ENUMERATION OF AFFINE PERMUTATIONS AVOIDING STANDARD PERMUTATIONS

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Abstract. Much work has been done in the classification and enumeration of standard permutations avoiding various patterns. We wish to find similar results in pattern avoidance among affine permutations. A few conjectures in this area are explained, and some progress toward their proofs is given.

Contents

1. Introduction

We begin by defining and describing affine permutations. For a fixed integer $n \geq 2$, let $\tilde{S}_n$ be the group of all bijections $\omega : \mathbb{Z} \to \mathbb{Z}$ such that for all $i \in \mathbb{Z}$,

$$\omega(i+n) = \omega(i) + n \tag{1}$$

and

$$\sum_{j=1}^{n} \omega(j) = \binom{n+1}{2} \tag{2}$$

with composition as the group operation. The elements of $\tilde{S}_n$ are called affine permutations.

The window notation of an affine permutation ω is the expression $[a_1, a_2, \ldots, a_n]$, where $a_i = \omega(i)$. By (1), a given window notation uniquely determines a corresponding affine permutation if and only if it satisfies (2) and the set of remainders $r_i \equiv a_i \pmod{n}$ equals the set $1, 2, \ldots, n$. This final requirement ensures that ω is a bijection onto the integers.

One set of generators of $\tilde{S}_n$ is $\{s_1, \ldots, s_n\}$, defined as follows:

For i = 1, ..., n-1, let

$$s_i = [1, 2, \ldots, i-1, i+1, i, i+2, \ldots, n].$$

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In addition, define

$$s_n = [0, 2, 3, \ldots, n-1, n+1].$$

Given a standard permutation $p \in S_m$ for some integer $m \geq 2$, an affine permutation ω is said to contain the "pattern" $p = [p_1, \ldots, p_m]$ if there exists some subsequence $\omega(i_1), \omega(i_2), \ldots, \omega(i_m)$ of ω with the same relative order as p. If no such subsequence exists, ω is said to avoid p. Note that the indices i_1 through i_m need not be in $\{1, 2, \ldots, n\}$, which makes permutation avoidance for affine permutations more difficult than that of regular permutations.

Finally, the graph of an affine permutation ω is the collection of points

$$\{(i, \omega(i)) : i \in \mathbb{Z}\}.$$

Based on (1), each group of n points must be to the lower left of the next n points for any affine permutation.

Example: [3, 6, 1, 0] 7, 10, 5, 4 is a valid permutation. This permutation avoids 3412, but it contains 3421 (use the entries 6, 7, 5, 4, for instance).

Our main goal is to enumerate as many classes of affine permutations avoiding certain patterns as possible. We begin by presenting relevant previous work done in this area, as well as results conjectured. Next, some useful tools for grouping affine permutations by similar pattern avoidance are discussed. Further progress toward proving the conjectures is presented in the final section.

2. Previous Work

Here we present previous results. The following theorem is critical to the theory of pattern avoidance for affine permutations.

Theorem 2.1. *Let $p \in S_m$. For any $n \geq 2$ there exist only finitely many $\omega \in S_n$ that avoid p if and only if p avoids the pattern* 321.

This result was stated and proved in [1].

For any $p \in S_m$, we define $\tilde{S}_n(p)$ to be the number of affine permutations in $\tilde{S}_n$ that avoid p. By Theorem 2.1, this value is infinite if and only if p contains the pattern 321.

Thus for any p avoiding 321, we define the generating function

$$\tilde{S}^p(t) = \sum_{n=2}^{\infty} \tilde{S}_n(p) t^n.$$

We will organize these generating functions into Wilf classes.

Definition 2.2. Two permutations $p, q \in S_m$ are Wilf-equivalent if both p and q avoid 321 and $\tilde{S}^p(t) = \tilde{S}^q(t)$, or if both p and q contain 321.

For a given m, all permutations in S_m can be organized into equivalence classes under the relation of Wilf-equivalence for affine permutations. These classes will be called Wilf classes.

Table 1 shows the known Wilf classes and the corresponding generating functions for S_3.

Similar classes have been nearly completed for permutations in S_4. See Table 2. The following conjectures (based on calculations with Sage) have been marked on the table in bold:

Conjecture 2.3.

$$\tilde{S}^{4123}(t) = \tilde{S}^{3412}(t) = \left(\sum_{n=2}^{\infty} \frac{1}{3} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \right) t^n \tag{3}$$

Note that both equalities above are conjectured.

