Uniqueness of Connections between Effectively Adjacent Circular Pairs

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Abstract

In this paper we investigate the sometimes pathological-seeming behavior of paths in CCCP graphs. We first demonstrate that there are in fact CCCP graphs which do not have any circular pairs which induce unique connections, something which seemed should have been true. However, in this paper we show that it is the case that if certain circular pairs which are "effectively adjacent" exist on a CCCP graph then it does in fact induce a connection which must be unique. We define and explore the conceps "minimal cycles" and "imposing sets of cycles" as well in the process, and we give some corollaries of this result and further directions for research.

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1 Introduction

Although we have seen that it is not true that there always exists unique paths for circular pairs, for special graphs for which special circular pairs we can get positive results. I use the definitions always adopted, with the notable exception that circular pair implies that there is a connection, and CCCP being Connected CCP.

- We will denote the set of vertices by V and the set of edges E, and further partition the vertices into ∂V, the set of boundary nodes, and $\text{int}V$ to be set of interior vertices.
- Two boundary nodes s, t are connected through the interior if there exists a set of edges $e_1, e_2, e_3 \cdots, e_n$ so that if $j \neq 1, n$ then e_j does not use any boundary nodes. Such a connection we call a path.
- It is rather difficult to talk about graphs without some convention of ordering. Let $\mathcal{E} = (E_1, E_2, E_3, \cdots, E_n)$, in ascending order, $S = (s_1, s_2, s_3, \cdots, s_n)$ in ascending clockwise order and $T = (t_1, t_2, t_3, \cdots, t_n)$, in counter-clockwise ascending order, such that we know just by the Jordan Curve Theorem that E_1 must be the connection between s_1 and t_1. We will consistently use this notation throughout this section.
- A circular pair (S, T) are two sets of nodes of the same cardinality such that $S, T \subset \partial V$ and $S \cap T = \phi$, S and T both form an arcs through the circle that do not intersect, and they are connected throught the interior, which means there exists a set of set of edges $\mathcal{E}$ such that the sets of edges are disjoint and for every $s_i \in S$ and $t_i \in T$ there exists a unique $E \in \mathcal{E}$ such that E forms a connection through the interior for s_i and t_i. We then denote $\mathcal{E}$ as a connection of (S, T), and the set of such connections $\mathcal{P}$. We say that (S, T) *induces* the connection $\mathcal{E}$. We also say that the connection is unique if there is only one member of the set $\mathcal{P}$. I use this definition because I only care about connected circular pairs for my spiel
- We then define a maximal circular pair to be a circular pair (S, T) such that if s' is any boundary node adjacent to S and t' is any boundary node adjacent to T, then $(S \cup s', T \cup t')$ is not a circular pair. We will omit the maximal part and assume that the circular pairs we talk about are maximal unless explicitly stated otherwise.

Now these are slightly non-standard definitions which are worth emphasizing a little more and therefore not putting in a list.

Definition Two boundary nodes $s_1, s_2 \in S$ in S *surround* $n \in \partial V$ with respect to S if by drawing an arc A from s_1 to s_k such that all the s_j's are on the arc, we can form a sub-arc $A' \in A$ such that s_1 and s_2 are on the boundary of that arc with respect to the embedding circle and $n \in A'$ as well. We do not consider s_1 and s_2 to be surrounded by s_1 and s_2 with respect to S.

Definition An *adjacent* circular pair (S, T) of a circular planar graph G is a circular pair for all adjacent nodes $s_i, s_{i+1} \in S$,(resp. $t_j, t_{j+1} \in T$) the only boundary nodes surrounded by s_i and s_{i+1} with respect to S (resp. t_j and t_{j+1} with respect to T) are s_i and s_{i+1} (resp. t_j and t_{j+1}) themselves.

Definition A *effectively adjacent* circular pair (S, T) of a circular planar graph G is a circular pair such that for all paths $\mathcal{P} = p_1, p_2, \cdots, p_k$, for all adjacent nodes $s_i, s_{i+1} \in S$ and $t_j, t_{j+1} \in T$, the paths p_i and p_{i+1} (resp. paths p_j and p_{j+1}) use all nodes (interior and boundary) which any boundary node n surrounded by s_i and s_{i+1} with respect to S (resp. surrounded by t_j and t_{j+1} with respect to T) has an edge to.

