Who wins in *To Knot or Not to Knot* played on Sums of Rational Shadows

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1 Introduction

The game *To Knot or Not to Knot* was introduced in a recent paper, *A Midsummer Knot's Dream* by Henrich et al. This game is played on a knot pseudodiagram, that is, a knot diagram with some crossings unresolved. The two players, King Lear and Ursula, alternatively resolve crossings, until all crossings are resolved. **K**ing Lear wins if the diagram is **k**notted, and **U**rsula wins if the diagram is equivalent to the **u**nknot. Henrich et al considered the game on shadows of twist knots, determining the winning player in this case. They also considered some rational knots, but as far as I know, nobody has determined who wins on all rational knot shadows. (A shadow is a pseudodiagram in which all crossings are unresolved.)

In another paper, I developed a convoluted theory for analyzing the game on sums of knot pseudodiagrams, where the connected sum of two knot pseudodiagrams is defined in an obvious way. The good thing about these sums is that we can tell which player won from knowledge of who won on each component. Unfortunately, knowing which player has a winning strategy is not as easy to predict from the sums. I showed that there was a certain 38-element monoid V, such that each pseudodiagram could be assigned a value in V, the player with the winning strategy could be determined from this value, and the value of a sum of two diagrams is the sum of the values of each diagram individually.

In this paper I apply this to shadows of rational knots, for which we have simple algorithms for determining whether the final diagram is knotted or not. I determine who wins in a given combination of rational knot shadows. The more general problem of determining who wins in a combination of rational pseudodiagrams seems much harder.

2 Rational Tangles and Knots

Rational tangels were probably invented by John Conway. They are exactly the tangles (with 4 loose ends) built up recursively from the tangle with two parallel strands that don't cross, by adding twists on the four sides. It turns out that we could equivalently only add twists on the right and the bottom. In this paper, I'll use the notation $[a_1, a_2, \ldots, a_n]$ to denote the tangle obtained by doing a_1 horizontal/vertical twists, a_2 vertical/horizontal twists, and so on, ending with a_n horizontal twists. (These numbers may be zero or negative, to indicate a negative twist or no twists at all.) I'll also abuse this notation to denote the knot or link obtained by joining the two strands on top and the two strands on bottom. The vast majority of the time I am more interested in knots than in tangles.

The following results are well known or easy to show:

Theorem 1. *Two tangles $[a_1, a_2, \ldots a_n]$ and $[b_1, b_2, \ldots b_n]$ are equivalent iff the continued fractions are equal:*

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \frac{1}{a_n}}}} = b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{\ddots + \frac{1}{b_n}}}}$$

Theorem 2. *The knot or link* $[a_1, a_2, \ldots a_n]$ *is a knot (as opposed to a link) iff*

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \frac{1}{a_n}}}} = \frac{p}{q}$$

with p odd.

Theorem 3. *The knot or link* $[a_1, a_2, \ldots a_n]$ *is equivalent to the unknot iff*

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \frac{1}{a_n}}}} = \frac{1}{q}$$

for some q.

Note that q can be 0 in the previous theorem!

I extend this notation to rational pseudodiagrams and shadows as follows: $[a_1(b_1), a_2(b_2), \ldots, a_n(b_n)]$ denotes a pseudodiagram in which there are $b_i \geq 0$ unresolved crossings, and resolved crossings totalling to a_i at the ith stage. If $a_i = 0$, we simply write $\ldots, (b_i), \ldots$, and if $b_i = 0$, we simply write $\ldots, a_i, \ldots$. So for example, the shadow of the trefoil can be written as $[(3)]$, while the shadow of the figure eight knot can be written as $[(2), (2)]$.

