THE DIRICHLET BOUNDARY VALUE PROBLEM ON A FINITE CIRCULAR NETWORK

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Abstract. Here the solution to a Dirichlet boundary value problem on a specific finite circular planar network is given. The methods involved use the formulation for the Green's function on a path [1], and eigenvalues and associated eigenfunctions on the cycles [2]. We also give an expression for the normal derivative of the Green's function, which may be used to form the Poisson kernel, and with the Poisson kernel one can form the Dirichlet-Nuemann kernel.

Contents

1. The expression for the Green's function

We use the same graph as in "The Dirichlet boundary value problem on a finite circular network using equilibrium measure", except we remove the center node and all the edges that touch the center node. Agian we assume unit conductace. From [1] we have that the expression for the Green's function:

$$G((x,y),(z,t)) = \sum_{l=0}^{n} G_{\lambda_l}(y,t)u_l(z)u_l(x)$$

where u_l is the eigenfunction associated with the cycles, and G_{λ_l} is the Green's function on a path (one sided boundary). The Green's function on a path is given by the expression (which can be found in [1]):

$$G_{\lambda_l}(y_k,y_s) = \frac{1}{v_{n+1}(q)} \begin{cases} v_k(q)u_{n-s}(q) & \text{if } k < s \\ v_s(q)u_{n-k}(q) & \text{if } s \le k \end{cases}$$

where $q = \lambda_j/2 + 1$, v is the 3rd order Chebyshev polynomial, and u is the 2nd order Chebyshev polynomial.

The eigenvalues associated with a cycle with $V = \{x_1, \dots x_n\}$, $0 = \lambda_0 \le \lambda_1 \dots \le \lambda_n$ are given by:

$$\lambda_l = 4\sin^2\left(\frac{\pi l}{n}\right),\ l = 1, \dots, n-1$$

The orthonormal eigenfunctions corresponding to these eigenvalues are:

$$u_l(x_i) = \frac{a}{n}\cos\left(\frac{2\pi l(i-1)}{n}\right),\ l = 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor$$
$$u_l(x_i) = \frac{2}{n}\sin\left(\frac{2\pi l(i-1)}{n}\right),\ \left\lfloor \frac{n}{2} \right\rfloor + 1, \dots, n-1$$

where $a = 1$ if n is even and $a = 2$ otherwise. We also state that $\lambda_0 = 0$ is the smallest eigen value, and $u_0 = \frac{1}{\sqrt{n}}$ is it's associated eigenfunction.

2. The expression for the normal derivative of the Green's function

We compute the normal derivative:

$$\frac{\partial G}{\partial \eta}\Big((x,y),\cdot\Big)\Big((n,j)\Big) = G\Big((x,y),(n,j)\Big) - G\Big((x,y),(n-1,j)\Big).$$

It has been shown ([1]) that the Green's function takes the following form:

$$\sum_{l=1}^{n-1} \bar{G}_l(y,t)u_j(z)u_i(x).$$

When $l = 1, \dots, \lfloor \frac{n}{2} \rfloor$, evaluating the Green's function at (n,j) we arrive at:

$$\sum_{l=1}^{\lfloor \frac{n}{2} \rfloor} \bar{G}_l(y,j)\frac{a}{n}\cos\left(\frac{2\pi l(i-1)}{n}\right)\cos\left(\frac{2\pi l(n-1)}{n}\right).$$

Then considering when $l = \lfloor \frac{n}{2} \rfloor + 1, \ldots, n-1$, evaluating the Green's function at (n, j) yields:

$$\sum_{l=\lfloor \frac{n}{2} \rfloor+1}^{n-1} \bar{G}_l(y,j)\frac{2}{n}\sin\left(\frac{2\pi l(i-1)}{n}\right)\sin\left(\frac{2\pi l(n-1)}{n}\right).$$

Now what we have found is that $G((x,y),(n,j))$ is the following expression:

$$\sum_{l=1}^{\lfloor \frac{n}{2} \rfloor} \bar{G}_l(y,j)\frac{a}{n}\cos\left(\frac{2\pi l(i-1)}{n}\right)\cos\left(\frac{2\pi l(n-1)}{n}\right)+\sum_{\lfloor \frac{n}{2} \rfloor+1}^{n-1} \bar{G}_l(y,j)\frac{2}{n}\sin\left(\frac{2\pi l(i-1)}{n}\right)\sin\left(\frac{2\pi l(n-1)}{n}\right).$$

We are required to evaulate $G((x,y),(n-1,j))$, and then subtract the two expressions to obtain the normal derivative of the Green's function.