Conjecture 2.4.

$$\tilde{S}^{3142}(t) = \left(\sum_{n=2}^{\infty} \sum_{k=0}^{n-1} \frac{n-k}{n} \binom{n-1+k}{k} 2^k \right) t^n \tag{4}$$

3. Additional Tools

3.1. **Involutions of $\tilde{S}_n$.** We begin this section by presenting two involutions in $\tilde{S}_n$ which preserve pattern avoidance in some sense. Note that because of the nature of affine permutations, a few involutions that apply to regular permutations are no

Table 1. Affine Wilf Classes in S_3

Wilf Class	Generating Function
123	0
132, 213	$\sum_{n=2}^{\infty} t^n$
231, 312	$\sum_{n=2}^{\infty} \binom{2n-1}{n} t^n$

Table 2. Affine Wilf Classes in S_4

Wilf Class	Generating Function
1234	0
1243, 1324, 2134, 2143	$\sum_{n=2}^{\infty} t^n$
1342, 1423, 2314, 3124	$\sum_{n=2}^{\infty} \binom{2n-1}{n} t^n$
3142, 2413	$\mathbf{\sum_{n=2}^{\infty} \left(\sum_{k=0}^{n-1} \frac{(n-k)}{n} \binom{n-1+k}{k} 2^k \right) t^n}$
3412, 4123, 2341	$\mathbf{\sum_{n=2}^{\infty} \left(\frac{1}{3} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \right) t^n}$

longer valid. First, we consider the map from an affine permutation ω to its inverse, which is equivalent to the reflection of the graph of ω about the line $y = x$. The following lemma was proved in [1].

Lemma 3.1. *For $\omega \in \tilde{S}_n$ and $p \in S_m$, ω avoids p if and only if ω^{-1} avoids p^-1.*

Proof. Suppose $\omega(i_1), \omega(i_2), \ldots, \omega(i_m)$ is an instance of $p = p(1), \ldots, p(m)$. Fill in later. ☐

We also define the reverse complement map, $rc : \tilde{S}_n \to \tilde{S}_n$, by

$$(rc(\omega))_i = n + 1 - \omega(n + 1 - i).$$

This map essentially takes the graph of ω and rotates it 180 degrees about the origin. The next result is similar.

Lemma 3.2. *For $\omega \in \tilde{S}_n$ and $p \in S_m$, ω avoids p if and only if $rc(\omega)$ avoids $rc(p)$.*

Proof. Suppose ω contains p, that is, $\omega(i_1)\omega(i_2) \cdots \omega(i_k)$ has the same relative order as $p_1 \cdots p_k$. We prove that $rc(\omega)(2n+1-i_k)rc(\omega)(2n+1-i_{k-1}) \cdots rc(\omega)(2n+1-i_1)$ has the same relative order as $rc(p)$.

Suppose $rc(\omega)(2n+1-i_a) < rc(\omega)(2n+1-i_b)$ for some $a, b \in \{1, 2, \ldots, k\}$. Then $n+1-\omega(i_a-n) < n+1-\omega(i_b-n)$, hence $\omega(i_b-n) = \omega(i_b)-n < \omega(i_a)-n = \omega(i_a-n)$ and $\omega(i_b) < \omega(i_a)$. Now ω contains the pattern p, so $p_{i_b} < p_{i_a}$. It then follows that $rc(p)(i_a) = n + 1 - p_{i_a} < n + 1 - p_{i_b} = rc(p)(i_b)$, as desired. ☐

Example: [4, 1, 7, -2] 8, 5, 11, 2 contains 2341, so its inverse,
[2, 8, -1, 1] 6, 12, 3, 5, and its reverse complement, [7, -2, 4, 1] 11, 2, 8, 5,
both contain 4123 (the inverse and reverse complement of 2341).

These results allow for the classification of many permutations into Wilf classes fairly trivially. For example, $\tilde{S}^{13245}(t) = \tilde{S}^{12435}(t)$ by the reverse complement involution.

3.2. **Inversion Tables.** An interesting feature of affine permutations are inversion tables. For $\omega \in \tilde{S}_n$ We define $\text{Inv}_i(\omega) = \#\{j \in \mathbb{N} : i < j, \omega_i > \omega_j\}$.