Lemma 1. *The following pairs of shadows are equivalent (up to planar isotopy).*

$$[(1), (a_1), \ldots, (a_n)] = [(a_1 + 1), \ldots, (a_n)]$$
$$[(a_1), \ldots, (a_n), (1)] = [(a_1), \ldots, (a_n + 1)]$$
$$[(0), (0), (a_1), \ldots, (a_n)] = [(a_1), \ldots, (a_n)]$$
$$[(a_1), \ldots, (a_i), (0), (a_{i+1}), \ldots, (a_n)] = [(a_1), \ldots, (a_i + a_{i+1}), \ldots, (a_n)]$$
$$[(a_1), \ldots, (a_n), 0, 0] = [(a_1), \ldots, (a_n)]$$
$$[(a_1), (a_2), \ldots, (a_n)] = [(a_n), \ldots, (a_2), (a_1)]$$

Proof. These are all clear from drawing pictures. Only the last one is nonobvious, since the isotopy involves turning everything inside out. (I guess this isn't really a planar isotopy, but its an isotopy if everything was embedded on a sphere instead of a plane). □

Definition 1. *A shadow* S *is obtained by a* phony Reidemeister 1 move *from a shadow* T *if* S *is obtained by removing a loop (with a precrossing) from* T*. We denote this* $T \to_1 S$.

Definition 2. *A shadow* S *is obtained by a* phony Reidemeister 2 move *from a shadow* T *if* S *is obtained by uncrossing two overlapping strands in* T*, just like a normal Reidemeister 2 move, except we don't care about which strands are on top. We denote this* $T \to_2 S$.

We also use the notation $T \to_1^* S$, $T \to_2^* S$, and $T \to_{1,2}^* S$ to denote that S is obtained from T by a sequence of applying the respective phony Reidemeister moves. $T \to_{1,2}^* S$ denotes a mix of both moves.

Lemma 2.

$$[(0), (a_1 + 1), (a_2), \ldots, (a_n)] \to_1 [(0), (a_1), (a_2), \ldots, (a_n)]$$
$$[(a_1), \ldots, (a_{n-1}), (a_n + 1), 0] \to_1 [(a_1), \ldots, (a_{n-1}), (a_n), 0]$$
$$[\ldots, (a_i + 2), \ldots] \to_2 [\ldots, (a_i), \ldots]$$

Proof. Again, these are all clear from drawing pictures. □

Lemma 3. *If T is a rational shadow, that resolves to be a knot (not a link), then $T \to^*_{1,2} U$, where U is the unknot.*

Proof. Let $T = [(a_1), \ldots, (a_n)]$ be a minimal counterexample. Then T cannot be reduced by any of the rules specified above. Since any $a_i \geq 2$ can be reduced by a phony Reidemeister 2 move, all $a_i < 2$. If $n = 0$, then $T = []$ which turns out to be the unknot. If $a_0 = 0$ and $n > 1$, then either a_1 can be decreased by 1, or a_0 and a_1 can be stripped off. On the other hand, if $n = 1$, then $T = [(0)]$, which is easily seen to be a link. So $a_0 = 1$. If $n > 1$, then T reduces to $[(a_2 + 1), \ldots, (a_n)]$. So $n = 1$, and T is $[(1)]$ which clearly reduces to the unknot via a phony Reidemeister 1 move. □

Later, we will see that this implies that no rational knot shadow can be a win for the knotter playing both first and second. That is, the unknotter always has a winning strategy, either moving first or moving second.

Definition 3. *A pseudodiagram T is* odd *or* even *if it has an odd or even number of precrossings. An* odd projection *of T is either T (if T is odd), or a T' with $T' \to_1 T$ if T is even. An* even projection *of T is either T (if T is even), or a T' with $T' \to_1 T$ if T is odd.*

The odd projection of a pseudodiagram is always odd, and the even projection is always even. From the point of view of *To Knot or Not to Knot*, all odd (even) projections of a pseudodiagram are equivalent. In fact, to use the notation of my previous paper, if $T' \to_1 T$, then the game associated with T' is equal to $*$ plus the game associated with T.

Lemma 4. *If T' and T are pseudodiagrams, with $T' \to T$, and T is a win for some player P moving second, then T' is a move for the same player moving first.*

Proof. Player P moves in T' to the loop that is missing in T, yielding a pseudodiagram equivalent to T. She then follows her strategy for T, as the second player. □

It follows that if T is any pseudodiagram, then the odd projection of T cannot be a win for P playing second if the even projection of T is a win for P's opponent playing first. Using this, we divide up pseudodiagrams into classes as follows:

