We describe a method for describing the inverse of the star-K transformation, given a response matrix Λ and the geometry of the original graph. Using some of the intuition from this method, we then inductively show that n-to-1 graphs can be constructed for any n.

Given an $n \times n$ Kirchhoff matrix K, we define the star $s(K)$ to be an $n-1 \times n-1$ matrix defined by

$$s(K)_{ij} = K_{ij} - \frac{K_{in}K_{jn}}{K_{nn}}.$$

Note that we assume the Kirchhoff matrices have off-diagonal entries nonnegative for this paper. The star map is then just taking the Schur complement with respect to the very last diagonal entry. Of course, we could take the star of any other vertex if we chose to, but unless otherwise specified, the bottom right diagonal entry will be the preferred vertex. If our graph has a single interior vertex, then $s(K)$ is the response matrix. The final goal of understanding the star map s is to understand how to invert a general response matrix, in the following sense. Suppose we are given Λ and the geometry of the network N. We form a set E of edges in N, so $ij \in E$ iff ij is an edge of K. We would like to totally describe the set of Kirchhoff matricies that agree with the geometry E and have $\Lambda_m(K) = \Lambda$, where we have m interior vertices. $\Lambda_m(\cdot)$ is the function that takes the Schur complement with respect to the final $m \times m$ block. This set is $\Lambda_m^{-1}(\Lambda) \cap K(E)$, $K(E)$ denoting all Kirchhoff matricies that satisfy E, in the sense that $K_{ij} = 0$ for $i \neq j$ if and only if $ij \notin E$.

Now, one can show that the above map Λ_m satsifies $\Lambda_m \circ \Lambda_n = \Lambda_{m+n}$, and so in particular $\Lambda_m = s^m$. Thus, if we can fully describe the inverse images of s, we can describe the inverse images of Λ_m. Our first step invovles a process called "localization". Observe that if a network with a single interior node n has some boundary vertex i such that $in \notin E$, then the star map does not change any of the edges incident to i, as $K_{in} = 0$. Furthermore, edges between vertices that do have edges to n are certainly not dependent on the vertex i. Thus, we can effectively ignore any vertex that does not have an edge between it and n and all edges incident to those vertices. For a proof, note that in the situation just described, our Kirchhoff matrix has form

$$K = \begin{bmatrix} A_{11} & B & A_{12} \\ B^T & D & 0 \\ A_{12}^T & 0 & C \end{bmatrix}.$$

The first row corresponds to those boundary vertices incident to the interior vertices. The second row is those boundary vertices not incident to the interior vertices, and the final row corresponds to boundary vertices. Taking the Schur complement results in the matrix

$$K/C = \begin{bmatrix} A_{11} - A_{12}C^{-1}A_{12}^T & B \\ B^T & D \end{bmatrix}.$$

But the upper left hand block is just the Schur complement of

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}$$

with respect to A_{22}. Thus, if we can recover the K from its Schur complement, then by reading off the proper entries, we can recover A from its Schur complement, and conversely. Therefore, it suffices to consider the case where for every boundary vertex v, there is an edge from v to n.

Now, we define a certain class of matricies that correspond to the terms $\frac{K_{in}K_{jn}}{K_{nn}}$ in the formula for s and demonstrate that there is a correspondence between such matricies and Kirchhoff matricies satisfying E.

Definition 1. A square matrix R is said to be residual if:

1. R is symmetric.
2. $R_{ij} \geq 0$ for all i, j.
3. $R_{ii}R_{jj} = R_{ij}^2$ for all i, j

Given an edge set E with n edges, an $n-1 \times n-1$ residual matrix R is said to satisfy E with respect to the $n-1 \times n-1$ matrix Λ if

1. $R_{ij} = \Lambda_{ij}$ if $ij \notin E$.
2. $R_{ij} < \Lambda_{ij}$ if $ij \in E$.