The inversion table for $\omega \in \tilde{S}_n$ is defined as

$$T(\omega) = [\text{Inv}_1(\omega), \text{Inv}_2(\omega), \ldots, \text{Inv}_n(\omega)].$$

Thus, the inversion table for the affine permutation with window $[-3, 8, 0, 6, 4]$ is $[0, 6, 0, 3, 1]$.

Some study of inversion table manipulation has been done. Let $\sigma \in \tilde{S}_n$ have window $[\sigma_1, \sigma_2, \ldots, \sigma_n]$ and corresponding inversion table. Consider $\sigma' \in \tilde{S}_n$ with the relation $\sigma'_i = \sigma_{i-1} + 1$. Its window is therefore
$[\sigma_0 + 1, \sigma_1 + 1, \sigma_2 + 1, \ldots, \sigma_{n-1} + 1] = [\sigma_n - n + 1, \sigma_1 + 1, \sigma_2 + 1, \ldots, \sigma_{n-1} + 1]$. The relative order of the terms of σ is preserved in σ', and $\text{Inv}_i(\sigma') = \text{Inv}_{i-1}(\sigma)$. Effectively, the inversion table of σ' is the result of "shifting" the inversion table of σ to the right. Since the relative order is preserved, so is any pattern containment (or avoidance).

If we let $\text{inv}(\omega) = \displaystyle\sum_1^n \text{Inv}_i(\omega)$ for $\omega \in \tilde{S}_n$, we have the remarkable property that $\ell(\omega) = \text{inv}(\omega)$.

We may combine this result with the proof (by Björner and Brenti, in [2]) that there is a bijection between affine permutations and inversion tables with at least one element 0. Because the length of an affine permutation is just the sum over the elements of the inversion table, we may easily generate all affine permutations of a given length using the algorithm – an easier feat than simplifying generating expressions!

The crux of the algorithm depends on manipulations E_i and D_i. E_i corresponds to adding a minimum positive amount to the i-th element of the permutation such that it is not congruent mod n to any elements of the window to its right, and then subtracting the same amount from the necessarily unique element to its left with the same mod n. We also define D_i :

$$D_i([b_1, \ldots, b_n]) = \begin{cases} [b_1, \ldots, b_{i-1}, b_{i+1}, b_i - 1, b_{i+2}, \ldots, b_n], & \text{if } b_i > b_{i+1}, \\ [(b_1, \ldots, b_{i-1}, b_{i+1} + 1, b_i, b_{i+2}, \ldots, b_n], & \text{if } b_i \leq b_{i+1}. \end{cases}$$

It can be shown that an unsorted tuple may be sorted using a sequence of D_i moves (one way is to use an Insertion Sort-like algorithm).

Given a sorted inversion table $(0, b_2, b_3, \ldots, b_n)$, we write

$$\mathcal{C}([0, b_2, b_3, \ldots, b_n]) = E_2^{b_2} E_3^{b_3} \cdots E_n^{b_n} [1, 2, \ldots, n].$$

Björner and Brenti prove that if a sequence $D_{i_1} \cdots D_{i_m}$ sorts a given inversion table T (with at least one element 0) into a tuple $[0, b_2, b_3, \ldots, b_n]$, then T is the inversion table of $\mathcal{C}(0, b_2, b_3, \ldots, b_n) s_{i_k} s_{i_{k-1}} \cdots s_{i_1}$.

This algorithm hints at the information encoded in an inversion table. Our current methods to find and count pattern-avoiding permutations involve generating all permutations of a given length using the above algorithm, and then works wit the recovered permutations. However, it might be possible to skip the middle step and discern pattern avoidance directly from inversion tables.

Currently, the recovery of the indices of the inversions is being analyzed. If implemented correctly, this has implications for pattern avoidance. For example, if a permutation σ contains the pattern 3142, with indices $i_3, < i_1, < i_4, < i_2$ corresponding to the respective relative positions of the pattern, then the permutation must have at least two inversions i_1 and i_2 with i_3, with at least one index i_4, $i_1 < i_4 < i_2$ and i_4 not being an inversion of i_3. Because of the periodicity of σ, we may assume that if σ contains 3142, the $i_3 \in [1, n]$ and need only check the above properties for the elements of the window.