- $T \in X_0$ if the even projection of T is U2 and the odd projection is U1
- $T \in X_1$ if the even projection of T is K1 and the odd projection is U1
- $T \in X_2$ if the even projection of T is K1 and the odd projection is K2
- $T \in Y_0$ if the odd projection of T is U2 and the even projection is U1
- $T \in Y_1$ if the odd projection of T is K1 and the even projection is U1
- $T \in Y_2$ if the odd projection of T is K1 and the even projection is K2

Here, K2 is short for "King Lear wins playing second," U1 is short for "Ursula wins playing first," and so on. Each pseudodiagram T belongs to exactly one of the X_i and one of the Y_i. Let $X(T) = i$ iff $T \in X_i$ and $Y(T) = i$ iff $T \in Y_i$. For reasons explained in my previous paper, T belongs to the same classes as its odd and even projections. (All this amounts to is the fact that if $T'' \to_1 T' \to_1 T$, then T'' and T are completely equivalent from a game-theoretic point of view.)

Lemma 5. *Let T and S be even pseudodiagrams, with $T \to_2 S$. Then if some player P wins moving second in S, she also wins moving second in T.*

Proof. Player P uses her strategy from S to play in T, responding to a move in the two new crossings by playing the countering move. She is never forced to be the first to play in the two new crossings herself, since S is even. □

Analogously we have the following:

Lemma 6. *If T and S are odd pseudodiagrams, $T \to_2 S$, and some player P wins moving first in Y, then he also wins moving first in X.*

The point of all this is the following:

Theorem 4. *If $T \to_{1,2}^{*} S$, then $X(T) \leq X(S)$ and $Y(T) \geq Y(S)$.*

Proof. We only need to show this in the case that $T \to_1 S$ or $T \to_2 S$. The first case follows from the fact that a game is in the same classes as its even and odd projections. That is, if $T \to_1 S$, then $X(T) = X(S)$ and $Y(T) = Y(S)$. So we only need to consider when $T \to_2 S$.

Suppose that $X(S) \leq 0$. Then an even projection of S is U2. We can take T' and S' to be even projections of T and S, such that $T' \to_2 S'$. Then by Lemma 5, T' is also U2. So T is in X_0, and $X(T) = 0 \leq X(S)$.

Next, suppose that $X(S) = 1$. Then an odd projection of S is U1. We can take T' and S' to be odd projections of T and S, such that $T' \to_2 S'$. Then by Lemma 6, T' is also U1. So T is in $X_0 \cup X_1$, and $X(T) \leq 1 = X(S)$.

Otherwise, $X(S) = 2$, and there is nothing to show. The fact that $Y(T) \geq Y(S)$ can be proven analogously, switching the roles of the two players. □

Corollary 1. *No rational knot shadow is a win for the knotter, i.e., is both K1 and K2. In fact, every rational knot shadow is in X_0. Every even rational knot shadow is a win for the unknotter if she plays second, and every odd rational knot shadow is a win for the unknotter if she plays first.*

Proof. If T is a rational knot shadow, then T is reducible by phony Reidemeister 1 and 2 moves to the unknot, which is clearly in X_0. So $T \in X_0$. If T is even, then T is its even projection, which is U2. If T is odd, then T is its odd projection, which is U1. □

On the other hand, it is easy to come up with rational pseudodiagrams which are wins for the knotter – for example, simply take a game that is already over! Unfortunately, the unknot is in Y_0, not Y_2, so it will take more work to classify the rational knots.

3 Odd-Even Shadows

Definition 4. *An odd-even shadow is a rational shadow of the form $[(a_1), (a_2), \ldots, (a_n)]$, where all $a_i \geq 1$, exactly one of a_1 and a_n is odd, and all other a_i are even.*

Note that these are all odd. It is straightforward to verify from the rules given above that every odd-even shadow reduces by phony Reidemeister moves to $[(1)]$, and so every odd-even shadow actually corresponds to a knot, not a link.