Note that the first and third conditions of the definition for a residual matrix together imply that every 2×2 submatrix of K has determinant zero, as $R_{ij}R_{kl} = \sqrt{R_{ii}R_{jj}R_{kk}R_{ll}} = R_{il}R_{kj}$.

Lemma 2. *There is a 1-1 correspondence between residual matrices that satisfy E with respect to Λ and Kirchhoff matricies K that satisfy E such that $s(K) = \Lambda$.*

Proof. Given such a Kirchhoff matrix, set $R_{ij} = \frac{K_{in}K_{ij}}{-K_{nn}}$. This matrix is residual and satisfies E with respect to Λ. Conversely, given a residual matrix, set $K_{in} = \sum_j R_{ij}$ and define for $j \neq n$ $K_{ij} := \Lambda_{ij} - R_{ij}$. We demonstrate here that K is a Kirchhoff matrix with the desired properties. Symmetry is easy to see. If $ij \notin E$, then $K_{ij} = 0$, as desired. If $ij \in E$, then $K_{ij} > 0$. Thus K satisfies the geometry and has nonnegative off diagonal entries. Also, note

$\sum_j K_{ij} = \sum_{j=1}^{n-1}(\Lambda_{ij} - R_{ij}) + \sum_j R_{ij} = 0$, and so row sums are zero. Finally,

$$\begin{aligned} s(K)_{ij} &= K_{ij} + \frac{K_{in}K_{jn}}{K_{nn}} \\ &= \Lambda_{ij} - R_{ij} + \frac{\sum_k R_{ik} \sum_l R_{jl}}{\sum_{lk} R_{lk}} \\ &= \Lambda_{ij} - R_{ij} + \frac{\sum_{kl} R_{ik} R_{jl}}{\sum_{lk} R_{lk}} \\ &= \Lambda_{ij} - R_{ij} + R_{ij} \\ &= \Lambda_{ij} \end{aligned}$$

Note the second to last identity used the determinant and symmetry properties of R. Namely, we have that $R_{ij} \sum_{lk} R_{lk} = \sum_{lk} R_{ij} R_{lk} = \sum_{lk} R_{ik} R_{lj} = \sum_{lk} R_{ik} R_{jl}$. Also, note that the correspondences defined above are inverses of one another. □

We have now turned the problem into one that involves finding residual matricies. First, note that the entire residual matrix is determined by its diagonal entires, and the first condition for satisfying E places relations between these values. That is, if $ij \notin E$, then we can express R_{jj} in terms of R_{ii} and Λ_{ij}: $R_{jj} = \frac{\Lambda_{ij}}{R_{ii}}$. Our next step, then, is to construct a new graph that will keep track of these relations between the diagonal entries. It will enable us to find all possible residual matricies R that satisfy the first condition of satisfying E with respect to Λ. To make this new graph, delete the interior vertex and all edges incident to it. Let this graph be called $\hat{G}$. Now, form graph G' by taking the vertices of $\hat{G}$ and placing an edge between two vertices if and only if there is not an edge between those two vertices in $\hat{G}$.

Now, given the diagonal entries of R, we define a function s on the vertices of G' by $s(i) = R_{ii}$. If we assign the values Λ_{ij} to the edges ij, then we have provided a solution to the following problem:

Problem 1. *Given some (not necessarily connected) graph G' whose edges are assigned stricly positive values λ_{ij}, find a vertex function s such that $s(i)s(j) = \lambda_{ij}^2$ for every edge ij of G'.*

Conversely, if we construct G' as above and find a solution to the stated problem, then letting $R_{ii} = s(i)$ and setting $R_{ij} = \sqrt{R_{ii}R_{jj}}$ yields a residual matrix that satisfies the first condition of satisfying E with repsect to Λ. Thus, we study the stated problem in the following. Below, the graph G' will be referred to as H and the assigned edge values will be called r_{ij}. First, we show that we may assume H is connected by showing we may consider the problem on each connected component separately:

