vertices, and —∂V—=r$\cdot 2^{n-1}$. We can create a table of the eigenvalues, λ_k, and their dimension.

Eigenvalue	Dimension
$\lambda_0 = 0$	1
$\lambda_1 = \sigma_1$	r$\cdot 2^{n-2}$
$\lambda_2 = \frac{1}{\frac{1}{\sigma_1}+\frac{2}{\sigma_2}}$	r$\cdot 2^{n-3}$
$\vdots$	$\vdots$
$\lambda_k = \frac{1}{\frac{1}{\sigma_1}+\frac{2}{sigma_2}+\cdots+\frac{2^{k-1}}{\sigma_k}}$	r$\cdot 2^{n-1-k}$
$\vdots$	$\vdots$
$\lambda_{n-2} = \frac{1}{\frac{1}{\sigma_1}+\frac{2}{sigma_2}+\cdots+\frac{2^{n-3}}{\sigma_{n-2}}}$	r$\cdot 2^1$
$\lambda_{n-1} = \frac{1}{\frac{1}{\sigma_1}+\frac{2}{sigma_2}+\cdots+\frac{2^{n-2}}{\sigma_{n-1}}}$	r$\cdot 2^0 =$ r
$\lambda_n = \frac{1}{\frac{1}{\sigma_1}+\frac{2}{sigma_2}+\cdots+\frac{2^{n-}}{\sigma_n}}$	r - 1

Totalling these dimensions up tells us the amount of the eigenspace that these eigenvalues cover. Totalling gives us

$$
\begin{aligned}
&1 + r[2^{n-2} + 2^{n-3} + \cdots + 2 + 1] + r - 1 \\
=&1 + r[\frac{2^{n-1}-1}{2-1}] + r - 1 \\
=&r \cdot 2^{n-1} - r + r \\
=&r \cdot 2^{n-1}.
\end{aligned}
$$

This total tells us that there are at least r$\cdot 2^{n-1}$ independent eigenvectors. The number of ∂-vertices, —∂V—=r$\cdot 2^{n-1}$, tells us that there are at most r$\cdot 2^{n-1}$ independent eigenvectors. Therefore, these λ_n and their corresponding eigenvectors form a complete set of independent eigenvectors.

■

4 Layered Ring Networks

A rooted layered ring network Γ with depth n and root degree r is a network with a specified root and *n - 1* conducting rings. The boundary nodes are of degree 1. We will layer the conductances in a similar manner as with the tree graphs, with the conductance σ_1 on the outermost edges and σ_n on the edges connected to the root. Each conducting ring will have a constant conductance μ_n, with the conductance of the outermost ring being μ_1 and the conductance of the innermost ring being μ_{n-1}. Figure 6 is an example of the type of graph we will be looking at.

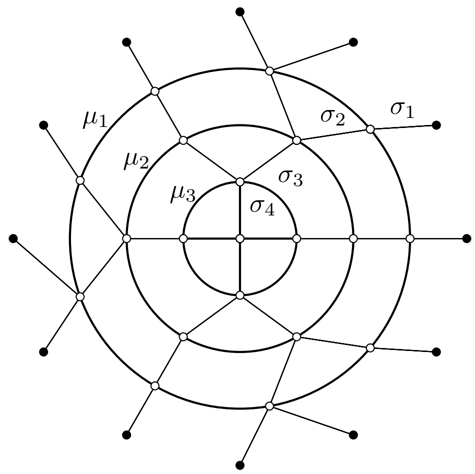


Figure 7.

Because these networks can get complicated quickly, we will look at two examples: the graph which we will call the Spider Web graph, where each interior vertex has degree 5, and the graph which we will call the Compass graph, where each interior vertex has degree 4.

4.1 Layered Spider Web Graph

The spider web graphs will have the same structure as the tree graphs with the exception of the conducting rings. There will be *n-1* conducting rings, which have constant conductivity, μ, at a given depth, as shown in Figure 7. Every interior vertex, excluding the root, will have degree 5.

n=2, r=3

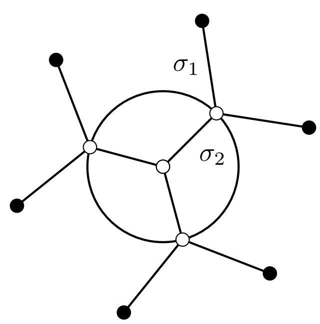


Figure 8.

