Characteristic Polynomial

Note Title 7/15/2009

Let A be an $n\times n$ matrix. $(A=(a_{ij}))$

Notation: Let $I=(i_1,i_2,\dots,i_k)$, $1\le i_1<i_2<\dots<i_k\le n$ be a k-multiindex. $|I|=k$

$A(I)$ is the $k\times k$ (sub) matrix consisting of the rows and columns indexed by I.

$A[I]$ is the $(n-k)\times(n-k)$ (sub) matrix with rows and columns I omitted

Thm: Let $p(\lambda)=|\lambda I-A|$.

Then $$p(\lambda)=\lambda^n-\left(\sum_{i=1}^n a_{ii}\right)\lambda^{n-1}+\dots+(-1)^k\sum_{|I|=k}|A(I)|\lambda^{n-k}+\dots$$
$$+(-1)^{n-1}\sum_{i=1}^n|A[i]|\lambda+(-1)^n|A|$$

Remark: $$\sum_{|I|=k}|A(I)|=\sum_{|J|=n-k}|A[J]|.$$

Proof. Let $p(\lambda)=\lambda^n+p_{n-1}\lambda^{n-1}+\dots+p_2\lambda^2+p_1\lambda+p_0$

$$p_0=p(0)=|-A|=(-1)^n|A|$$

The proof will go by induction.

(1) $p'(\lambda)=n\lambda^{n-1}+(n-1)p_{n-1}\lambda^{n-2}+\dots+2p_2\lambda+p_1$

(2) $p'(\lambda) = \frac{d}{d\lambda}|\lambda I - A| = \sum_{i=1}^{n} |(\lambda I - A)[i]|$

by the product rule (or chain rule). (Or directly by computing.)

Hence $p_1 = p'(0) = \sum (-1)^{n-1} |A[i]|$, immediately verifying the p_1 term.

By induction, each coefficient of λ^k in $|(\lambda I - A)[i]|$ is the (signed) sum of the principal minors of the form $|A[J_i]|$, where $|J_i| = k+1$ and includes i. Each such J_i occurs $k+1$ different times in the sum (2). So the coefficient of λ^k in $p'(\lambda)$ is $(k+1) \sum_{|J|=k+1} |A[J]|$, up to sign.

Using (1) gives the result.