RECOVERY OF NON-LINEAR CONDUCTIVITIES FOR CIRCULAR PLANAR GRAPHS

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Abstract. We consider the problem of recovering nonlinear conductances in a circular planar graph. If the graph is critical (in the sense of [2]), and the conductance functions satisfy some weak conditions (such as being bijective), we show that the conductance functions are completely recoverable from the Dirichlet-to-Neumann relationship. This result is general enough that it also demonstrates the recoverability of conductances in critical circular planar linear networks with negative or complex (but nonzero) conductances, extending previous work in [3].

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1. Introduction

In a previous paper, I showed that if an electrical network has non-linear but monotone and continuous conductance functions, then the Dirichlet-to-Neumann map is well-defined. This raises issues of recovery. When the conductance function is allowed to be bounded (so that a maximum current can flow through some conductors), then recovery will generally be impossible, because by surrounding some conductor c with bounded-current conductors, part of the conductance function of c could be concealed. Therefore, in this paper, we impose an additional constraint: the conductance functions must be surjective. In order to have a sort of duality, it also seems good to require that the conductance functions be strictly monotone, that is, injective. With both constraints, there is a bijection between current and voltage along each edge, which allows planar graph duality to be used to its full potential.

The recovery problem was solved for critical circular planar graphs with these assumptions. However, much weaker assumptions were actually used by this algorithm. This paper presents these results, which also turn out to be useful for the case of negative conductivities. Of course, for non-critical circular planar graphs, recovery is already known to be impossible, by Chapter 8 of [2].

2. Preliminaries

Definition 2.1. *A* simple undirected graph with boundary *is a triple* $\Gamma = (V, \partial V, E)$*, where* V *is a set of* nodes*,* $\partial V \subseteq V$ *is a set of* boundary nodes*, and* $E \subseteq V \times V$ *is a set of* edges*, subject to*

$$(i,i) \notin E$$
$$(i,j) \in E \iff (j,i) \in E$$

for all $i, j \in V$*. In other words,* E *is a symmetric irreflexive relationship on* V*.*

Definition 2.2. *A* circular planar graph *is a graph with boundary* $(V, \partial V, E)$ *which can be embedded on a disk* D*, such that each boundary vertex* $v \in \partial V$ *is on the boundary of* D*, and no edges cross* [1]*.*

In this paper, we will only consider nonlinear electrical networks of the following type:

Definition 2.3. *A* bijective nonlinear network *is a pair* (Γ, γ)*, where* Γ *is a simple undirected graph with boundary, and* γ *is a map which assigns to each* $(i,j) \in E$ *a function* $\gamma_{ij} : \mathbb{R} \to \mathbb{R}$*, such that*

- $\gamma_{ij}(0) = 0$
- $\gamma_{ij}(-x) = \gamma_{ji}(-x)$
- $\gamma_{ij}(x)$ *is a bijection from* $\mathbb{R}$ *to* $\mathbb{R}$*.*

The function γ_{ij} *is the* conductance function on edge (i,j)

Note that there are *no* requirements that the γ_{ij} are monotone or continuous. In fact, bijective nonlinear networks generalize both the symmetric networks of [1], and the negative conductivities on linear networks, considered by Michael Goff in [3].

Definition 2.4. *A* voltage function *on* (Γ, γ) *is a function* $x : V \to \mathbb{R}$*. A* current function *on* (Γ, γ) *is a function* $c : E \to \mathbb{R}$ *such that* $c(i,j) = -c(j,i)$ *for all* $(i,j) \in E$*.*

The value $c(i,j)$ is interpreted as the current flowing from node i to node j.

Definition 2.5. *A current function* c *and a voltage function* x *on* (Γ, γ) *are* compatible *if for every* $(i,j) \in E$*,* $c(i,j) = \gamma_{ij}(x(i) - x(j))$*.*

For c a current function, the total current flowing out of a node i is given by

$$\phi_c(i) = \sum_{j,\ (i,j) \in E} c(i,j).$$

Definition 2.6. *A current function* c *on* (Γ, γ) *satisfies* Kirchhoff's Current Law *(KCL) if* $\phi_c(i) = 0$ *for all* $i \in \operatorname{int} V$*.*

Definition 2.7. *Given a voltage function* x*, the* boundary voltage function u_x *is the restriction of* x *to* ∂V*. Given a current function* c*, the* boundary current function ψ_c *is* ϕ_c *restricted to* ∂V*.*

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[1] In other words, we have an injection $f : V \to D$, such that $f(\partial V) \subseteq \partial D$, and for each edge $(i,j) \in E$ a continuous injection $f_{ij} : [0,1] \to D$ such that $f_{ij}(0) = i$, $f_{ij}(1) = j$, and $f_{ij}(\alpha) \neq f_{i'j'}(\beta)$ for any i, j if $\alpha, \beta \in (0,1)$

Definition 2.8. *For a fixed bijective nonlinear network* (Γ, γ)*, the* Dirichlet-to-Neumann relationship Λ *is the relationship between the possible boundary voltages and boundary currents:*

$$\Lambda = \{(u_x, \psi_c) \ : x \text{ is a voltage function, } c \text{ is a current function}$$

$$\text{satisfying the KCL, and } x \text{ and } c \text{ are compatible}\}.$$

Note that Λ is not necessarily a function, because the individual conductance functions may be very ill-behaved. On the other hand, if the γ_{ij} are all strictly increasing, then Λ is almost a bijection, as I showed in another paper.

3. Covoltage

If Γ is a circular planar graph, we can consider the dual graph $\Gamma^\dagger$. (For definitions, see, e.g., §5 in [4].) We now make the following definition.

Definition 3.1. *A* covoltage function *on* Γ *is a voltage function on* $\Gamma^\dagger$*.*

To each covoltage function $x^\dagger$ on Γ, we associate a current function c on Γ satisfying the KCL, by

$$c(i,j) = x^\dagger(a) - x^\dagger(b),$$

where a and b are the faces on the left and right sides of the edge (i,j).