For $G((x,y),(n-1,j))$ we have the expression:

$$\sum_{l=1}^{\lfloor \frac{n}{2} \rfloor} \bar{G}_l(y,j)\frac{a}{n}\cos\left(\frac{2\pi l(i-1)}{n}\right)\cos\left(\frac{2\pi l(n-2)}{n}\right)+\sum_{\lfloor \frac{n}{2} \rfloor+1}^{n-1} \bar{G}_l(y,j)\frac{2}{n}\sin\left(\frac{2\pi l(i-1)}{n}\right)\sin\left(\frac{2\pi l(n-2)}{n}\right).$$

The final expression for the normal derivative of the Green's function then becomes the differene of equations (1) and (2) respectively:

$$\begin{aligned}
-2a\sum_{l=1}^{\lfloor \frac{n}{2} \rfloor} \frac{\bar{G}_l(y,j)}{n}\cos\left(\frac{2\pi l(i-1)}{n}\right)\sin\left(\frac{\pi l(2n-3)}{n}\right)\sin\left(\frac{\pi l}{n}\right)+\\
4\sum_{\lfloor \frac{n}{2} \rfloor+1}^{n-1} \frac{\bar{G}_l(y,j)}{n}\sin\left(\frac{2\pi l(i-1)}{n}\right)\cos\left(\frac{\pi l(2n-3)}{n}\right)\sin\left(\frac{\pi l}{n}\right).
\end{aligned}$$

□

Encinas [3] and Carmona [4] give a simplification of this expression that we will use. The simplification for the eigenfunctions is as follows:

For n odd we have:

$$\begin{aligned}
u_0(x_i) &= \sqrt{\frac{1}{n}}\\
u_l(x_i) &= \sqrt{\frac{2}{n}}\cos\left[\frac{2\pi(n-l)(i-1)}{n}\right]; l = 2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor\\
u_l(x_i) &= \sqrt{\frac{2}{n}}\sin\left[\frac{2\pi(n-l)(i-1)}{n}\right]; l = 2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor\\
&= \sqrt{\frac{2}{n}}\sin\left[\frac{(2\pi l)(i-1)}{n}\right]
\end{aligned}$$

For n even we have:

$$\begin{aligned}
u_0(x_i) &= \sqrt{\frac{1}{n}} \\
u_l(x_i) &= \sqrt{\frac{2}{n}}\cos\left[\frac{2\pi l(i-1)}{n}\right]; l = 2, \dots, \left\lfloor \frac{n}{2} \right\rfloor - 1 \\
u_{\lfloor \frac{n}{2} \rfloor}(x_i) &= \sqrt{\frac{1}{n}}\cos\left[\frac{\pi(i-1)}{n}\right] = (-1)^{i-1}\sqrt{\frac{1}{n}} \\
u_l(x_i) &= \sqrt{\frac{2}{n}}\sin\left[\frac{2\pi(n-l)(i-1)}{n}\right] = -\sqrt{\frac{2}{n}}\sin\left[\frac{2\pi l(i-1)}{n}\right]; l = 2, \dots, \left\lfloor \frac{n}{2} \right\rfloor - 1
\end{aligned}$$

Now we simplify the eigenvalues for n odd we have:

$$\begin{aligned}
\lambda_l &= 4\sin^2\left(\frac{l\pi}{n}\right); l = 2, \dots, \left\lfloor \frac{n}{2} \right\rfloor \\
\lambda_{n-l} &= 4\sin^2\left(\frac{\pi(n-l)}{n}\right) = 4\sin^2\left(\pi - \frac{\pi l}{n}\right) = 4\sin^2\left(\frac{\pi l}{n}\right); l = 2, \dots, \left\lfloor \frac{n}{2} \right\rfloor
\end{aligned}$$