Definition A *open subgraph* G' of a graph G is a subgraph with boundary such that if $v \in \text{int}G'$, then all the neighbors of v in G are in G'.

Will noticed that if G is recoverable, then imagining G' as a little black box inside G, it must follow that G' is recoverable as well. Peter has some more technical proof but this makes sense and Will's never been wrong before.

It will also be useful to notate consistently cycles within graphs, as non-unique paths can only exist if there are cycles present. We will consider cycles to be subgraphs without boundary. Suppose E_k has a cycle for which it is the lowest bound. Then we will associate that cycle below with E_k, and if E_k is the upper bound of a cycle, we will associate that cycle above with E_k. Perhaps with a slight abuse of notation, we will use $C \cap E_k$ to denote the largest subgraph of C that is in E_k, and $C \setminus E_k$ to be the largest subgraph of C which does not intersect E_k.

In section 2 we demonstrate rather briefly that there is a CCP graph which does not induce unique connections. In section 3 we develop the tools necessary for the proof for the claim that all effectively adjacent circular pairs induce unique connections which use all of thei interior nodes given in this paper, while in section 3 we proceed with the proof for well-connected graphs and the inductive step, and in section 5 we discuss easy implications of this result and possible future research.

2 Non-Unique Connections on CCP Graphs

In the spirit of bad new first, we present this following theorem which I hoped to be false.

Theorem 2.1 *There exist critical circular planar graphs which do not have any circular pairs which admit unique connections.*

Proof We can directly produce a counter-example: take the rectangular graph with five vertical boundary spikes and five horizontal boundary spikes and contract the middle boundary spikes on the top, bottom, left and right. We will call this grid with contracted spikes G_c. Because we could create this graph bounary spike contraction from a critical graph, it is critical as well. We will show that there is no maximal circular pair which admits a unique connection on this graph. The most straightforward way to do this is to notice that the four interior nodes circled in Figure 2 are "disconnecting;" that is, if we delete any one of these interior nodes, it disconnects the interior completely.

We define the four quadrants of boundary nodes; that is, the four on the top left, the four on the top right, etc., which do not include the middle boundary

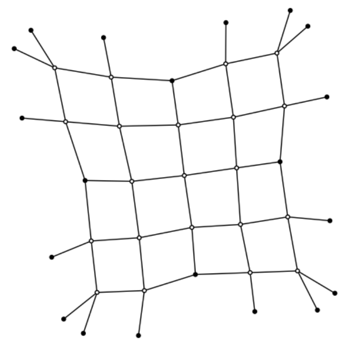

Figure 1: G_c with boundary nodes colored in and interior nodes hollow

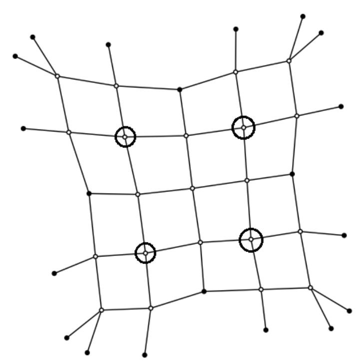

Figure 2: G_c with four disconnecting interior nodes circled

nodes on the top, left, bottom, or right. Furthermore, we define the middle boundary nodes to be the ones in between the quadrants. Consider a circular pair (S, T) for which we need to connect one boundary node v_1 in one quadrant Q_1 to another boundary node v_2 in a different quadrant Q_2. Because those four nodes are disconnecting these quadrants from each other, the path which connects v_1 to v_2 must go through two of those four nodes; furthermore, if there were no other paths starting or ending in either of those quadrants, it is evident that this path cannot be unique. Therefore, (S, T) must have at least one path starting or ending in Q_1 and starting or ending in Q_2; however, then as the connection from v_1 to v_2 used the disconnecting nodes for Q_1 and Q_2, these other paths can at most start and end at the middle boundary nodes for that quadrant, but then (S, T) cannot be a circular pair.