Claim 3.2. *This construction yields a current function satsifying the KCL. Conversely, any current function c satisfying the KCL comes from a covoltage function, which is determined uniquely, up to the addition of a constant.*

This can be proven using arguments similar to those used in the linear case in §6 and §7 of [4].

Consequently, we can speak of a voltage function and a covoltage function being *compatible*, if the voltage function is compatible with the covoltage function's associated current function. The covoltage function is somewhat like the harmonic conjugate of the voltage function; not all voltage and covoltage functions will have compatible partners.

Definition 3.3. *The* voltage-covoltage *relationship* Θ *is the relationship between possible boundary voltages* u *and boundary covoltages* $u^\dagger$*, i.e.,*

$$\Theta = \{(u, u^\dagger) \ : \ u \text{ and } u^\dagger \text{ are compatible}\}$$

Theorem 3.4. *The voltage-covoltage relationship* Θ *contains the same information as the Dirichlet-to-Neumann relationship* Λ*.*

Proof. Given the boundary covoltages of a current function on Γ, the boundary currents can be determined: the boundary current at a boundary node i is just the difference in covoltages of the two boundary faces on either side of i. Conversely, given the boundary currents, the boundary covoltages are determined up to addition by a constant, because the differences between covoltages in adjacent boundary faces is known. ∎

4. Boundary-to-Boundary Edges and Boundary Spikes

In this section, we show why the layer-stripping approach that is used to recover linear networks in [2] is also applicable for the nonlinear case. That is, we show that if boundary-to-boundary edges and boundary spikes are recoverable, then the entire graph is recoverable. The theorems in this section are the analogs of those in Chapter 6 of [2].

Suppose we take a circular planar graph Γ with conductivity function γ, and we create a new graph Γ' by adjoining a boundary-to-boundary edge (i,j) with conductivity function γ_{ij}. Then the voltage-covoltage relationship Θ changes in a predictable way. Let a denote the boundary face between i and j on the old graph, and a' denote the boundary face between i and j on the new graph. Then $(u,u^\dagger) \in \Theta$ iff $(u,v^\dagger) \in \Theta'$, where

$$v^\dagger(a') = u^\dagger(a) + \gamma_{ij}(u(i) - u(j)),$$

$$v^\dagger(b) = u^\dagger(b) \text{ for } b \neq a'.$$

Importantly, the map from Θ to Θ' is a bijection. We have shown:

Theorem 4.1. *Given the voltage-covoltage relationship Θ, or, equivalently, the Dirichlet-to-Neumann relationship Λ of a bijective nonlinear network (Γ,γ), and given γ_{ij} for some boundary edge (i,j), the voltage-covoltage and Dirichlet-to-Neumann relationships of (Γ',γ') are determined, where Γ' is the graph obtained by deleting (i,j), and γ' is the restriction of γ to the edges of Γ'.*

In other words, once we have determined the conductance function of a boundary-to-boundary edge, we can remove it from the picture, reducing to a simpler problem.

Similarly, suppose we take a circular planar graph Γ and adjoin a boundary spike at node i (making i an interior node and adding a new node i'). If we extend γ with a conductance function on (i',i) given by $\gamma_{i'i}$, then $(u,u^\dagger) \in \Theta$ iff $(v,u^\dagger) \in \Theta'$, where

$$v(i') = u(i) + \gamma_{i'i}^{-1}(u^\dagger(a) - u^\dagger(b)),$$

$$v(j) = u(j) \text{ for } j \neq i',$$

where a and b are the faces on either side of the boundary spike. Again, the map that sends Θ to Θ' is a bijection. Consequently,

Theorem 4.2. *Given the voltage-covoltage relationship Θ of the Dirichlet-to-Neumann relationship Λ of some bijective nonlinear network (Γ,γ), and given $\gamma_{i'i}$ for some boundary spike (i',i), the voltage-covoltage and Dirichlet-to-Neumann relationships of (Γ',γ') are determined, where Γ' is the graph obtained by contracting (i',i), and γ' is the restriction of γ to the remaining edges of Γ'.*

In other words, once we have determined the conductance function of a boundary spike, we can remove it from the picture, reducing to a simpler problem.

Moreover, it is known (Lemma 8.6 of [2]) that every *critical* circular planar graph has at least one boundary-to-boundary edge or boundary spike, and removing it retains criticality. It follows that we can recover the conductivity function from the Dirichlet-to-Neumann relationship as long as we can recover the conductivity of an arbitrary boundary-to-boundary edge or boundary spike.

5. Convex Sets

We now develop the technical machinery that will enable recovery. Assume throughout that Γ is a critical circular planar graph. Some results do not require criticality – I leave to the reader the task of sorting them out. For definitions of *medial graphs*, *criticality*, and *geodesics*, see chapter 8 of [2]. Let S be the set of all cells in the medial graph of Γ.

Definition 5.1. *Two cells x and y in the medial graph of Γ are* adjacent *if they share an edge. A* connected *set of cells X is one that is connected through adjacency.*

Definition 5.2. *A* corner *of a set $X \subseteq S$ is a pair (v, x) where v is a vertex, and x is a cell in X touching v, such that neither of the cells adjacent to x and touching v are in X. (That is, if some cell of X other than x touches v, it is the one opposite x). An* anticorner *of X is a vertex v such that exactly three of the cells around v are in X.*

Intuitively, a corner is a place on the boundary of X where the boundary turns inwards, and an anticorner is a place where the boundary turns outwards. If two cells of X come together on opposite sides of a vertex, there are two corners there; otherwise, corners can be identified with vertices.

