The Story of Pseudodiagrams and Knot Games

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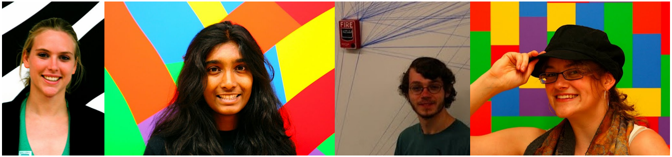
Seattle University

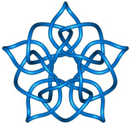
June 28, 2010

Joint work with Nöel MacNaughton, Sneha Narayan, Oliver Pechenik, Jennifer Townsend (SMALL 2009)



Outline

A **knot** is an embedding of the circle into Euclidean 3-space.



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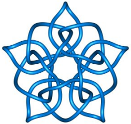

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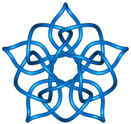
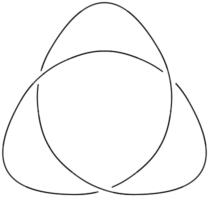


Equivalently, we can thing of a knot as a knotted piece of rope with its ends glued together.

Two knots are the same if we can deform one into the other by bending and stretching the rope without cutting it at any point.

Knots are often represented by diagrams in the plane.



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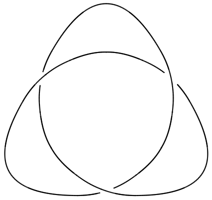
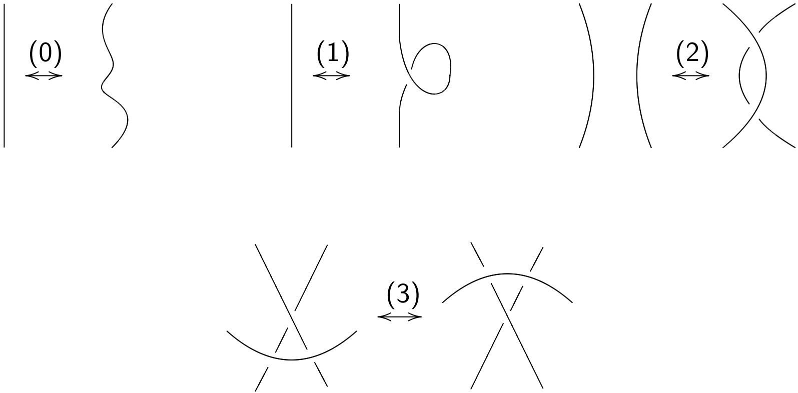


Two knot diagrams represent the same knots, i.e. are *equivalent*, if and only if they can be related by a sequence of diagrammatic moves.

Introduced by Kurt Reidemeister in the 1930s, these moves generate the equivalence relation on ordinary knot diagrams.



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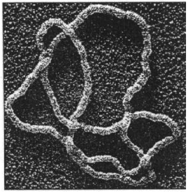

Problem: Crossings cannot always be determined in a picture of a DNA strand.

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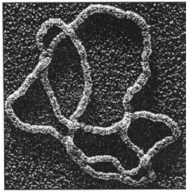
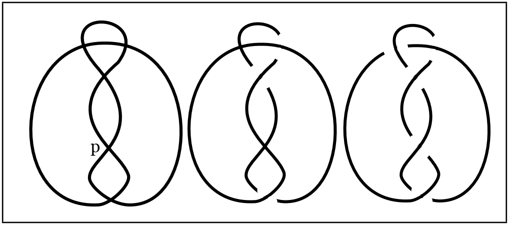


Problem: Crossings cannot always be determined in a picture of a DNA strand.

Solution: We need a better model!

A **pseudodiagram** is a diagram of a knot with some crossing information missing.



