THE CHARACTERISTIC POLYNOMIAL OF AN ELECTRICAL NETWORK

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Abstract. We discuss random stuff about characteristic polynomials.

Contents

1. Introduction

Consider an electrical network $\Gamma = (G, \gamma)$, where G = (V,E) is a graph with the vertices partitioned into two sets, called $intV$ and ∂V respectively. We number the vertices starting at 1 such that the all the boundary vertices are numbered before any of the interior vertices. Let $\gamma : E \to \mathbb{R}$ be termed the conductivity of an edge where γ_{ij} is the conductivity of the edge between vertex i and vertex j. We create a matrix K such that:

$$K(i,j) = \left\{ \begin{array}{ll} -\gamma(i,j) & \text{if there is an edge from i to j} \\ 0 & \text{if no such edge exists} \\ \sum_{i \neq j} \gamma_{ij} & \text{i=j} \end{array} \right\}$$

This is called the Kirkhoff matrix of Γ. It's useful to write the Kirchhoff matrix in the following block form:

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

This is done such that all the indices in the A block consist of of boundary vertices and all the indices in the C block consist of interior vertices.

This paper will discuss multiple aspects of the characteristic polynomial of K and related matrices. In doing so, we will frequently be using results from Matthew Lewandowski's paper **??**on tree diagrams.

Definition 1.1. A tree diagram T, is a forest that spans the interior of a network such that each component contains at most 1 node in ∂V. We will refer to T interchangeably as both the tree digram itself and its edge set. The value of a tree diagram is $val(T) = \prod_{e \in T} \gamma(e)$

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In that paper he proves the following theorem which will prove useful in this discussion.

Theorem 1.2. *Let Ω be the set of all tree diagrams of an electrical network. If $K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ is the Kirkhoff matrix of that electrical network, then:*

$$\sum_{T\in\Omega} val(T) \tag{1}$$

2. Characteristic Polynomials and Principle Proper Submatrices

In this section we relate the characteristic polynomial and related polynomials in any number of variables to the principle proper submatrices of a matrix. The following results will apply to any matrix, making no assumptions about its form. If M is a matrix then for $A \subseteq \{1, 2, ..., |M|\}$, $M(A)$ is the principle proper submatrix obtained by deleting the rows and columns associated with the indices in A from M. Similarly, $M[A]$ is obtained by deleting all rows and columns not in A from M. The same can be used analagously for a single integer a, i.e $M(a)$ is obtained by deleting the row and column associated with a.

Lemma 2.1. *Let M be a matrix with real valued entries. Let $A \subseteq \{1, 2, ..., |M|\}$. Let I_A be a matrix where off diagonal entries are 0 and diagonal entries are 1 if their index is in A and 0 otherwise. Define the function $f_A(\lambda) = det|M - \lambda I_A|$. In the case that A consists of all the indices of M we have $f(\lambda) = f_A(\lambda)$ is the characteristic polynomial of M. Then*

$$f'_A(\lambda) = \sum_{i\in A} det|M(i) - \lambda I_M(i)| \tag{2}$$

Proof. □

Theorem 2.2. *Let M be a matrix and $A_1, ..., A_n$ be disjoint subsets of $\{1, 2, ..., |M|\}$ Define $f_{A_1,...,A_n}(\lambda_1, ..., \lambda_n) = det|M - \lambda_1 I_{A_1} - ... - \lambda_n I_{A_n}|$. Let B_{a_k} be an a_k element subset of A_k. Let $B = \bigcup B_{a_k}$. Let $\Theta_{a_1,...a_n}$ be the set of all such B for the exponents $a_1, ...a_n$. Then the coefficient of the term $\lambda_1^{a_1}...\lambda_n^{a_n}$ in $f_{A_1,...,A_n}(\lambda_1, ..., \lambda_n)$ is $\sum_{B\in\Theta_{a_1,...a_n}} det|M[B]|$.*

Proof. □

This gives a nice result for the characteristic polynomial.

Corollary 2.3. *If $A = \{1, 2, ..., |M|\}$ then*

$$f(\lambda) = \sum_{n=0}^{n=|M|} \sum_{B\subset A, |B|=n} det|M[B]| \tag{3}$$

This gives a nice interpretation of the characteristic polynomial of C using tree diagrams. We can think of deleting a row and column from C as turning that node into a boundary node. Thus, the coefficient of λ^n can now be seen to be the sum of all tree diagrams resulting from graphs with n interior nodes converted into boundary nodes.

3. Tree Diagrams and the Characteristic Polynomial of the Kirkhoff Matrix

Now we interpret the coefficients of the characteristic polynomial of the Kirkhoff Matrix in terms of Lewandowski's tree diagrams. In order to do this, consider a network Γ. Associate with it a network Γ', which is constructed by letting its interior contain all the vertices in Γ and adding boundary spikes with edge weights $-\lambda$. Clearly if $K' = K = \begin{bmatrix} A' & B' \\ B'^T & C' \end{bmatrix}$, then $detC' = f(\lambda)$, in other words, the determinant of the principle proper submatrix of Γ' associated with its interior is equal to the characteristic polynomial of Γ.

Theorem 3.1. *Consider the characteristic polynomial $f(\lambda)$ of the Kirkhoff matrix of a network Γ. Let $comp(T)$ be the number of components in T. Then the coefficient of λ^k in $f(\lambda)$ is*

$$\sum_{T \in \Omega, comp(T)=k} g(T)val(T)$$

. Where $g(T)$ is the product of the component sizes of T.

Proof. □

An important idea will be to think of the characteristic polynomial as a polynomial not in 1 variable, but as a polynomial in $|E| + 1$ variables, where we use the conductivity of each variable as a seperate indeterminate. The following results utilize this construction.

Lemma 3.2. *Consider a subset F of E consisting of i edges $e_1, ..., e_i$. If the characteristic polynomial (in the edges and λ) has degree n (in λ) then the m edges form a star if and only if the coefficient of $e_1, ..., e_i, \lambda^{n-i}$ is $n - i + 1$, and the coefficient of any j element subset of F multiplied by λ^{n-j} is $n - j + 1$.*

Proof. □

Corollary 3.3. *The n-star with edges $e_{1,2}, ..., e_n$ has characteristic polynomial*

$$f(\lambda) = \sum_{i=0}^{i=n} \sum_{F \subset E, |F|=i} \prod_{e \in F} \gamma(e)$$

Theorem 3.4. *The characteristic polynomial of the Kirkhoff Matrix with all conductivities given as indeterminates contains enough information to recover the underlying graph.*

4. Characteristic Polynomials of Response Matrices

The response matrix of an electrical network is found by taking the Schur complement, i.e $\Lambda = A - BC^-1B^T$. (Derivation of response matrix equation, comment on inapplicability of the results of previous section, interpretations as tree diagrams)

5. Conclusion

Conclusion goes here.

References

[1] Matthew Lewandowski. "The Determinant of a Principle Proper Submatrix of a Kirkhoff Matrix" 2003.

[2] Burtis, B., and James A. Morrow. "Inverse Problems for Electrical Networks." Series on applied mathematics – Vol. 13. World Scientific, ©2000.

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