Theorem 5. *Every odd-even shadow T is a win for the unknotter, playing first or second.*

Proof. Since $T = [(a_1), (a_2), \ldots, (a_n)]$ is odd, by Corollary 1 it is a win for the unknotter playing first. So we only need to show that it is a win for the unknotter if she plays second. Suppose we have a minimal counterexample (n is as small as possible). Without loss of generality, a_1 is odd and all other a_i are even. If a_1 is 1, then T is the same as $[(a_2 + 1), \ldots,]$, a smaller odd-even shadow. So a_1 is at least 3. Since none of the a_i are zero, and the others are all even, all $a_i \geq 2$. Now suppose the knotter makes a move in the component (a_i). Then the unknotter can reply with a canceling move, effectively decreasing a_i by 2. If the new value of a_i is nonzero, then we still have an odd-even shadow, which the unknotter can now win in, by choice of T. Otherwise, a_i was decreased by 2 to yield 0, so $i > 1$, and the new shadow T' is

$$T' = [(a_1), \ldots, (0), \ldots, (a_n)] = [(a_1), \ldots, (a_{i-1} + a_{i+1}), \ldots, (a_n)],$$

another odd-even shadow, unless $i = n$. In this case,

$$T' = [(a_1), \ldots, (a_{n-2}), (a_{n-1}), (0)] \to_1 [(a_1), \ldots, (a_{n-2}), (a_{n-1} - 1), (0)] \to_1 \ldots$$
$$\to_1 [(a_1), \ldots, (a_{n-2}), 0, 0] = [(a_1), \ldots, (a_{n-2})] .$$

So $T' \to_1^* T''$ for some odd-even shadow T'. By choice of T, T'' is U2, and so $Y(T') = Y(T'') = 0$. The shadow T' is also odd, so this means that T' is U2 too. Therefore, the unknotter can win in T'.

In summary, then, the unknotter can respond to any move in T, by simply playing in the same component. Therefore, T is in U2, and so it is a win for the unknotter both ways. □

In fact, something more is true - all odd-even shadows are equivalent (modulo $*$) to the *zero game,* the game where nobody moves and the unknotter always wins. (The unknot shadow, for example, is a zero game). Such games are identities under addition. In the notation of my previous paper, we want to show that every odd even shadow is in the class Z_0. It turned out that a game G was in Z_0 iff $G + E \in Y_0$, where E is the game $\{\{\{u\}, \{k\}\}\}$. The game E is an odd game, of length 3, which is a win for whoever moves second. On the first move there are no options. After the first move, there are two options - a move to $*$ and a move to $k*$. The second move determines who wins within E. The third move has no effect, but is a spare move that can come in handy.

Theorem 6. *If T is an odd-even shadow, then T is in Z_0.*

Proof. We need the game $T + E$ to be in Y_0. Since T and E are both odd, $T + E$ is even, and what we really need is $T + E + *$ to be U2. Here is the strategy for the unknotter, playing second, in this combination of games:

- At all costs, we never make the first move in E, since the knotter can immediately respond with a killer move. We are never forced to make this move in E, since if that was the sole move remaining, the unknotter would be faced with an odd position, which shouldn't happen since initially the knotter was faced with the odd position $T + E + *$.
- If the knotter every plays in E (making the first move), we respond immediately by moving to $*$, since otherwise the knotter can make his killer move and we lose.
- If the knotter plays in T, we respond in the same component. This is only problematic if T has been entirely resolved, or if the knotter plays in an odd component with only one twist. The latter is not really a problem, since from a different point of view, the one twist can be seen as a part of the next component. On the other hand, if the knotter has just finished T, and there are still moves remaining, we make any of them, subject to the preceding caveats.
- If the knotter plays in $*$, or makes the third move of E, and we are left to play somewhere else, then play in the odd component of T, if it exists. (Otherwise, the game is close to being over).
- If for some reason we are ever forced to move in T for a second time, simply undo the move that we made there in the first place.

The reader can convince himself that this strategy actually works. Another way of saying this is to note that all of the following are safe moves for the unknotter to move to[1]:

- Positions of the form $[(a_1), (a_2), \ldots, (a_n)] + E + *$ where a_1 is odd and the other a_i are even.
- Positions of the form $[(a_1), (a_2), \ldots, (a_n)] + * + *$ where a_1 is odd and the other a_i are even.
- Positions of the form $[(a_1), (a_2), \ldots, (a_n)]$ where a_1 is odd and the other a_i are even.
- Positions of the form $[1(a_1), (a_2), \ldots, (a_n)] + E$, where all the a_i are even.

[1]This is assuming that initially, a_1 was odd

- Positions of the form $[1(a_1), (a_2), \ldots, (a_n)] + *$, where all the a_i are even.