Eigenvector	Eigenvalue
(1,1,1,1,1,1)	$\lambda_0 = 0$
(1,-1,0,0,0,0)	$\lambda_1 = \sigma_1$
(0,0,1,-1,0,0)	
(0,0,0,0,1,-1)	
(1,1,-1,-1,0,0)	$\lambda_2 = \frac{\sigma_1\sigma_2+3\sigma_1\mu_1}{2\sigma_1+\sigma_2+3\mu_1}$
(0,0,1,1,-1,-1)	

The eigenvectors for this network are the same as the eigenvectors for the layered tree graph without conducting rings. The eigenvalues are the same with the exception of λ_2, which is nearly the same.

The eigenvectors and eigenvalues are fairly simple to find for small n, but as n grows, they become much more complicated to find.

n=3, r=3

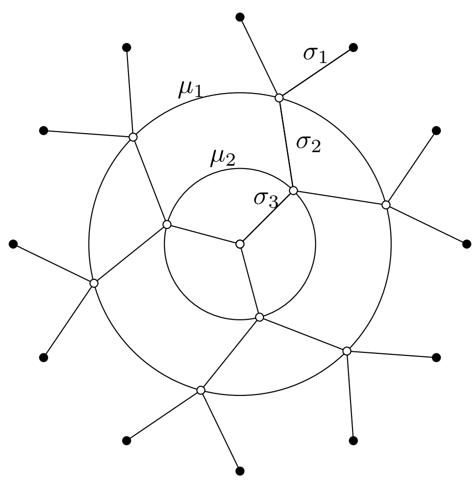


Figure 9.

Eigenvector	Eigenvalue
(1,1,1,1,1,1,1,1,1,1,1,1)	$\lambda_0 = 0$
(1,-1,0,0,0,0,0,0,0,0,0,0)	$\lambda_1 = \sigma_1$
(0,0,1,-1,0,0,0,0,0,0,0,0)	
(0,0,0,0,1,-1,0,0,0,0,0,0)	
(0,0,0,0,0,0,1,-1,0,0,0,0)	
(0,0,0,0,0,0,0,0,1,-1,0,0)	
(0,0,0,0,0,0,0,0,0,0,1,-1)	
(1,1,-1,-1,1,1,-1,-1,1,1,-1,-1)	$\lambda_2 = \frac{\sigma_1\sigma_2+4\mu_1\sigma_1}{2\sigma_1+\sigma_2+4\mu_1}$
(1,1,b,b,a_+, a_+, a_+, a_+,b,b,1,1)	λ_3
(1,1,b,b,a_-, a_-, a_-, a_-,b,b,1,1)	
(b,b,a_+, a_+, a_+, a_+,b,b,1,1,1,1)	
(b,b,a_-, a_-, a_-, a_-,b,b,1,1,1,1)	

λ_3 is not a simple eigenvalue and yields no useful information. In the four eigenvectors corresponding to λ_3, $a_\pm = \frac{-\sigma_2^2 \pm \sqrt{\sigma_2^4+\mu_1(2\sigma_2+\sigma_3+3\mu_2)(\sigma_2^2+2\sigma_2\mu_1+\sigma_3\mu_1+3\mu_1\mu_2)}}{\sigma_2^2+2\sigma_2\mu_1+\sigma_3\mu_1+3\mu_1\mu_2}$ and b = -1$-a_\pm$.

4.2 Layered Compass Graph

The layered compass graph consists of a root vertex with r branches and *n-1* conducting rings, which start at the end of the root branches continue outward, as seen below. Each of these root branches extends directly to the boundary without splitting. This forces every interior vertex, excluding the root, to have degree 4.

n = 3, r = 4

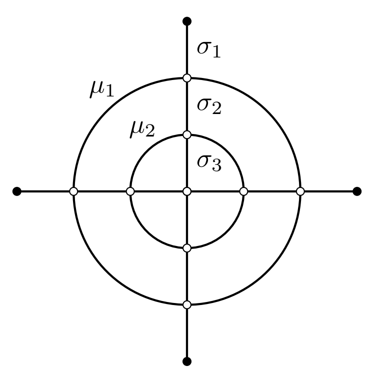


Figure 10.