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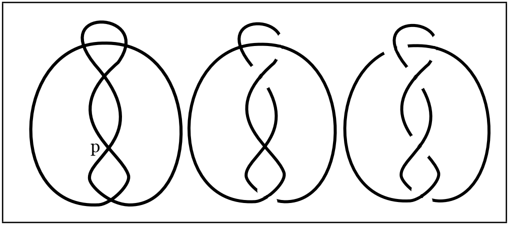

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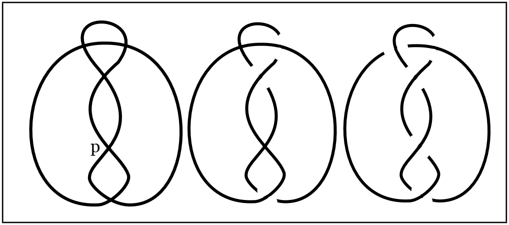

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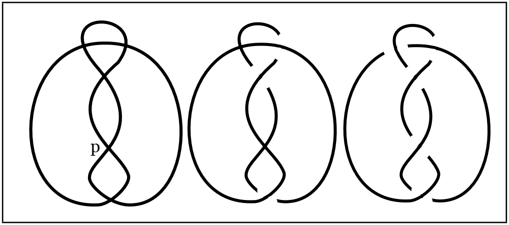

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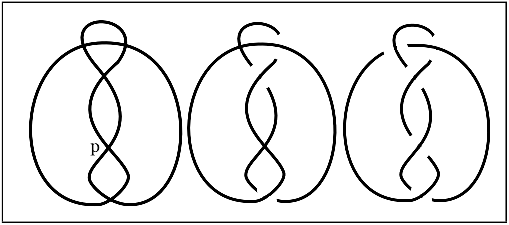

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Pseudodiagram theory was introduced recently by Ryo Hanaki.

BIG question for pseudodiagrams:

How much more crossing information do we need before we know if the pseudodiagram is knotted or unknotted?

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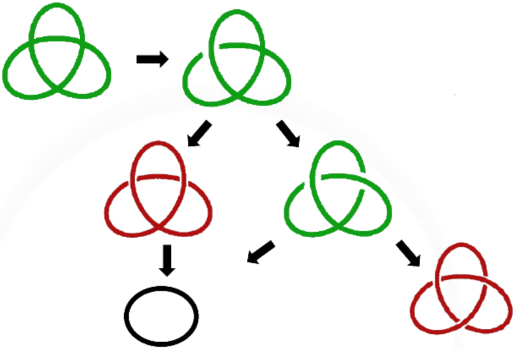
How much more crossing information do we need before we know if the pseudodiagram is knotted or unknotted?

The **trivializing number**, $tr(P)$, is the minimum number of crossings that need to be resolved to guarantee P is the unknot.

The **knotting number**, $kn(P)$, is the minimum number of crossings that need to be resolved to guarantee P is knotted.

We see with the following resolving tree that, if P is a trefoil shadow, then $tr(P) = 2$ and $kn(P) = 3$.



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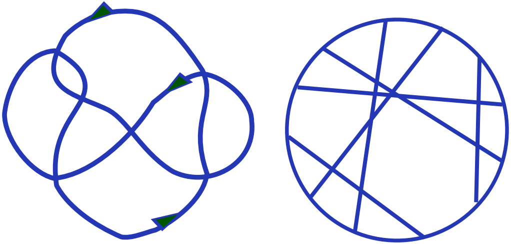
2. Is there a relationship between the **unknotting number** of a knot and $tr(S)$ for a shadow of that knot?

Before we proceed, we introduce one of our tools.

The following is an example of a shadow and corresponding **chord diagram**:



Note: If the chord diagram contains only parallel (i.e. non-intersecting) chords, the shadow is necessarily trivial.

Lemma (Hanaki)

For shadow S, $tr(S)$ is equal to the minimum number of chords that must be removed from the chord diagram to produce a diagram with non-intersecting chords.

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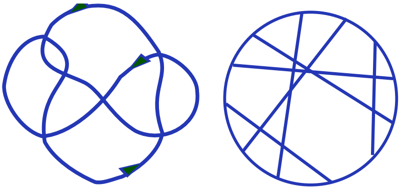
Example:



$$tr(S) = 4$$

To answer our first question, we start with a fact...

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Any chord in the chord diagram of a classical knot shadow intersects an even number of chords in the diagram.

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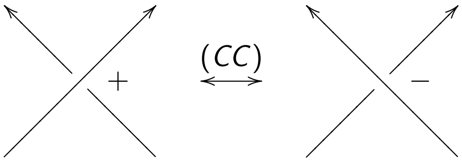
Now for our second question...