Therefore no circular pair which admits an unique connection can have a path which must start and stop in different quadrants. Now consider the circular pairs which have paths which start at middle boundary nodes, and assume without loss of generality we are dealing with a path which starts at the top middle boundary node. If it had to terminate at a boundary node at the bottom, then it could potentially pass through both top disconnecting nodes, so paths which use those disconnecting nodes or other nodes in the part which would be disconnected by removing the disconnecting nodes must exist. These nodes cannot be forced to pass below the top half of the graph, as they cannot terminate below the middle nodes by above, and if they terminate at the middle nodes there is no way the path could be forced to descend below. But then we see we cannot add any paths which start and end in the bottom without breaking circularity, so the path starting at the top middle node will be non-unique.

Suppose it must terminate at a boundary node in the upper quadrants or the left central or right central boundary node. This path would if unfettered be able to go through the disconnecting nodes for the upper left and the upper right quadrants, so clearly some other paths must prevent that from happening. The path on the left msut start in the top left quadrant, so can at most terminate at the left-middle node, and similarly for the path on the right. But this means that without breaking circularity we cannot add any paths starting and stopping in the bottom half of the graph, so the bottom-most path starting or stopping in the top-left hand quadrant can always have an alternate path.

Thus, the only circular pairs which can admit unique connections have paths which all start and stop in the same quadrant. However, that means paths can only involve two of the quadrants, but we can see that then we can add another pair of boundary nodes; namely one middle node between the two quadrants and the other middle node adjacent to one of the quadrants while maintaing circularity and still having a path; therefore this final type of circular pair is not maximal, and to make it maximal we must make the paths it induces to be non-unique. ∎

3 Minimal Cycles and Imposing Sets

The more annoying type of cycle for our purposes will be the type of cycle C associated with E_k such that $C \backslash E_k$ is not part of any other cycle, as it introduces no new nodes. We will call this type of cycle a *minimal cycle* associated with the connection $\mathcal{C}$ and we require some ideas which in conjunction allow us to ignore them.

Definition Let (S, T) be a maximal circular pair which induces a connection $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$. Then a cycle K_1 whose lower bound is C_k will *impose* a cycle K_2 whose upper bound is C_k if $K_2 \cap C_k \subset K_1 \cap C_k$, and a set of cycles $\{K_r, K_{r+1}, \ldots, K_p\}$ in ascending order are *imposing* if the upper bound of K_i is C_{i+1} which is the lower bound of K_{i+1} and K_{i+1} imposes on K_i for all $r \ leqi \leq p-1$, and K_p is minimal with respect to C_p.

Lemma 3.1 *Suppose there is a set of cycles $\mathcal{K} = \{K_r, K_2, \ldots, K_p\}$ which are emposing on a connection $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ for an effectively adjacent maximal circular pair (S, T). Then it cannot be that there is a boundary node $n \notin S$ surrounded by nodes in S with respect to S (or in T with respect to T, but it is symmetric) such that it has an edge to $K_i \cap C_{i-1}$.*

Proof Suppose such a node existed between C_{j-1} and C_j. Then, for all $p \geq i \geq j$, take C_i until it hits K_i, at which point follow K_i; these paths are disjoint with each other and the original paths which we did not change because $\mathcal{K}$ is imposing; and furthermore, this connection allows n to have an edge to an node not used in the connection from (S, T); hence (S, T) is not effectively adjacent, which contradicts the hypothesis. ∎

Similarly, we can also easily prove the following lemma, along the same lines, so we omit the proof.

Lemma 3.2 *Suppose there is a set of cycles $\mathcal{K} = \{K_1, K_2, \ldots, K_p\}$ which are emposing on a connection $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ for a maximal circular pair (S, T). Then it cannot be that there is a boundary node $n \notin S, T$ between S and T which has a connection to another boundary node $n' \notin S, T$ which requires only nodes not used in $\mathcal{C}$ and $K_1 \cap C_1$.*

These are all the tools necessary for our proof.

4 Proof by Hugging Paths

The idea of a hugging path I think goes back to Tom. It is constructed for our purposes thusly: given an effectively adjacent circular pair (S, T), we will construct the *bottom hugging connection* by, after adopting the conventions from section 1, for s_1 to t_1, take the path such that if we remove all the nodes and edges above this path, then there is no path from s_1 to t_1 through the interior. Then, we proceed downwards: after having constructed the path from s_i to t_i, take the path from s_{i+1} to t_{i+1} such that if we remove all the nodes below and including the path from s_i to t_i previously constructed, and all the edges that lead into it, as well as all the nodes above the path from s_{i+1} to t_{i+1}, there is no other path through the interior from s_{i+1} to t_{i+1}. Each of these we will call a *bottom hugging path*. It is quite evident that this notion is well-defined.