Lemma 3. *Let H be a graph with non-negative values assigned to each edge, as above. Suppose H has connected componenents $H_1, \ldots, H_m$. Then every solution to $s(v_i)s(v_j) = r_{ij}^2$ on H restricts to a solution on $H_1, \ldots, H_m$. Conversely,* *given solutions $s_1, \ldots, s_m$ to $H_1, \ldots, H_m$, we can define a solution s on H that restricts to those.*

Proof. The first statement is obvious. For the second one, we define $s(v_i) = s_j(v_i)$ for $v_i \in H_j$, and make it zero elsewhere. If v_i, v_j are two given vertices, connected by an edge ij, then the both must lie in the same component H_k and so $s(i)s(j) = s_k(i)s_k(j) = r_{ij}^2$. □

So we assume H is connected in the following.

Lemma 4. *Suppose a solution s to the problem exists. Then if there are any odd cycles, it is unique. Otherwise, there is a one parameter family of solutions.*

Proof. Suppose H has an odd cycle and a solution s. We will write $s(i)$ as s_i below. Let $v_1, v_2, \ldots, v_{2m+1}, v_1$ be an odd cycle. Then we have

$$\begin{aligned}
s_2 &= \frac{r_{12}^2}{s_1} \\
s_3 &= \frac{r_{23}^2}{s_2} = \frac{s_1 r_{12}^2}{r_{23}^2} \\
s_4 &= \frac{r_{34}^2 r_{12}^2}{r_{23}^2 s_1} \\
s_5 &= \frac{r_{45}^2 r_{23}^2 s_1}{r_{34}^2 r_{12}^2} \\
&\cdots \\
s_{2n+1} &= \frac{r_{2n,2n+1}^2 r_{2n-2,2n-1}^2 \cdots r_{23}^2 s_1}{r_{2n-1,2n}^2 r_{2n-3,2n-2}^2 \cdots r_{12}^2} \\
s_1 &= \frac{r_{2n+1,1}^2 r_{2n-1,2n}^2 \cdots r_{12}^2}{r_{2n,2n+1}^2 \cdots r_{23}^2 s_1} \\
\Rightarrow s_1 &= \frac{r_{2n+1,1} r_{2n-1,2n} \cdots r_{12}}{r_{2n,2n+1} \cdots r_{23}}
\end{aligned}$$

and so every solution must have that value for s_1. But since H is connected and has nonzero edges, all other values of s can be found from that one, and so there is a unique solution.

Now, suppose that H has no odd cycles. Choose some particular vertex, say v_1. Then we can partition the rest of the graph's vertices into what we will call even and odd sets. A vertex v_i is called odd if every path $v_1, v_{i_1}, v_{i_2}, \ldots, v_i$ from v_1 to v_i has an odd number of jumps, and a vertex is called even similarly. Note that our condition on having no odd cycles means that no vertex can be in both groups, as if there was a path of odd length and a path of even length, reversing one and concatenating results in a cycle of odd length from vertex v_1. By convention, we say that v_1 is even. If there is a solution s to the problem, then we define another solution s_t as follows: if v_i is an odd vertex, then set

$s_t(v_i) = \frac{s(v_i)}{t}$. If v_i is even, then set $s_t(v_i) = ts(v_i)$. We easily verify that this is a solution, as if there is an edge from v_i to v_j, then one is odd and one is even, so $s_t(v_i)s_t(v_j) = s(v_i)s(v_j) = r_{ij}^2$ as the t terms cancel. We would now like to show that every solution can be written as s_t for some t. Let s, s' be solutions solution. Now $s(v_1)$ cannot be zero, as this would imply the edges from v_1 have value zero. Set $t = \frac{s'(v_1)}{s(v_1)}$. Then the solutions s_t, s' agree at v_1 and so they must agree everywhere by connectedness. □