Proposition

Any knot can be unknotted if you are allowed to do the following move in addition to Reidemeister moves.

The Changing Crossing (CC) move:



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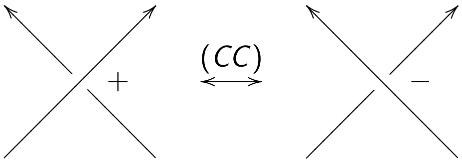
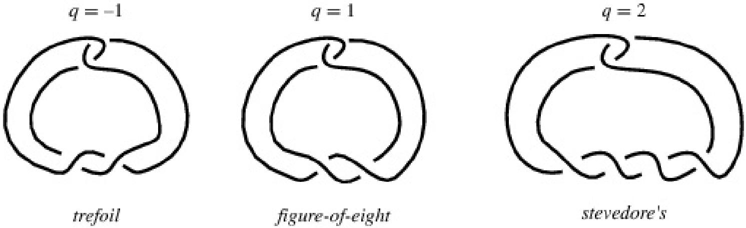


The **unknotting number**, $\mathbf{u}(K)$, of a knot K is the minimum number of Changing Crossing moves required in a sequence of Reidemeister and (CC) moves between K and the unknot.

The following is a family of knots called **twist knots** that all have unknotting number 1.



Theorem (JANOS)

For any knot K,

$$u(K) \leq \min_{S_K} \left[\frac{tr(S_K)}{2} \right]$$

where S_K denotes a shadow of a diagram D_K of K.

Theorem (JANOS)

For any knot K,

$$u(K) \leq \min_{S_K} \left\lceil \frac{tr(S_K)}{2} \right\rceil$$

where S_K denotes a shadow of a diagram D_K of K.

Idea of proof: Let S_K be a shadow of K corresponding to diagram D_K. Suppose $c_1, c_2, ..., c_n$ is a trivializing set of crossings in S_K. Note that, if resolving the crossings in S_K one way produces the unknot, then resolving them the opposite way does too. But one of these two resolutions of $c_1, c_2, ..., c_n$ agrees with D_K for at least $1/2$ of the crossings. Q.E.D.

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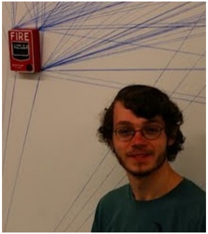
For instance, thinking about pseudodiagrams gave Oliver an idea...



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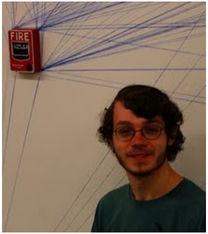


...why don't we **play games** on pseudodiagrams?

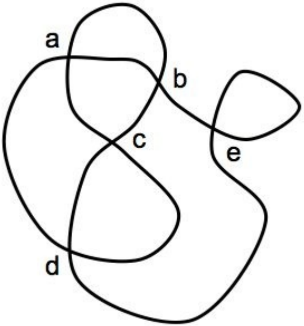
Game: To Knot or Not to Knot

Players:

Ursula
(**u**nknots)

King Lear
(**k**nots)

The starting shadow...



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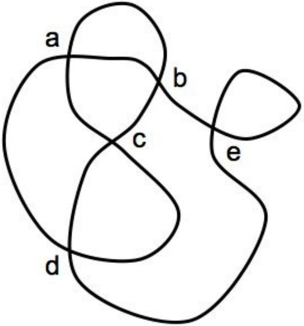

What if we start with a different shadow (e.g. a twist knot)?

Putting our game theorist hats on, we want to ask the following questions of To Knot or Not to Knot.

1. For which starting shadows does Player 1 have a **winning strategy**?

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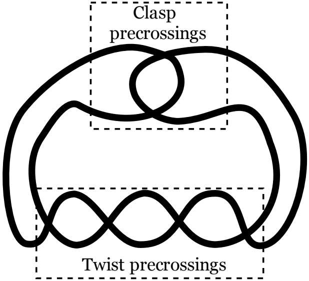
2. For which starting shadows does Player 2 have a winning strategy?

Theorem (JANOS)

Suppose To Knot or Not to Knot is played on a standard shadow of a twist knot with n twists. Then, King Lear has a winning strategy if and only if n is even and King Lear plays second.