Definition 5.3. *A set $X \subseteq S$ is* simply connected *if it is connected and every component of the complement $S \setminus X$ is connected to a boundary cell.*

Note that if X is not simply connected, then we can "fill in" the isolated components of $S \setminus X$ yielding an $X' \supseteq X$ which is simply connected.

Lemma 5.4. *Let $X \subseteq S$ be simply connected, and suppose X contains no boundary cells of the medial graph. Then, X has at least three corners.*

Proof. Let X be a counterexample containing as few cells as possible. Suppose first that there is at least one anticorner v of X. Choose one of the geodesics which originate at v that heads into the interior of X. Follow it until it reemerges from the interior of X. This geodesic segment divides X into two disjoint sets X_1 and X_2, as shown in Figure 1, because X is simply connected. Both X_1 and X_2 are easily seen to be simply connected. (The main difficulty is showing that they are *connected*, but this follows from the fact that the set of cells in each X_i which lie along the side of the geodesic segment are connected.)

The total number of corners introduced by this partition is at most 3, one at v and one or two at the site of reëmergence. To see this, suppose (v, x) is a new corner of, say, X_1. Then since (v, x) was not a corner of X, we must have $x \in X_1$ and some of the neighbors of x that touch v in X_2 (see Figure 2). Both cannot be in X_2 since the geodesic dividing X_1 and X_2 cannot make a sharp turn or intersect itself. Therefore, there are three consecutive cells x, y, and z, around v, with $x \in X_1$, $y \in X_2$, and $z \notin X$. This must be one of the end points of the geodesic segment. And (v, x) can only be a new corner if x is one of the two cells which flank the last edge along the geodesic segment. Thus there are at most two at each end of the geodesic segment. But because one end was at an anticorner, there is only one new corner on that side, so at most 3 new corners were added.

Consequently, the total number of corners among X_1 and X_2 is at most $2+3 = 5$, so either X_1 or X_2 is a smaller counterexample.

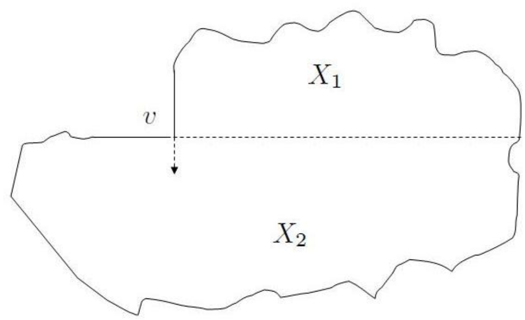

FIGURE 1. How we divide up X in Lemma 5.4

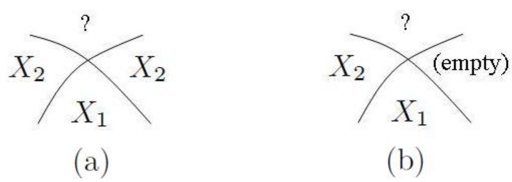

FIGURE 2. The possible situations in the neighborhood of (v, x): (a) cannot happen, so (b) must happen.

Otherwise, we can assume that X contains no anticorners. If X contains no corners, then the entire boundary of X is a loop, contradicting the criticality of Γ. If there is exactly one corner, then the boundary is a self-intersecting geodesic, which is impossible. Finally, if there are exactly two corners, then the boundary of X consists of two geodesics which intersect at two points, so the medial graph contains a lens, and Γ is not critical. (All of these conclusions require the fact that X does not contain any boundary cells.) □

Lemma 5.5. *Let $X \subseteq S$ be connected, and suppose X contains no boundary cells of the medial graph. Then, X has at least three corners.*

Proof. Let $X' \supseteq X$ be the simply connected set obtained by "filling in" any holes in X. X' certainly has no more corners than X, and does not contain any boundary cells of the medial graph, because X does not. Then by Lemma 5.4, X' has at least three corners. Therefore, so does X. □

Every geodesic in the medial graph divides S into two pieces.

Definition 5.6. *A* half-disk *is the subset of S on one side of a geodesic g. A set $X \subseteq S$ is* convex *if it is an intersection of half-disks.*

For critical graphs, it will turn out that convex sets are connected.

Lemma 5.7. *Suppose x and y are two cells inside some cells of a critical medial graph M. If x and y are on the same side of every geodesic in M, then $x = y$.*

Proof. We proceed by induction on the number of geodesics in M. If there are no geodesics, then there is just one cell in M, so $x = y$.

Otherwise, let g be a geodesic in M. Construct a new medial graph M' by removing the side of g that does not contain x and y, and making g part of the boundary of M' (see Figure 3). Now, any geodesic h in M' will not separate x and y, because if it did, it would cross g and then necessarily separate x and y in the original medial graph M (see Figure 4). Therefore, by induction, $x = y$.

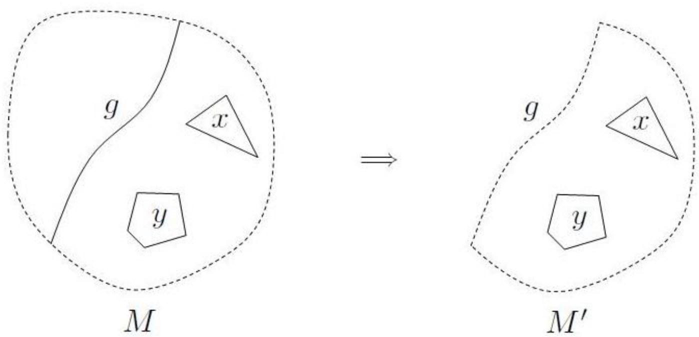

FIGURE 3. Constructing a new medial graph M' from one side of the geodesic g.

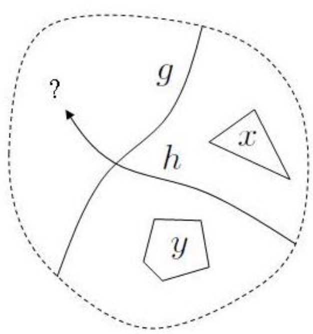

FIGURE 4. If h separates x and y in M', then it does in M. Here is the case where h crosses g.