Eigenvector	Eigenvalue
(1,1,1,1)	$\lambda_0 = 0$
(1,0,-1,0)	$\lambda_1 = \frac{\sigma_1[(\sigma_2+2\mu_1)(\sigma_2+\sigma_3+2\mu_2)-\sigma_2^2]}{(\sigma_1+\sigma_2+2\mu_1)(\sigma_2+\sigma_3+2\mu_2)-\sigma_2^2}$
(0,1,0,-1)	
(1,-1,1,-1)	$\lambda_2 = \frac{\sigma_1[(\sigma_2+4\mu_1)(\sigma_2+\sigma_3+4\mu_2)-\sigma_2^2]}{(\sigma_1+\sigma_2+4\mu_1)(\sigma_2+4\sigma_3+2\mu_2)-\sigma_2^2}$

4.3 General Layered Ring Network

General layered ring networks are much harder to work with because of their lack of symmetry. For the other cases, there was a systematic way to find the eigenvectors, but that doesn't seem to be the case for any general ring network. Some of the eigenvectors are easily found, while some we were unable to find.

Example: n=3 r=4

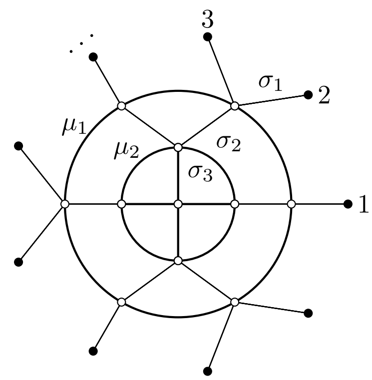


Figure 11.

We can easily get the eigenvectors that correspond to $\lambda_0 = 0$ and $\lambda_1 = \sigma_1$, but we could not see a systematic way of finding the other five eigenvectors.

Eigenvector	Eigenvalue
(1,1,1,1,1,1,1,1,1)	$\lambda_0 = 0$
(0,1,-1,0,0,0,0,0,0)	$\lambda_1 = \sigma_1$
(0,0,0,0,1,-1,0,0,0)	
(0,0,0,0,0,0,0,1,-1)	

4.4 Some Additional Theorems

These last theorems address the reverse problem of what has been covered in this paper: Given an eigenvector or an eigenvalue of a network, can we glean what the network looks like?

Theorem 4.1. *If a layered network with at least four boundary vertices has an eigenvalue $\lambda_1 = \sigma_1$ then there exists a boundary antenna on the graph.*

Proof. The eigenvalue $\lambda_1 = \sigma_1$ implies that the current at a boundary vertex is a multiple of the voltage, v, at the vertex. The equation for the current at the boundary node then becomes

$$(v - w)\sigma_1 = \sigma_1 v.$$

This forces the voltage, w, at the interior node which connects the boundary node to the interior of the graph to be zero. The zero voltage at an interior node, which has zero net current, implies that there exists another edge which has a negative voltage at its second node. Because the conductances on the edges are layered, then the current equation for the second node becomes

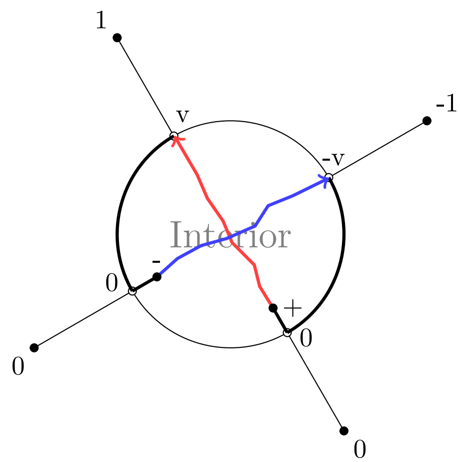
$$(0 - x)\sigma_1 = \sigma_1 v.$$

So x = -v because $\lambda_1 = \sigma_1$ is an eigenvalue. So $\lambda_1 = \sigma_1$ implies there is a boundary antenna with voltages v and -v.

■

Theorem 4.2. *If a layered network with at least four boundary vertices has an eigenvector with sequential (1,-1) and other entries zero then there exists a boundary antenna at the nodes corresponding to the (1,-1).*

Proof. The voltages of 1 and -1 on sequential vertices cause at least two sequential vertices with zero voltages on the boundary. The nodes connecting the boundary nodes with voltage 1 and -1 to the interior of the graph have voltage v and -v respectively, because the conductances are layered. Take the node with voltage v. There is a path from the node with voltage v to a node with voltage zero, and from there to a node with a negative voltage, w. By the Jordan Curve Theorem, there is a continuous path with negative voltage, denoted in blue, to the node with voltage -v. The same reasoning can be used to find a node in the graph with a positive voltage, x, which must have a continuous positive path, denoted in red, to the node with voltage v. Because our graphs are circular planar, these two paths must cross at some point, which would have both positive and negative voltage, which is a contradiction. Therefore, the nodes with voltage v and -v must be the same node.


1
v
-1
-v
Interior
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0
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■