Then it is straightforward to check (using techniques similar to the proof of Theorem 5) that $U2$ can remain within this safe set. □

4 The other games

The next lemma unfortunately has no clear proof that I know of, other than a verification by computer:

Lemma 7. *The following rational shadows are in Y_2:*

$$[(3),(1),(3)]\,,[(2),(1),(2),(2)]\,,[(2),(2),(1),(2)]\,,[(2),(1),(1),(2)]\,,[(2),(2),(1),(2),(2)]\,,[(2),(2)]$$

Lemma 8. *If $T = [(a_1), \ldots, (a_n)]$ is a non-trivial rational shadow corresponding to a knot (not a link), then either $T \rightarrow_1^* O$ for some odd-even shadow O, or $T \rightarrow_{1,2}^* A$, where A is equivalent to one of the six shadows in Lemma 7.*

Proof. Without loss of generality, T is irreducible as far as phony Reidemeister-1 moves go. Then we can make the assumption that all $a_i > 0$. If all of the a_i are even, then by stripping applying phony Reidemeister 2 moves, we can reduce T down to either $[(2),(2)]$ or $[(2)]$. But the second of these is a link (draw a picture to check this), so $T \rightarrow_{1,2}^* [(2),(2)]$. Otherwise, at least one of the a_i is odd. If the only odd a_i are $i = 1$ and/or $i = n$, then either T is an odd-even shadow, or a_1 and a_n are both odd. But if both a_1 and a_n are odd, then by applying phony Reidemeister two moves, we can reduce to one of the cases $[(1),(0),(1)]$ or $[(1),(1)]$. Both of these are equivalent to $[(2)]$, which is not a knot.

This leaves the case where at least one a_i is odd, $1 < i < n$. Let T be (a) not reducible by phony Reidemeister 1 moves, and (b) as reduced as possible by phony 2 moves, without breaking the property of having one of the a_i be odd, for $1 < i < n$. If $a_j > 2$ for any $1 < j < n$, then we can reduce a_j by two. So for every $1 < j < n$, $a_j \leq 2$. Similarly, a_1 and a_n must be either 2 or 3. (They cannot be 1 or else T would be reducible by a phony Reidemeister 1 move.) If $a_1 = 3$ and $i > 2$, then we can reduce a_1 by two and combine it into a_2 to yield a smaller T. So if $a_1 = 3$, then $a_2 = 1$ and $a_j \neq 1$ for $j > 2$ (or else we could have chosen a different i and reduced). Similarly, if $a_n = 3$, then $a_{n-1} = 1$ and $a_j \neq 1$ for $j < n - 1$. Thus, if a sequence begins with (3), the next number must be (1), and the (1) must be unique. For example, the sequence $[(3),(1),(1),(3)]$ can be reduced to $[(1),(1),(1),(3)]$ and thence to $[(2),(1),(3)]$.

On the other hand, suppose $a_1 = 2$. If $i > 4$ then we can reduce T farther by decreasing a_1 by a phony Reidemeister 2 move, and then decreasing a_2 one by one (by phony Reidemeister 1 moves) until both a_1 and a_2 are zero. Then both can be removed, yielding a smaller T. Moreover, this also works if $i = 4$, unless $a_3 = 1$.

Therefore, what precedes $a_i = 1$ must be one of the following:

- (3)
- (2)
- (2)(2)
- (2)(1)
- (2)(2)(1)
- (2)(1)(1)

(Only the first three of these can precede the first (1)). The same sequences reversed must follow any (1) in sequence. Then the only combinations which can occur are (this takes some checking):

- $[(3),(1),(3)]$

- $[(3),(1),(2)]$ and its reverse
- $[(3),(1),(2),(2)]$ and its reverse
- Not $[(3),(1),(1),(2)]$ because more than just (3) precedes the second (1).
- $[(2),(1),(2)]$
- $[(2),(1),(2),(2)]$ and its reverse
- $[(2),(1),(1),(2)]$
- $[(2),(1),(1),(2),(2)]$ and its reverse
- $[(2),(1),(1),(1),(2)]$
- $[(2),(2),(1),(2),(2)]$
- $[(2),(2),(1),(1),(2),(2)]$
- Not $[(2),(2),(1),(1),(1),(2)]$ because too much precedes the last (1).