□

Lemma 5.8. *Let x and y be two cells in a critical medial graph M. Let n be the number of geodesics which separate x from y. Then there is a path of adjacent cells from x to y of length n (We can get from x to y in n moves).*

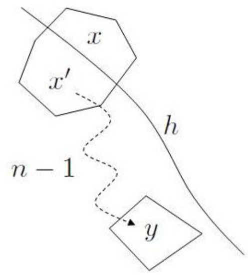

FIGURE 5. Getting from x to y in n steps.

Proof. Proof by induction on n. Suppose $n = 0$. Then by the previous lemma, x and y are the same cell so we are done.

Otherwise, consider all the geodesics around the boundary of x. If some geodesic h separates x from y, then we move from x to the neighboring cell x' across h. Then $n-1$ geodesics separate x' from y, so there is a path from x' to y of length $n-1$. Combining this with the step from x to x', we get a path of length n (Figure 5).

On the other hand, if all geodesics around the edge of x do not separate x from y, then consider a new critical medial graph consisting of only these geodesics, and let x' and y' be the cells containing x and y. Then no geodesics separate x' and y', so by Lemma 5.7 $x' = y'$. Therefore, there is a path from x to y which stays entirely in $x' = y'$, i.e., which does not cross any of the geodesics around the edge of x. This is only possible if $x = y$, contradicting the $n > 0$ geodesics separating x and y. □

Note that if x, y, n are as in the lemma, then a path from x to y must take at least n steps, to cross the n geodesics which separate x from y. Therefore, any path which uses only n steps crosses only the geodesics which separate x and y.

Theorem 5.9. *Every convex set X of cells is connected.*

Proof. Let x and y be two cells in X. Then by the previous lemma, there is a path from x to y which only crosses geodesics which separate x from y. If this path ever leaves X, it must exit one of the half-disks which are used to define X. It would then cross a geodesic which does not separate x from y, a contradiction. So there is a path between any x and y in X, so X is connected. □

Suppose that four cells x_1, x_2, x_3, x_4 come together at some interior vertex v of the medial graph. If we know the voltage/covoltage at three of these four cells, then we can determine the voltage or current at the edge of Γ associated with v, allowing us to determine the fourth voltage/covoltage. Note that this occurs precisely when the set of cells with known voltage/covoltages has an anticorner.

Definition 5.10. *A set $X \subseteq S$ is* closed *if it has no anticorners. In other words, whenever it contains three of the cells which come together at an interior vertex v, it also contains the fourth.*

If $X \subseteq S$ is arbitrary, the closure $\overline{X}$ *is the smallest closed set containing X.*

The definition of closure makes sense because

Lemma 5.11. *The intersection of two closed sets is closed.*

Proof. (Clear from the definition) □

So the closure of X is the intersection of all closed sets containing X. Alternatively, the closure $\overline{X}$ can be constructed by adding one cell at a time to X, removing anticorners, until X becomes closed (this process terminates because S is closed). Consequently, we have

Lemma 5.12. *The closure of a connected set is connected.*

Lemma 5.13. *X is closed iff each connected component of X is closed.*

Proof. Suppose X is closed, and let X' be a connected component of X'. If four cells x_1, x_2, x_3, x_4 come together at a vertex, and three are in X', then the fourth is in X, but it is also connected to the cells in X', so it is in X' itself. So X' is closed.

Conversely, suppose that every connected component of X is closed. Then if four cells x_1, x_2, x_3, x_4 meet at a vertex, and three are in X, they are necessarily in the same connected component of X, so the fourth is also in X. Thus X is closed. □

Theorem 5.14. *If X is convex, then X is closed.*

Proof. Since the intersection of two closed sets is closed, it suffices to show that a half-disk H is closed. Suppose that four cells x_1, x_2, x_3, x_4 meet at a vertex, $x_4 \notin H$, but $x_1, x_2, x_3 \in H$, and x_1 is adjacent to x_4 (see Figure 6). Then H must come from the geodesic which separates x_1 from x_4. But this geodesic also separates x_3 from x_2, so they cannot both be in H. □

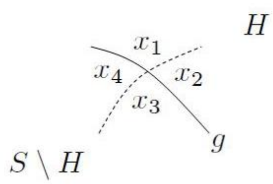

FIGURE 6. Why a half-disk H is closed: if $x_1 \in H$ but $x_4 \notin H$, then $x_3 \notin H$

The converse is true for connected sets, but requires some lemmas first.

Lemma 5.15. *If X is closed and connected, then X is simply connected. Also, there are no vertices at which X has two corners.*

Proof. Suppose X is not simply connected. Then there is some connected component Y of $S \setminus X$ which contains no boundary cells of the medial graph. Then by Lemma 5.5 Y has at least three corners (v, y). Thus $y \in Y$ but the two cells next to v and y are in X. The fourth cell at v must be in Y or else X is not closed. Thus Y and some component of $S \setminus X$ touch at v, from opposite sides. Construct a multigraph on the connected components of $S \setminus X$ by having an edge between two if they share a vertex. Multiple edges correspond to multiple vertices. Identify all the vertices corresponding to components touching the boundary of the medial graph. Any cycle in this graph will disconnect X (Figure 7). Because every vertex save one has degree at least 3, there must be a cycle, unless there is only one vertex, the exterior one. In this case, X is simply connected.