Now that the above is dealt with, we turn our attention to figuring out exactly when a connected graph H has a solution. We fix a base vertex v_1 in H. Let r_{ij} be the value on the edge ij. We take a path $p = (v_1, v_2, \ldots v_k)$ of vertices, allowing any repetitions as needed. We define the value of the path $V(p)$ to be

$$V(p) = \frac{r_{k-1,k}^2 r_{k-3,k-2}^2, \cdots r_{2,3}^2}{r_{k-2,k-1}^2 r_{k-3,k-4}^2 \cdots r_{1,2}^2} \text{ for } k \text{ odd}$$
$$V(p) = \frac{r_{k-1,k}^2 r_{k-3,k-2}^2, \cdots r_{1,2}^2}{r_{k-2,k-1}^2 r_{k-3,k-4}^2 \cdots r_{2,3}^2} \text{ for } k \text{ even}$$

Note that we simply walk backwards along the path, alternating placing the term for that edge in the numerator or denominator. That is, $r_{k-1,k}^2$ goes in the numerator, $r_{k-2,k-1}^2$ in the denominator, and on until the list is exhausted. Also, note that if we take a path p and extend it by a single edge ij to form a path q, then $V(q) = \frac{r_{ij}^2}{V(p)}$. Also, by induction we can show that if we take a path p followed by a path q, then $V(qp) = \frac{V(q)}{V(p)}$ if q has an odd length, while $V(qp) = V(q)V(p)$ if q has an even length. Furthermore, if we take $\overline{p}$, the reversal of p, then $V(\overline{p}) = V(p)$ if p is odd, and $V(\overline{p}) = V(p)^{-1}$ if p is even. We use this below to find solutions.

Lemma 5. *There is a solution s to $s(v_i)s(v_j) = r_{ij}^2$ for each edge ij in H if and only if there is a constant C that depends on v_1 so that every odd cycle c based at v_1 has $V(c) = C$ and every even cycle has $V(c) = 1$.*

Proof. (⇐) Suppose there is no odd cycle in H. Set $s(v_1) = 1$ and then we define $s(v_i) = V(p)$ for some path p from v_1 to v_i. This is uniquely defined, since all paths give the same value. This is because if p, q are paths from v_1 to v_i, then they must both be odd or both be even, else there would be an odd cycle. Then $\overline{q}p$ is an even cycle based at v_1. We then have $V(\overline{q}p) = 1$. But $V(\overline{q}p) = \frac{V(q)}{V(p)}$ or $\frac{V(p)}{V(q)}$, depending on which has which parity, but in either case $V(p) = V(q)$. Now, if e_{ij} is an edge from i to j, then we extend the path p from v_1 to v_i by adding the edge e_{ij}, forming the path q from v_1 to v_j. Then $s(v_j) = V(q)$, and $s(v_i)s(v_j) = V(p)V(q) = r_{ij}^2$.

If there is an odd cycle, then let c be such a cycle, say from vertex v_1 to itself. Let C be the common value for all odd cycles from v_1. We define $s(v_1) = \sqrt{C}$. If

p is a path from v_1 to v_k, then let pc denote the concatenation of c and p, taking the cycle c first. Define $s(v_k) = \frac{V(p)}{\sqrt{C}}$ for an odd path p from v_1 to v_k. (This is consistent with our definition of $s(v_1)$.) Note that there must always be such an odd path, as by connectedness we know there is at least one path, and if it is even, concatenating with c gives an odd path. We must show that this definition does not depend on which odd path we used. Suppose that p, q are odd paths from v_1 to v_k. Then $\overline{q}p$ is an even cycle based at x_1, and so $1 = V(\overline{q}p) = \frac{V(q)}{V(p)}$. We now must show the s defined this way provides a solution. Let p be an odd path from v_1 to v_i, so $s(v_i) = V(p)$. Suppose we have edge e_{ij} from v_i to v_j. We form an odd path q from v_i to v_j by taking $e_{ij}pc$, the concatenation of c, then p, and finally the edge e_{ij}. Then $s(v_j) = \frac{V(e_{ij}pc)}{\sqrt{C}} = \frac{r_{ij}^2}{V(pc)\sqrt{C}} = \frac{r_{ij}^2\sqrt{C}}{V(p)} = \frac{r_{ij}^2}{s(v_i)}$. Thus $s(v_i)s(v_j) = r_{ij}^2$.