4. Progress toward Proof of Conjectures

Lemma 4.1. *Given a 3412-avoiding affine permutation $\omega \in \tilde{S}_n$, we must have $\omega(i) - \omega(j) \leq 2n$ for all $i \in \{1, 2, \ldots, n\}$ and any positive integer $j > i$.*

Proof. If $\omega(i) - \omega(j) > 2n$, $\omega(i-n)\omega(i)\omega(j)\omega(j+n)$ forms a 3412 pattern. □

Corollary 4.2. *Denoting the maximum entry in the window of a permutation ω by ω_α, it follows that $n \leq \omega_\alpha \leq 2n + \alpha - 2$ for all ω that avoid 3412.*

Proof. If $\omega_\alpha > 2n + \alpha - 2$, the smallest entry of the window to the right of ω_α, denoted ω_β must be less than or equal to $\alpha - 1$, forcing $\omega_\alpha - \omega_\beta > 2n$. Note that the upper bound is actually attained by the following affine permutation: for $i < \alpha$, define $\omega(i) = \omega_\alpha - 3n + (\alpha - i)$; for $i > \alpha$, let $\omega(i) = \omega_\alpha - n - i$. □

We may find the following alternate form of the summation in Conjecture 2.3 more useful.

Proposition 4.3.

$$\frac{1}{3}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k} = \frac{1}{3}\sum_{p+q+r=n}\left(\frac{n!}{p!q!r!}\right)^2,$$

where the second sum is taken over non-negative integers p, q, and r.

Proof. By comparing coefficients of x^k in the expansions of the left and right hand sides of $(1+x)^{2k} = (1+x)^k(1+x)^k$, we find the identity

$$\sum_{p=0}^{k}\binom{k}{p}^2 = \binom{2k}{k}.$$

Then

$$\sum_{p+q+r=n}\left(\frac{n!}{p!q!r!}\right)^2 = \sum_{k=0}^{n}\sum_{p+q=k}\left(\frac{n!}{p!q!(n-k)!}\right)^2$$
$$= \sum_{k=0}^{n}\sum_{p+q=k}\binom{n}{k}^2\left(\frac{k!}{p!q!}\right)^2 = \sum_{k=0}^{n}\binom{n}{k}^2\sum_{p+q=k}\binom{k}{p}^2$$
$$= \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}.$$

□

Other progress is related to the involutions discussed earlier. First, it is not hard to see that the reverse complement preserves length.

Lemma 4.4. *For any $a, b \in \tilde{S}_n$, $rc(ab) = rc(a)rc(b)$.*

Proof. We verify this with direct computation.

$$rc(ab)(i) = n+1-ab(n+1-i).$$

Working from the right hand side,

$$rc(a)rc(b)(i) = rc(a)(n+1-b(n+1-i)) = n+1-a(b(n+1-i)),$$

thus the expressions are equal. □

Corollary 4.5. *The reverse complement operation preserves length.*

Proof. For all generators s_i, $rc(s_i) = s_{n-i}$. Given $\omega \in \tilde{S}_n$, we can find a reduced expression for ω, say $s_{i_1}\cdots s_{i_k}$, where k is the length of ω. Applying the reverse complement and Lemma 4.4, $rc(\omega) = rc(s_{i_1}\cdots s_{i_k}) = s_{n-i_1}\cdots s_{n-i_k}$. Thus the length of $rc(\omega)$ is at most the length of ω. Applying this argument again beginning with $rc(\omega)$ shows that the length of ω is at most the length of $rc(\omega)$. Thus these lengths are equal. □

Because 3412 is both its own inverse and its own reverse complement, Lemmas 3.1 and 3.2 imply that ω avoids 3412 if and only if all three of ω^{-1}, $rc(\omega)$, and $rc(\omega^{-1})$ avoid 3412. Note that Corollary 4.2 implies $rc(\omega)^{-1} = rc(\omega^{-1})$. By then shifting the inversion table of these four permutations, up to $4n$ distinct affine permutations which avoid 3412 could be generated based on one affine permutation that avoids 3412. Because both taking the inverse and reverse complement of an affine permutation are involutions, no further information can be gleaned. These operations could still prove to be useful, however.

5. Conclusion

We have made some progress toward proving the difficult conjectures mentioned. Hopefully more results will follow.

References

[1] Crites, Andrew. "Enumerating Pattern Avoidance For Affine Permutations." arXiv:1002.1933v1. [math.CO] 9 Feb 2010
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