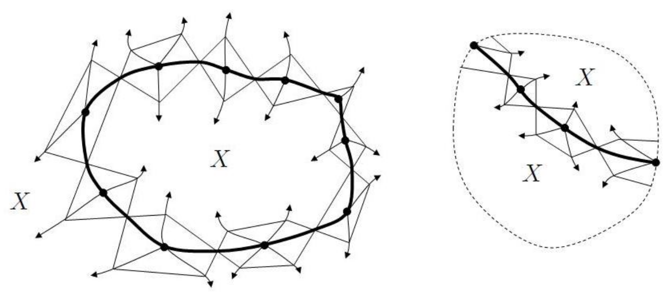

FIGURE 7. A cycle in the multigraph means that X is not connected. Note that a path that starts and ends on the medial graph boundary counts as a cycle, because all boundary components have been identified.

To prove the second assertion, note that if there was such a vertex, where exactly two cells were in X and the two were not adjacent, then this would yield another edge in the auxiliary graph from the first part of this proof. Since there is only vertex in this auxiliary graph, this would be a self-loop, which would divide X. $\square$

Let g be a fixed geodesic with orientation, and let X be a closed set of cells. We can partition the edges that constitute g into four classes:

- Interior edges, which lie between two cells of X
- Exterior edges, for which both of the adjacent cells are not in X.
- Left edges, for which the cell on the left of the edge is in X but not the one on the right.
- Right edges, for which the cell on the right of the edge is in X but not the one on the left.

Let the *components* of g be the maximal contiguous series of edges which are classified in the same way. See Figure 8, for example. Since X is closed, an interior component can not be adjacent to a right component or a left component.

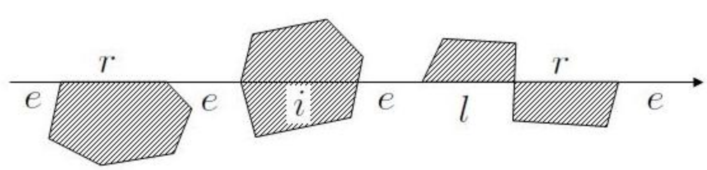

FIGURE 8. Eight components of g. From left to right, there are exterior, right, exterior, interior, exterior, left, right, and exterior components.

Lemma 5.16. *If X is closed and connected and is on only one side of g, then there is at most one non-exterior component.*

Proof. Without loss of generality, X is entirely on the right side of g. Then there are only right and exterior components. Suppose s_1 and s_2 are two right components, chosen as close as possible, and d is the exterior component between them. Let H be the half-disk on the side of g containing X, and consider the connected component Y of $H \setminus X$ containing d. If Y contains some boundary cells of the medial graph, then X is not connected. If Y touches any other segment of g other than d, then X is not connected. We thus have a sitution similar to the one in Figure 9.

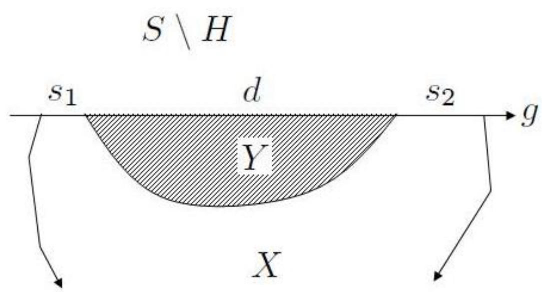

FIGURE 9. The scenario of Lemma 5.16

Now suppose (v, y) is a corner of Y. Then $y \in Y$ but y's neighbors around v are not in Y. Therefore, they are either in $S \setminus H$ or in X. If both are in X, then since X has no anticorners, the fourth cell at v must not be in X. But this cannot happen by Lemma 5.15. So one of the cells around v must be in $S \setminus H$. By a similar argument, one of the cells must be in X. This forces v to be along g, since $S \setminus H$ and X are on opposite sides. In fact, v must be one of the two endpoints of d, since every other point along d is adjacent to two cells of $S \setminus H$ and two of Y.

Therefore, Y has too few corners, and we have a contradiction. □

Lemma 5.17. *If X is closed and connected, then there is only one non-exterior component.*

Proof. Let H_l and H_r be the two half-disks on the left and right sides of g. Both are closed, so by Lemma 5.11 $H_l \cap X$ and $H_r \cap X$ are both closed. The components of g along $H_l \cap X$ will come from the left and interior components of g along X, and the components of g along $H_r \cap X$ will come from the right and interior components of g. By Lemma 5.16, there will only be one component of g along each connected component of $H_l \cap X$ or $H_r \cap X$.

Now x and y are two cells in X. They are connected by some path of adjacent cells which remains inside X. If this path ever crosses g it does it at some interior component of g. Now suppose it crosses g more than once. At some point it enters a connected component X' of $H_r \cap X$ or of $H_l \cap X$, and then it leaves X'. By Lemma 5.16, there is only one component of g along X'. This component comes from an interior component s of g along X. The set of cells along each side of s is connected, so we could modify the path in question by skipping the detour into X'

entirely, and just travel along the side of s (Figure 10). By repeating this process, we can ensure that any path from x to y only crosses g at most once.

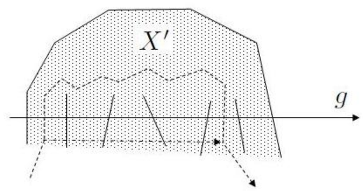

FIGURE 10. If a path enters and exits some component X' of $H_r \cap X$, we can remove the detour into X' by traveling along the side of g.

It follows that there is at most one component of $H_r \cap X$ and at most one component of $H_l \cap X$ (otherwise, we could not get between two components on the same side of g). If either $H_r \cap X$ or $H_l \cap X$ vanishes, then we are done by Lemma 5.16. Otherwise, there is certainly an interior component of g, because X is connected. This interior component yields a non-exterior component of g on each of $H_r \cap X$ and $H_l \cap X$ – then by Lemma 5.16 there can be no other non-exterior components of $H_r \cap X$ or $H_l \cap X$. Therefore, there can be no more non-exterior components of X save the one interior component.