($\Rightarrow$) The above shows that path agreement gives a solution. Now, we suppose there is a solution s. Let $s(v_1) = t$. Then by induction, we see that if $p = (v_1, \ldots, v_k)$ is a path from v_1 to v_k, then $V(p) = s(v_k)s(v_1)$ if the path has an odd length and $V(p) = \frac{s(v_k)}{s(v_1)}$ if the path has an even length. The statement immediately follows from letting $v_k = v_1$. $\square$

Now, the cycles above have arbitrary length, and so are not suitable for use in any sort of algorithm. We would like to demonstrate that the properties are only needed on cycles of a bounded length to provide a solution.

Lemma 6. *Let H be given and have diameter D. Choose a vertex v_0. Then if all odd cycles of length less than $2D+2$ based at v_0 have the same value and all even cycles of length less than $2D+2$ based at v_0 have value 1, there is a solution.*

Proof. Define $s(v_0)$ to be 1 if there are no odd cycles of length less than $2D+2$, else set $s(v_1) = \sqrt{C}$, where C is the value of the odd cycles based at v_0 of length less than $2D+2$. Set $t := s(v_1)$. Let v_i be any vertex and let p be a shortest path from v_0 to v_i. Set $s(v_i) = tV(p)$ if p is even, $s(v_i) = \frac{V(p)}{t}$ if p is odd. Note that this does not depend on the choice of the shortest path, as any two such paths will both be even or odd (else one is longer than the other!), and so the values will agree as the cycle they form will have length less than $2D+2$.

Now consider v_i, v_j joined by an edge e_{ij}. Let p be a shortest path from v_0 to v_i, q be a shortest path from v_0 to v_j. Then $e_{ij}p$ is a path from v_0 to v_j and $c = \overline{q}e_{ij}p$ is a cycle of length less than $2D+2$. Our first case is that p and q both have odd length. Then the cycle c has odd length, and we have $C = V(c) = \frac{V(q)V(p)}{r_{ij}^2} \Rightarrow r_{ij}^2 = \frac{V(q)}{\sqrt{C}}\frac{V(p)}{\sqrt{C}} = s(v_i)s(v_j)$. If p and q both have even length, then c is still odd and so $C = V(c) = V(\overline{q})V(e_{ij}p) = \frac{r_{ij}^2}{V(q)V(p)} \Rightarrow e_{ij}^2 = s(v_i)s(v_j)$. Now suppose that p has odd length and q has even length. Then c has even length.

Then $1 = V(c) = V(\overline{q})V(e_{ij}p) = \frac{r_{ij}^2}{V(q)V(p)} \Rightarrow s(v_i)s(v_j) = V(q)V(p) = r_{ij}^2$. The case that q has odd length and p has even length is handled similarly. $\square$

Let $S(G, \Lambda)$ be the set of all solutions to the above problem on the graph $\hat{G}$ with edge values taken from Λ.

Theorem 7. *The set of residual matricies R satisfying E with respect to Λ are in correspondence with solutions $s \in S(G, \Lambda)$ such that $\sqrt{s(i)s(j)} < \Lambda_{ij}$ for $ij \notin E$.*

Proof. If R is such a residual matrix, we have already shown that setting $s(i) = r_{ii}$ gives an element of $S(G, \Lambda)$ and this s does indeed satisfy $\sqrt{s(i)s(j)} < \Lambda_{ij}$ for $ij \in E$. Conversely, given $s \in S(G, \Lambda)$ with $\sqrt{s(i)s(j)} < \Lambda_{ij}$, we set $R_{ii} = s(i)$ and form the residual matrix by taking $R_{ij} = \sqrt{R_{ii}R_{jj}}$. By construction of $\hat{G}$, we have $R_{ij} = \sqrt{s(i)s(j)} = \Lambda_{ij}$ for $ij \notin E$. By assumption, $R_{ij} < \Lambda_{ij}$ for $ij \in E$, and so R is a residual matrix satisfying E with respect to Λ. $\square$