Now we show that there cannot be any minimal cycles on this path.

Lemma 4.1 *If (S,T) is an effectively adjacent circular pair on a CCCP graph G, then there exist no minimal cycles associated with the bottom hugging connection embedded below any of the bottom hugging paths.*

Proof Take the bottommost instance such a minimal cycle could appear. Then the bottom hugging connection would have used that, so it could not have been a bottom hugging connection. ∎

Although this may seem obvious, this is the key to the next, more difficult, lemma, about minimal cycles on the other side.

Lemma 4.2 *If (S,T) is an effectively adjacent circular pair on a CCCP graph with convention as above, then if $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ is the bottom hugging connection in ascending order, then there exist no minimal cycles embedded above C_j, for any $1 \geq j \geq n$.*

Proof Suppose there was a cycle K_j embedded above C_j. By criticality, there exists some other circular pair (S', T') which induces a connection $\mathcal{C}' = \{C'_1, C'_2, \ldots, C'_\nu\}$ which must use $K_j \setminus C_j$. Denote the path which goes through $K_j \setminus C_j$ C'_{m_j}. Then another path $C'_{m_{j-1}}$ must have forced it to use this minimal cycle by using nodes in $K_j \cap C_j$. This path must intersect C_{j-1} to enter $C_{m_j} \cap C_j$: if it starts above or at s_j, it cannot use a minimal cycle on C_j to enter $K_j \cap C_j$ as then by the Jordan Curve Theorem that minimal cycle would have to be embedded below C_j, which by 4.1 cannot happen. If it starts at a node surrounded by s_j and s_{j-1} with respect to S then again by 4.1 it cannot use a minimal cycle to enter, and if it had an edge directly into $K_j \cap C_j$, by treating $\{K_j\}$ as an imposing set, by 3.1 this cannot happen, and if it starts at or below s_{j-1} again by 4.1 it cannot enter by a minimal cycle. By this same logic, to exit $K_j \cap C_j$, it must intersect C_{j-1}, and at a different node. Now we will begin to construct an imposing set, with K_j being the cap. The next one starts where $C'_{m_{j-1}}$ intersects C_{j-1} to enter $K_j \cap C_j$, (which we just showed must happen) and is composed of the path on C_{j-1} between the two nodes used to enter and exit $K_j \cap C_j$, as well as the path that $C_{m_{j-1}}$ takes after it enters $K_j \cap C_j$ until it exits, and denote this K_{j-1}. This is imposed on by K_j by construction.

We now proceed inductively downwards. Suppose we have made this imposing set capped by K_j and so far with its bottom at K_k, which is bounded above by C_k. Suppose also that there is a path C'_{m_k} which must use $K_k \setminus C_k$. Then we will show there is another one so embedded below C_k. We have already shown the base case. Now now show the inductively case. The only reason why C'_{m_k} could not have used $K_k \cap C_k$ is because the next one $C'_{m_{k-1}}$ must have used some nodes in it. By the same logic as above if it starts at or above s_k or at or below s_{k-1} because there are no minimal cycles below C_k it must enter $K_k \cap C_k$ by C_{k-1}; if it starts embedded between s_k and s_{k+1} by again this same logic it could not use a minimal cycle to enter, and by 3.1 it could not use an edge directly into $K_k \cap C_k$, so it too must intersect C_{k-1} once to enter, and again once to exit, so constructing the cycle as we did before, we see that we have proven the inductive step.

Hence, by this logic, there is an imposing set $Q = \{K_1, K_2, \ldots, K_j\}$ which is capped by K_j, and whose bottom-most member is bounded below by C_1, and C'_{m_1} must use $K_1 \setminus C_1$. The only reason it must use that is that part of $K_1 \cap C_1$ must be used by another path C'_f. This new path cannot start at or above s_1, as then it would have to use a minimal cycle below C_1 to enter $K_1 \cap C_1$, which it can't. It must therefore start and end (the logic is symmetric) below s_1 and t_1, respectively. Furthermore, C'_f cannot intersect C_1 at any point outside of $K_1 \cap C_1$ because again, if it did, it would have to use a minimal cycle to enter $K_1 \cap C_1$. Hence by 3.2 (S, T) cannot be maximal, which is a contradiction. ∎

Now as there cannot be any minimal cycles, we have reduced the pathologies which can occur on these hugging paths.