□

Lemma 5.18. *Let X be a connected, closed set, and let e be an edge along the boundary of X. If e is along a geodesic g (as opposed to the boundary of the medial graph), then X is entirely on one side of g.*

Proof. By Lemma 5.17, there is only one non-exterior component of g along X. It is not an interior component, since e is not between two cells of X. Therefore, it is a right or left component. Suppose without loss of generality that it is a right component. Then there are no cells of X touching the left side of g. Since X is connected, this can only happen if X is entirely on one side of g. □

Theorem 5.19. *If X is connected and closed, then it is convex.*

Proof. Let X' be the convex closure of X, i.e., the intersection of all half-disks containing X. X' is clearly convex, so it is connected. Also, if e is any edge along the boundary of X', and e is on a geodesic g, then X is entirely on one side of g, by Lemma 5.18, so X' is too. Suppose some $x \in X'$ is not in X. Then by the connectedness of X', there is a path from some $y \in X$ to x. By Lemma 5.8, there is a path (not necessarily in X') from x to y which does not cross any geodesic which does not separate x and y. This would include every geodesic along the boundary of X. Therefore, the path from x to y does not actually cross the boundary of X, so $x \in X$. □

Corollary 5.20. *Let X be connected. Then the closure of X is the intersection of all half-disks containing X.*

Proof. Let $\overline{X}$ be the closure of X. As noted above, $\overline{X}$ will be connected because X is. By the preceding theorem, $\overline{X}$ is convex, so it is the intersection of all half-disks containing $\overline{X}$. Let X' be the intersection of all half-disks containing X. Any half-disk which contains $\overline{X}$ contains X, so $X' \subseteq \overline{X}$. On the other hand, X' is a convex set containing X, so it is also a closed set containing X. Therefore, $\overline{X} \subseteq X'$. □

6. Consistent Data and Nice Extensions

Fix Γ a critical circular planar graph and γ a conductivity function on Γ. Let X be a set of cells in the medial graph of Γ.

Definition 6.1. *Data on X is a function f from the cells of X to $\mathbb{R}$, interpreted as voltage or covoltage data, as appropriate. The data f is* consistent (with γ) *if whenever four cells in X meet at a vertex, the voltages and covoltages at that vertex are consistent with the conductance function at the associated edge in Γ. If the cells a, b correspond to faces in Γ and i, j correspond to vertices, some equation like*

$$f(a) - f(b) = \gamma_{ij}(f(i) - f(j))$$

must be true.

Definition 6.2. *If X is a set of cells in the medial graph, and X' is obtained from X by adding a single cell a, such that a and three other cells meet at an anticorner of X, then X' is a* simple extension *of X. If a only touches one anticorner of X (i.e., a is adjacent to only two cells of X), then X' is a* nice simple extension*. If X'' is obtained from X by a series of simple extensions, then X'' is an* extension *of X, and if it is obtained by a series of nice simple extensions, then it is a* nice extension *of X.*

It is clear that the maximal extension of any set X is simply the closure $\overline{X}$, and that any extension of X is a set Y satisfying $X \subseteq Y \subseteq \overline{X}$.

The motivation for nice extensions (and closure and convexity, in fact) is the following theorems:

Theorem 6.3. *Suppose f is consistent data on a set of cells X, and X'' is an extension of X. Then f can be extended in at most one way to be consistent data f'' on X''.*

Proof. It suffices to show this for the case of X'' a simple extensions. But this is obvious, since a simple extension X' is obtained from X by adding a cell a which is the missing cell at some anticorner of X. The value of f'' at a is uniquely determined by the consistency requirement, since all conductivity functions are bijections. □

Theorem 6.4. *Suppose f is consistent data on a set of cells X, and X'' is a nice extension of X. Then f can be extended in a unique way to be consistent data f'' on X''.*

Proof. As before, we only need to consider nice simple extensions. Here, the only thing to prove is that the addition of the cell a does not violate any consistency requirements. Since this is a nice simple extension, there is only one vertex v which touches a and three other members of X''. Therefore, there is only one relationship imposed on $f''(a)$, and it can be satisfied. □

If X is a set of cells in the medial graph, an *interior vertex* of X is a vertex whose four neighboring cells are all in X. Specifying consistent data on X amounts to specifying the value of $|X|$ variables subject to n equations, where n is the number of interior vertices in X.

Definition 6.5. *If X is a set of cells in the medial graph, the* rank *of X, denoted* $\text{rank}(X)$, *is the number of cells in X minus the number of interior nodes.*

The following lemma is obvious from the definitions:

Lemma 6.6. *Let X' be a simple extension of X. Then* $\text{rank}(X') \leq \text{rank}(X)$ *with equality iff X' is a nice simple extension.*

Theorem 6.7. *Let X' be an extension of X. Then* $\text{rank}(X') \leq rank(X)$, *with equality iff X' is a nice extension of X.*

Proof. Obvious from the lemma, using induction. □

It turns out that there is a different way of understanding the rank:

Theorem 6.8. *Let X be a convex set of cells. Then* $\text{rank}(X)$ *is one more than the number of geodesics g which pass through the interior of X (i.e., have interior components along X).*

Proof. Consider the subgraph of the medial graph containing the cells of X and all the edges around them (including the edges along the boundary of the medial graph, not corresponding to any geodesics). Since X is simply connected (Lemma 5.15), the leftover areas are all connected, and can be replaced by one exterior face. This graph will have $n_2 + n_3 + n_4 + n_i$ vertices, where

- n_2 is the number of 2-valent vertices, which are all corners.
- n_3 is the number of 3-valent vertices, which all occur along the boundary.
- n_4 is the number of boundary 4-valent vertices, which are all anticorners (by Lemma 5.15).
- n_i is the number of interior nodes, which are all 4-valent.