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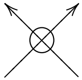
WHAT is a virtual knot???

Virtual knots can be defined in terms of knot diagrams that have an extra type of crossing:



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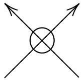
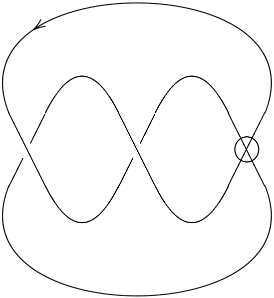
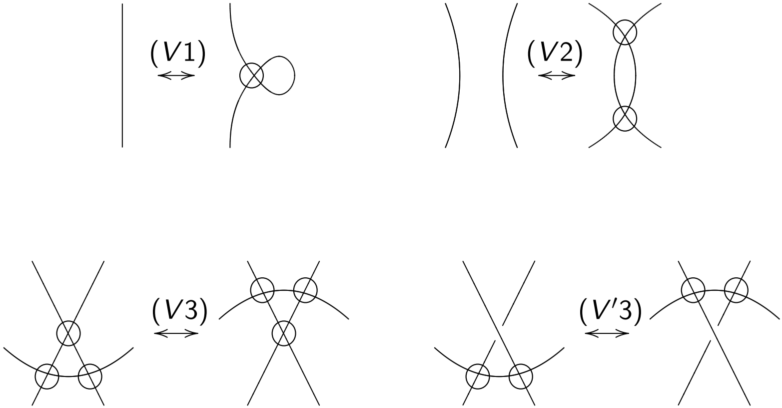


Using this definition, here is an example of a virtual knot:



Virtual knot diagrams are equivalent under the ordinary Reidemeister moves and the following *virtual* Reidemeister moves.



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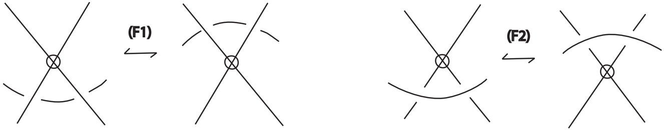

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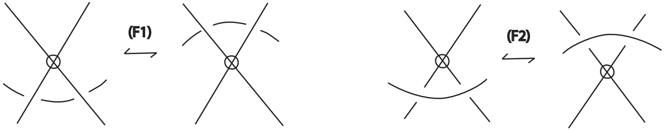

If we include one of these moves into our set of allowed Reidemeister moves, we get a definition for **welded knots**. If *both* forbidden moves are allowed, the resulting theory is trivial.

BIG question for virtual knots:

When is a virtual knot actually a classical knot?
In other words, what virtual knot diagrams
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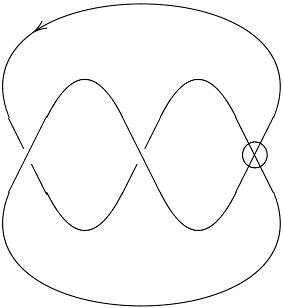

A classical knot

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This question is related to **games on virtual knots**.

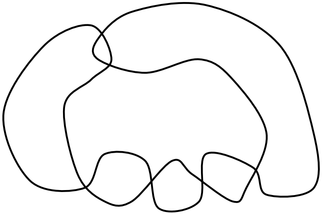
Game: A Virtual Battle

Players:

Eric Cartman
(**c**lassical)

Principal Victoria
(**v**irtual)

The starting shadow...



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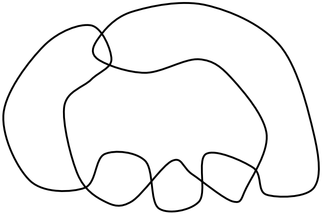

What if we have a starting shadow with a different number of twists?

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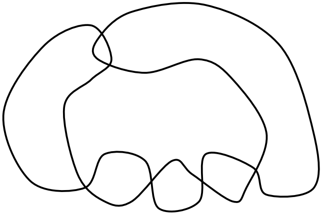

What if we have a starting shadow with a different number of twists? In the case of twist knots, when does one of the players have a winning strategy?

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I hope to work with students over the next few years to learn more about winning strategies for knot games and to extend our results about virtual pseudodiagrams.

THANK YOU!!!

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