And for n even we have:

$$\begin{aligned}
\lambda_l = \lambda_{n-l} &= 4\sin^2\left(\frac{l\pi}{n}\right); l = 2, \dots, \left\lfloor \frac{n}{2} \right\rfloor \\
\lambda_{\frac{n}{2}} &= 4
\end{aligned}$$

Next using these simplifications for the eigenfunctions and eigenvalues we may rewrite the Green's function in terms of these simplifications. For n odd the Green's function becomes:

$$G((x_i, y_h), (x_j, y_k)) = \frac{1}{n}G(y_h, y_k) + \frac{2}{n}\sum_{l=1}^{\lfloor \frac{n}{2} \rfloor} G_{\lambda_l}(y_h, y_k)\cos\left(\frac{2\pi l(i-j)}{n}\right)$$

For n even we have:

$$G((x_i, y_h), (x_j, y_k)) = \frac{1}{n}G(y_h, y_k) + \frac{(-1)^{i-j}}{n}G_4(y_h, y_k) + \frac{2}{n}\sum_{l=1}^{\lfloor \frac{n}{2} \rfloor - 1} G_{\lambda_l}(y_h, y_k)\cos\left(\frac{2\pi l(i-j)}{n}\right)$$

Now we write the Green's function for every n:

$$\begin{aligned}
G((x_i, y_h), (x_j, y_k)) = {} & \frac{1}{n}G(y_h, y_k) + \frac{2}{n}\sum_{l=1}^{\lfloor \frac{n}{2} \rfloor} G_{\lambda_l}(y_h, y_k)\cos\left(\frac{2\pi l(i-j)}{n}\right) \\
& + \frac{(-1)^{i-j+1}[1 + (-1)^n]G_4(y_h, y_k)}{2n}
\end{aligned}$$

$\square$

Now that we have the simplified form of the Green's function we may obtain the normal derivative of said function:

$$\begin{aligned}\frac{\partial}{\partial \eta} G(x_i, y_h)(x_n, y_k) &= G((x_i, y_h), (x_j, y_k)) - G((x_i, y_h), (x_{n-1}, y_k)) \\ &= \frac{2}{n} \sum_{l=1}^{\lfloor \frac{n}{2} \rfloor} G_{\lambda_l}(y_h, y_k) \left[\cos\left(\frac{2\pi l(i-n)}{n}\right) - \cos\left(\frac{2\pi l(i+1-n)}{n}\right) \right] \\ &+ \frac{[1+(-1)^n]}{n} (-1)^{i-n+1} G_4(y_h, y_k)\end{aligned}$$

If we use the trigonometric identity $\cos a - \cos b = 2\sin((a+b)/2)\sin((a-b)/2)$ we arrive at the final expression for the normal derivative is:

$$\begin{aligned}\frac{\partial}{\partial \eta} G(x_i, y_h)(x_n, y_k) &= G((x_i, y_h), (x_j, y_k)) - G((x_i, y_h), (x_{n-1}, y_k)) \\ &= \frac{4}{n} \sum_{l=1}^{\lfloor \frac{n}{2} \rfloor} G_{\lambda_l}(y_h, y_k) \sin\left(\frac{\pi l}{n}\right) \sin\left(\frac{\pi l(2i+1)}{n}\right) \\ &= (-1)^{i-1} \frac{[1+(-1)^n]}{n} G_4(y_h, y_k)\end{aligned}$$

Thus we have obtained the normal derivative of the Green's function.

□

Appendix A. REFERENCES

References

[1] E. Bendito, A. M. Encinas, A. Carmona: *Eigenvalues, eigenfunctions and Green's Functions on a Path via Chebyshev Polynomials* Appl. Anal. Discrete Math. **3** (2009), 282-302.
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