Likewise the number of edges will be given by

$$E = \frac{2n_2 + 3n_3 + 4n_4 + 4n_i}{2},$$

and the interior faces will be given by

$$F = 1 + E - V = 1 + n_2 + \frac{3}{2}n_3 + 2n_4 + 2n_i - n_2 - n_3 - n_4 - n_i = 1 + \frac{n_3}{2} + n_4 + n_i,$$

because there is exactly one exterior face. Now F is just the number of cells in X, so

$$\text{rank}(X) = F - n_i = 1 + \frac{n_3}{2} + n_4.$$

Now each geodesic that passes through the interior of X terminates at a 3-valent or 4-valent boundary vertex. A 3-valent boundary node is the endpoint of one geodesic while a 4-valent boundary node is the endpoint of two. Therefore, if g is the number of geodesics,

$$2g = n_3 + 2n_4.$$

We are not overcounting any geodesics, because each geodesic which passes through the interior of X only has one component, by Lemma 5.17.

$$\text{rank}(X) = 1 + g.$$

□

We now come to the key results that will be needed for recovery:

Theorem 6.9. *Let X be a convex set which touches the boundary of the medial graph and let a be some cell on the boundary of the medial graph, $a \notin X$ but a touches X (see Figure 11). Then the closure of $X \cup \{a\}$ is a nice extension of $X \cup \{a\}$.*

Proof. The closure $\overline{X \cup \{a\}}$ is certainly an extension of $X \cup \{a\}$, so it suffices to show that

$$\operatorname{rank}(\overline{X \cup \{a\}}) \geq \operatorname{rank}(X \cup \{a\}).$$

Now the number of interior nodes in $X \cup \{a\}$ must be the same as the number of interior nodes in X, since X is closed. Therefore, $\operatorname{rank}(X \cup \{a\}) = \operatorname{rank}(X) + 1$. So it suffices to show that

$$\operatorname{rank}(\overline{X \cup \{a\}}) > \operatorname{rank}(X).$$

But this is simple, since both X and $\overline{X \cup \{a\}}$ are convex sets, and more geodesics pass through the latter. Specifically, the geodesic which separates a from X (which cannot pass through the interior of X, because it is part of the boundary of X) now passes through the interior of $\overline{X \cup \{a\}}$. □

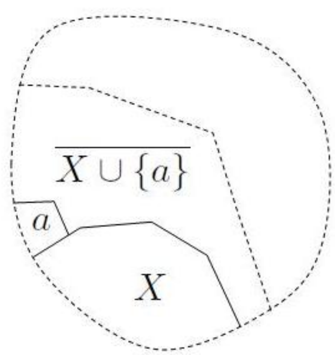

FIGURE 11. Theorem 6.9.

Theorem 6.10. *Let ∂M be the set of all boundary cells in the medial graph. If X is a convex set which is the closure of $X \cap \partial M$, and $X \cap \partial M$ is connected, then there is some $X' \subseteq X \cap \partial M$ such that X is a nice extension of X'.*

Proof. Let S be a subset of $X \cap \partial M$ of maximal size such that $\overline{S}$ is connected and a nice extension of S. Clearly, any one-element subset of X has this property, so $|S| \geq 1$. If $\overline{S} = X$, then we are done, so suppose $\overline{S} \subset X$. Because X is the closure of its intersection with the boundary, $\overline{S} \neq X \cap \partial M$. Let a be a cell in $(X \cap \partial M) \setminus \overline{S}$ which is adjacent to $\overline{S}$. Then by Theorem 6.9, $\overline{\overline{S} \cup \{a\}}$ is a nice extension of $\overline{S} \cup \{a\}$. Then a is not next to an anticorner of $\overline{S}$, so it is not next to an anticorner of S either. Therefore,

$$\operatorname{rank}(S \cup \{a\}) = \operatorname{rank}(S) + 1,$$

$$\text{rank}(\overline{S} \cup \{a\}) = \text{rank}(\overline{S}) + 1.$$

But we also have

$$\text{rank}(\overline{S}) = \text{rank}(S),$$
$$\text{rank}(\overline{\overline{S} \cup \{a\}}) = \text{rank}(\overline{S} \cup \{a\}),$$

because of the nice extensions. It follows that

$$\text{rank}(\overline{\overline{S} \cup \{a\}}) = \text{rank}(S \cup \{a\}).$$

But $\overline{\overline{S} \cup \{a\}}$ is just $\overline{S \cup \{a\}}$. So if we take $S' = S \cup \{a\}$, then $\text{rank}(\overline{S'}) = \text{rank}(S')$, so S' is a subset of $X \cap \partial M$ whose closure is connected (because $\overline{S} \cup \{a\}$ is connected) and is a nice extension of S. Also, $|S'| > |S|$, contradicting the choice of S. So we are done. □

Theorem 6.11. *Let X be a convex set which touches the boundary of the medial graph ∂M. Then there is some set of cells $S \subseteq \partial M \setminus X$ such that the entire medial graph is a nice extension of $X \cup S$.*

Proof. Let Q be a maximal convex set which is a nice extension of X and some boundary cells. If Q contains all of ∂M, then Q is already the entire medial graph, since the closure of ∂M is the entire medial graph (because no geodesic can be entirely on one side of ∂M). So some boundary cells are not in Q. Choose $a \in \partial M \setminus Q$ such that a is adjacent to Q. By a Theorem 6.9, $Q' = \overline{Q \cup \{a\}}$ is a nice extension of $Q \cup \{a\}$. Now Q itself is a nice extension of $X \cup S'$ for some $S' \subseteq \partial M \setminus X$. Therefore,

$$\text{rank}(Q') = \text{rank}(Q \cup \{a\}) = \text{rank}(Q) + 1 = \text{rank}(X \cup S') + 1.$$

But $\text{rank}(X \cup S') + 1 = \text{rank}(X \cup S' \cup \{a\})$, because a does not complete an anticorner in Q, and therefore does not complete an anticorner in $X \cup S' \subseteq Q$ either. Also, $Q' = \overline{Q} = \overline{X \cup S'} \cup \{a\} = \overline{X \cup S' \cup \{a\}}$, so Q' is a nice extension of $X \cup S$, where $S = S' \cup \{a\}$, contradicting the choice of Q. □

7. Recovery

As noted in a previous section, we can recover critical circular planar graphs as long as we can recover the conductivity on individual boundary-to-boundary edges and boundary spikes. Because of the duality between voltage and covoltage (which works because conductance functions are bijections), we only focus on boundary-to-boundary edges here. The case for boundary spikes is completely analogous.