With all this done, we can describe what the algorithm must essentially do:

1. Localize the problem to reduce the complexity.
2. Using the localized graph G, construct G' and find all components.
3. If $H_1, \ldots, H_n$ are the components, then choose a base vertex v_i in each and write out equalities that the edges must satisfy so that cycles of length less than $2D(H_i) + 2$ based at v_i have the proper values for a solution to exist.
4. Write out the residual matrix in terms of the possible parameters introduced for each component.
5. Write out a set of inequalities that must be true for R to satisfy the second condition of satisfying E with respect to Λ.
6. "Delocalize". That is, simply replace all information that may have been removed in localization.

All of the above theorems and lemmas taken together demonstrate that every Kirchhoff matrix that satisfies E and maps to Λ under s will be described by the constraints output by this algorithm. This will essentially describe the space of all possible solutions. We could also ask the algorithm to check a particular solution by simply ensuring that the inequalities and equlities hold.

We now describe the method for constructing an n-to-1 graph. The process will rely upon a single important graph, which I have named "the kite". The kite has four boundary and one interior vertex. Label the boundary vertices $1, 2, 3, 4$ and let 5 denote the interior vertex. Draw edge $i, 5$ for each i, and also add edges between $1, 2$ and $2, 4$. Then, given some Λ as a response matrix, we find that we obtain a unique solution if and only if

$$\lambda_{14}\lambda_{23} < \lambda_{13}\lambda_{24}$$
$$\lambda_{14}\lambda_{23} < \lambda_{12}\lambda_{34}$$

The residual matrix is:

$$R = \begin{bmatrix} \frac{\lambda_{14}\lambda_{13}}{\lambda_{34}} & \frac{\lambda_{14}\lambda_{23}}{\lambda_{34}} & \lambda_{13} & \lambda_{14} \\ \frac{\lambda_{14}\lambda_{23}}{\lambda_{34}} & \frac{\lambda_{23}^2\lambda_{14}}{\lambda_{34}\lambda_{13}} & \lambda_{23} & \frac{\lambda_{23}\lambda_{14}}{\lambda_{13}} \\ \lambda_{13} & \lambda_{23} & \frac{\lambda_{13}\lambda_{34}}{\lambda_{14}} & \lambda_{34} \\ \lambda_{14} & \frac{\lambda_{23}\lambda_{14}}{\lambda_{13}} & \lambda_{34} & \frac{\lambda_{14}\lambda_{34}}{\lambda_{13}} \end{bmatrix}$$

The idea is that the kite allows us to "test" some of the values of the response matrix, failing if the inequalities are not met.

Theorem 8. *There exists a network N and response matrix Λ for N such that there are exactly n Kirchhoff matricies K that could have produced Λ that satisfy the geometry of N.*

Proof. We actually wish to prove something slightly stronger. Namely, we want some boundary to bounday edge of the graph to take on a different value for each of the n possible Kirchhoff matricies. Our base case is trivial. Suppose we have a network N that has n different Kirchhoff matricies corresponding to response matrix Λ_1. Let e be the edge, and let $r_1, \ldots, r_n$ be the resistances it takes for K_i. Suppose $r_1 < r_2 < \ldots r_n$ WLOG. We now choose another network, D with reponse matrix Λ_2 such that there are exactly two Kirchhoff matricies producing Λ_2 and an edge f that takes resistances w_1, w_2 such that $\frac{r_i}{r_j} \neq \frac{w_1}{w_2}$. That such a network can be constructed is shown elsewhere. Note that this condition shows that the $2n$ values $w_i r_j$ are all distinct. WLOG, we will assume that $r_1 w_1 < r_2 w_1 < r_3 w_1 < \ldots r_n w_1 < r_1 w_2 < \ldots < r_n w_2$. Now, choose c so that $r_1 w_2 < c < r_2 w_2$. There are then $n+1$ different $r_i w_j$ less than c, the others being greater.