So either T is one of the combinations in Lemma 7 or one of the following happens:

- $[(3),(1),(2)]$ reduces by a phony Reidemeister two move to $[(1),(1),(2)] = [(2),(2)]$. So does its reverse.
- $[(3),(1),(2),(2)]$ reduces by two phony Reidemeister two moves to $[(3),(1),(0),(0)] = [(3),(1)] = [(4)]$ which is a link, not a knot. Nor is its reverse.
- $[(2),(1),(2)]$ reduces by a phony Reidemeister 2 move to $[(0),(1),(2)]$, which in turn reduces by a phony Reidemeister 1 move to $[(0),(0),(2)] = [(2)]$ which is a link, not a knot. So this case can't occur.
- $[(2),(1),(1),(2),(2)]$ reduces by phony Reidemeister moves to $[(2),(1),(1),(0),(2)] = [(2),(1),(3)]$ so it isn't actually minimal.
- $[(2),(1),(1),(1),(2)]$ likewise reduces by a phony Reidemeister two move and a one move to

$$[(0),(0),(1),(1),(2)] = [(1),(1),(2)] = [(2),(2)]$$

- $[(2),(2),(1),(1),(2),(2)]$ reduces by a phony Reidemeister two move to $[(2),(0),(1),(1),(2),(2)] = [(3),(1),(2),(2)]$, so it isn't actually minimal.

In summary then, every T that does not reduce by phony Reidemeister one moves to an odd-even shadow reduces down to a finite set of minimal cases. Each of these minimal cases is either reducible to one of the six shadows in Lemma 7, or is not actually a knot. □

5 Sums of Rational Knot Shadows

Putting everything together, then, we have

Theorem 7. *Let T be a rational knot shadow, and let $T' = [a_1, a_2, \ldots, a_n]$ be the smallest T' such that $T \to_1^* T'$. Then if T' is an odd-even shadow, $T \in Z_0$, and otherwise, $T \in X_0 \cap Y_2$.*

Proof. It turns out that the class Z_0 is also closed under phony Reidemeister 1 moves, so if $T' \in Z_0$, then $T \in Z_0$. We already know that if T' is an odd-even shadow, then $T' \in Z_0$. So suppose that T' is not an odd even shadow. Then by Lemma 8, T' must reduce by phony Reidemeister 1 and 2 moves to a rational shadow T'' that is one of the six shadows in Lemma 7. By Lemma 7, $T'' \in Y_2$. Then by Theorem 4, $T \in Y_2$, since $T \to_1^* T' \to_{1,2}^* T''$. Also, by Corollary 1, $T \in X_0$. □

Definition 5. *A rational knot shadow* reduces to an odd-even shadow *if it reduces to an odd-even shadow via phony Reidemeister one moves.*

The previous theorem can be restated to say that a rational knot shadow is in Z_0 iff it reduces to an odd-even shadow, and is in $X_0 \cap Y_2$ otherwise.

Theorem 8. *If $T_1, T_2, \ldots T_n$ are rational knot shadows, and $T = T_1 + T_2 + \ldots + T_n$ is their connected sum, then T is a win for the unknotter (T is U1 and U2) if all of the T_i reduce to odd-even shadows, and otherwise, T is a win for the second player if T is even, and a win for the first player if T is even.*

Proof. The class X_0 is closed under addition, so $T \in X_0$. Also, the class Y_2 is closed under addition by anything, so if any $T_i \in Y_2$, so is T. By the previous lemma, this happens unless every $T_i \in Z_0$. Now the class Z_0 is also closed under addition, as shown in my previous paper. So in this case, when all the games are reducible to odd-even shadows, the sum is also in Z_0. But $Z_0 \subseteq Y_0$, so in this case $T \in Y_0$.

Now it is clear from the definitions of X_i and Y_i that the outcomes work out in the way stated. For example, if T is odd, and in $X_0 \cap Y_2$, then the odd projection of T, which is simply T, must be U1 (because of X_0), and K1 (because of Y_2). So T is a win for the first player. □

6 Computer Experiments

To be continued. . . [2]

[2] The computer experiments to be described here are already done, but I haven't written them up yet.