Let (i, j) be a boundary-to-boundary edge. Let a be the cell in the medial graph between the cells i and j. There will be two geodesics g and h which originate on either side of a and immediately cross, making a into a three-sided cell. Assume g separates a from i and h separates a from j. We will construct a mixed problem with a uniquely determined solution that forces all voltages and covoltages on one side of g to be 0, and the covoltage at a to be anything we please. It is clear then that the voltage at j will be given by $\pm\gamma^{-1}(x)$, where x is the covoltage at a, γ is the conductance function along the boundary-to-boundary edge and the sign is determined by the orientation of things.

Lemma 7.1. *If there is a boundary-to-boundary edge (i, j), then for any x there is a mixed problem (a set of voltages and covoltages along the boundary of the medial graph) whose unique solution takes the value $\gamma^{-1}(x)$ at j.*

Proof. Begin by setting all of the voltages and covoltages on the side of g opposite a to be 0. This will force all voltages and covoltages to vanish in the closure of those boundary cells, which is just the half-disk determined by g. For suppose some other half-disk H contains all the boundary cells on that side of g: then the geodesic k which determines H must have both endpoints on the same side of g as a, and so by criticality, it cannot cross g at all. So the half-disk H already contains the half-disk determined by g.

Now, by Theorems 6.4 and 6.9, if we additionally set the value of the covoltage at a to be x, then this will uniquely determine all the voltages and covoltages in some bigger convex set X. Then by Theorem 6.11, there is some set of boundary nodes S such that $X \cup S$ nicely extends to the entire medial graph. So by taking arbitrary values on S, x at a, and all zeros on the far side of g, we get a mixed problem with a unique solution f. This mixed problem is pictured in Figure 12. It is clear from the consistency requirement that the voltage at j, $f(j)$ will end up being something like $f(i) + \gamma^{-1}(f(a) - f(b))$, where b is the cell opposite a. Since b and i are both on the far side of g, $f(i) = 0 = f(b)$, and so $f(j) = \gamma^{-1}(f(a)) = \gamma^{-1}(x)$. □

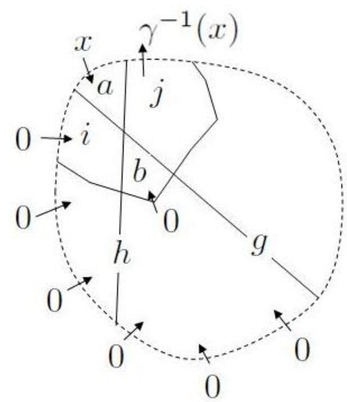

FIGURE 12. The mixed problem to recover boundary-to-boundary edge (i, j). There will also be some arbitrary additional boundary constraints on the right side of this diagram.

Theorem 7.2. *If* Γ *is a critical circular planar graph, and* γ *is an arbitrary conductivity function on* Γ*, then* γ *is recoverable from the Dirichlet-to-Neumann relationship* Λ*.*

Proof. By the preceding, we know that we can recover the conductivity function along any boundary-to-boundary edge from Λ. Boundary spikes can be recovered similarly (the situation is perfectly dual). By Theorem 4.1, knowledge of the conductivity function along a boundary-to-boundary edge along with the Dirichlet-to-Neumann relationship determines the Dirichlet-to-Neumann relationship of the graph obtained by deleting the boundary-to-boundary edge. Similarly, boundary spikes can be contracted. Also, it is previously known (for example, in Chapter 8 of [2]) that every critical circular planar graph contains at least one boundary-to-boundary edge or boundary spike, and deleting or contracting them (respectively), preserves criticality. Therefore, all the conductivities in Γ can be recovered by layer stripping. That is, if some Γ was not recoverable, then choose a minimal such one, recover a boundary-to-boundary edge or boundary spike, reduce the graph to a smaller Γ', obtain the Dirichlet-to-Neumann relationship Λ' for Γ', and recover the conductivity functions in Γ'. The end. □

Corollary 7.3. *Suppose a critical circular planar graph is given, with non-zero but possibly negative conductivites on each edge. Then the conductivities are recoverable from the Dirichlet-to-Neumann relationship, even if it is not a function.*

Proof. Negative conductivities satisfy the requirements of bijectivity and zero-preservation. □

Previously, Michael Goff [1] had shown this in the case where the Dirichlet-to-Neumann map exists (is a function). This work extends his result.

8. Future Work

The requirement that condunctance functions be strictly monotone lacks any real motivation. Therefore, one idea to explore is nonlinear conductance networks in which the conductances are (monotone and continuous) surjections.

It would also be good to have a theory similar to this one, but for nonplanar graphs. There might be some worth in considering the ideas of nice extensions and convex sets for the case of graphs of low genus, where we can make some kind of medial graph. For example, if a graph is not circular planar, but can be embedded on a torus, then we can construct a medial graph on the torus. It is conceivable that some of the results from this paper might carry over into that context, though it is not clear in that case even what a "critical" graph would be, or how emptying lenses should work.

One of the most disappointing things observed while writing this paper is the complexity of the lemmas in §5. The basic idea of medial-graph convexity seems simple enough that it should not require complicated arguments to establish the main results of §5. There are probably much simpler proofs to be found. At any rate, it seems like there should be some theory lurking behind the lemmas in §5.

9. References

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