Now, recall the kite. If we took the star-K transformation of it, we would end up with a complete graph on four vertices. The idea is to "plug" the edge e into the edge 14 of the complete graph and edge f into edge 23. Thus, we have n choices for λ_{14} and two four λ_{23}. If we then chose $\lambda_{13}\lambda_{24} = c$ and $\lambda_{12}\lambda_{34}$ large, then it would follow that $\lambda_{14}\lambda_{23} < \lambda_{13}\lambda_{24}$ would hold if and only if $r_i w_j < c$, which happens for exactly $n+1$ choices. Choosing $\lambda_{12}\lambda_{34}$ large ensures the other inequality always holds.

Rigorously, we form G as follows. Delete the edges e and f from their respective graphs. Identify the endpoints of e and f with vertices $2, 3$ and $1, 4$ of the kite, respectively. Let i_N denote the number of interior vertices of N and i_D the number of interior vertices of D. Then G has $i_N + i_D + 1$ interior nodes. Perform the first star-K transformation by removing the interior node belonging neither to N nor D. Doing this results in a graph formed by taking N, D separately and then connecting the endpoints of e and f. Call this graph $\hat{G}$. Let's consider this graph in its own right. Order the vertices of $\hat{G}$ with the boundary of N first, then the boundary of D, then the interior of D, and finally the interior of N. We write a Kirchhoff matrix for $\hat{G}$ as

$$K = \begin{bmatrix} K_{11} & K_{12} & 0 & K_{14} \\ K_{12}^T & K_{22} & K_{23} & 0 \\ 0 & K_{23}^T & K_{33} & 0 \\ K_{14}^T & 0 & 0 & K_{44} \end{bmatrix}$$

Taking the Schur complement with respect to the bottom corner (which is just taking the star-K transformation by removing the interior vertices of N first results in the matrix

$$K' = \begin{bmatrix} Sc(N) & K_{12} & 0 \\ K_{12}^T & K_{22} & K_{23} \\ 0 & K_{23}^T & K_{33} \end{bmatrix}$$

Where $S(N) = K_{11} - K_{14}^T K_{44} K_{14}$ is the matrix that would be found by taking the star-K transformation of N by itself, as the matricies here only use edges between vertices of N. Defining $S(D) = K_{22} - K_{23}^T K_{33} K_{23}$, we take the Schur completement of K' with respect to K_{33} to get

$$K'' = \begin{bmatrix} Sc(N) & K_{12} \\ K_{12}^T & Sc(D) \end{bmatrix}$$

This computation means that if we are given the response matrix

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_{22} \end{bmatrix},$$

then we can find all possible K that will map to that via repeated star-K by simply finding all possible Kirchhoff matrices of N, D that result in Λ_{11} and Λ_{22} respectively. This means that if we used $\Lambda_{11} = \Lambda_1$ and $\Lambda_{22} = \Lambda_2$ from above, we would get $2n$ possible Kirchhoff matrices, whose values are taken from the matricies K_i and the two possible Kirchhoff matrices for D.

But now, when we attempt to invert the first star-K transformation for each of these $2n$ Kirchhoff matrices, by localization we need only to look at e, f and the edges connecting their endpoints. The values for the edges that are not e, f are given by Λ_{12} above, while $e = r_i, f = w_j$. Chosing the entries of Λ_{12} appropriately, as described above, we can force exactly $n-1$ of the $2n$ choices of Kirchhoff matrices to not have any inverse and the rest to have a unique inverse, therefore producing an $n+1$ to one graph. Note that one of the boundary to boundary edges takes on $n+1$ distinct values, as can be seen from the computation of the residual for the kite, above.

□