Corollary 4.3 *If (S, T) is an effectively adjacent circular pair on a CCCP graph, then the bottom hugging connection is the only connection between S and T.*

Proof If there is any other path, by taking the symmetric difference between the two we see that this can only occur if this new path uses cycles. There are no minimal cycles, so all these cycles must intersect at least two bottom hugging paths. In particular, there are no cycles below the bottom-most hugging path and no cycles above the topmost bottom hugging path, so the alternate path from s_1 to t_1 must intersect C_2, so s_2 to s_2 in this alternate connection must intersect C_3, and continuing inductively s_{n-1} to t_{n-1} must intersect s_n to t_n, so the alternate path from s_n to t_n must use some cycle above that, which does not exist. ∎

Hence we have uniqueness of connections. Now it is easy to show that every interior node must be used.

Corollary 4.4 *If (S, T) is an effectively adjacent circular pair on a CCCP graph then the unique connection uses every interior node.*

Proof Suppose there are some nodes not used by that connection. By partitioning the graph into pieces that are between the paths used by $\mathcal{E}$, for each j create the open subgraph whose interior nodes are the interior nodes bounded and not used by E_j and E_{j+1} and which does not have any boundary-to-boundary edges. Call that N_j, with N_n being the open subgraph whose interior vertices are all which are above E_n, and N_0 the open subgraph whose interior vertices are all below E_1, both times excluding boundary-to-boundary nodes. By considering connected subgraphs of this open subgraph N_j we will assume that N_j is connected.

If $j \neq 1, n$, we know that N_j must have at least two boundary nodes, otherwise there would be a resistor attached to nothing which we know is not recoverable. Because the maximal pair is effectively adjacent, N_j cannot connect to any boundary nodes in G other than possibly the ones in E_j and E_{j+1}. If N joins E_j twice, then there is a cycle of nodes not used by $\mathcal{E}$, so the connection is not unique, so N must join E_j once and E_{j+1} once. Furthermore, we know that it cannot join E_j or E_{j+1} anywhere else for again we would have a cycle (as N_j is connected and has no boundary-to-boundary edges) which would imply non-uniqueness, which by 4.3 cannot happen, hence N_j has at exactly two boundary nodes. But as N_j is recoverable and the only critical graphs with only two boundary nodes have only two nodes total. Therefore N_j cannot have any interior nodes at all, which contradicts the construction.

If $j = 0, n$, note that no two boundary vertices not in S or T can be connected through N_j by maximality, so we can assume by taking connected subgraphs that N_j only uses one of the boundary nodes not in (S, T). Now if N_j had three boundary vertices, again there would not be a unique connection, so it can have at most two, and again this implies that N_j has no interior vertices, so the partitioning (that is, the path from (S, T)) must use all the interior vertices. ∎

5 Conclusion

There is an obvious consequence of this, which is what I had wanted to show initially.

Corollary 5.1 *All adjacent circular pairs induce unique connections which use all the interior nodes.*

Proof All adjacent circular pairs are effectively adjacent. ∎

Furthermore, we can finally show that minimal cycles, which cause so much pain, do not exist on effectively adjacent circular pairs.

These two corollaries were very cathartic for the author. It is unclear at the moment exactly what implications this has for recoverability, still, it is an interesting result. It is unclear exactly how to generalize these results. The author would be curious if there were a condition on critical graphs which would guarantee the existance of a unique connection; this remains to be seen. If it is a condition on the existance of circular pairs it must be stronger than just effective adjacency; many graphs for instance that are Y-Δ equivalent to those with effectively adjacent circular pairs only have circular pairs whcih are not effectively adjacent but do induce unique connections.

Another possible area of research would be non-circular pairs; this seems much harder as the crucial tool of the Jordan Curve Theorem goes away. Naive suggestions of generalizing this notion hugging paths which literally just hug the top do not work unfortunately to produce nice behaving connections, and it is unclear at least to the author how then one should proceed.