· Quick refresher — rigid homology of hush complexes

Definition: n-hush
there is a combinatorial description of the n-hush which is completely accurate but completely opaque — it is in the paper
for $n=0$, the 0-hush is just a point
for $n>0$, the n-hush is made up of $n+1$ $n-1$-hushs whose faces are identified in a certain canonical way together with an extra node which is connected to all the corners

Pictures:

the 0-hush A_0

the 1-hush A_1

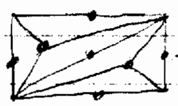
the 2-hush A_2

the 3-hush A_3

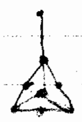
the ith face of an n-hush is the map $\partial^i: A_{n-1} \longrightarrow A_n$ which maps the n-1-hush to the n-1 hush opposite the ith corner

Definition: hush complex
there is an explicit setwise definition of a hush complex which is completely accurate but again completely opaque — it too is in the paper
informally, a hush complex is a collection of hushs of various dimension (up to some N) some of whose faces of equal dimension are identified

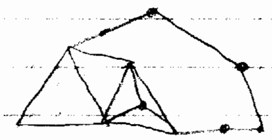
Examples:



: G_1



: G_2



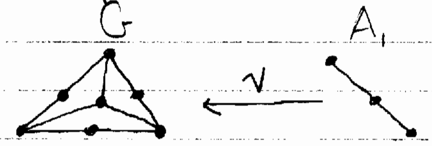
: G_3

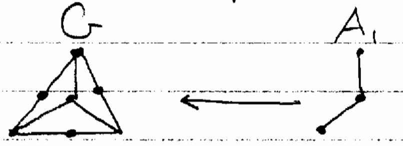
Definition: rigid n-hush in a hush complex G
a rigid n-hush is a canonical inclusion map $\gamma: A_n \longrightarrow X$ which maps the n-hush onto one of the building blocks

Definition: rigid n-hush (contd.)

or alternatively onto the face of one of the building bloc[ks]

Example: Non-example:





we denote the set of rigid n-hushs by $A_n^R(G)$

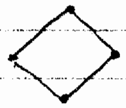
Definition: rigid chain complex of a hush complex G

The sequence of groups $C_*^R(G) = (C_n^R(G))_{n\in\mathbb{N}}$ where $C_n^R(G) = \langle A_n^R(G)\rangle$ equipped with the boundary map $\partial : C_n^R(G) \to C_{n-1}^R(G)$ which is the alternating sum of the faces and is given on basis elements by $\partial f = \sum (-1)^i f \circ \partial^i$

Definition: rigid homology of a hush complex G

the sequence of abelian groups $H_*^R(G) = (H_n^R(G))_{n\in\mathbb{N}}$ where $H_n^R(G) = Z_n^R(G)/B_n^R(G)$

Examples:

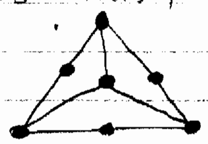
G:
G: 
G: 

$H_*^R(G) = (\mathbb{Z}, 0, \dots)$ $H_*^R(G) = (\mathbb{Z}, \mathbb{Z}, 0, \dots)$ $H_*^R(G) = (\mathbb{Z}, \mathbb{Z}\oplus\mathbb{Z}, \mathbb{Z}, 0, \dots$

Discussion

rigid homology has serious shortcomings: in particular it is only defined on hush complexes which are very special kinds of graphs. we would like to define a homology theory which uses the ideas of hushs (without using them explicitly) to notice topological properties in certain dimension[s]

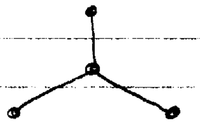
to do this, we look at a 2-hush:



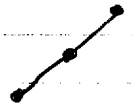
we notice that it has 3 corners and that there

Discussion: contd

is a tree contained in the hush which contains all 3 namely



we notice that the same is true for the faces of the hush; so, for example, consider the top and left nodes — there is a tree in the hush that connects them, namely



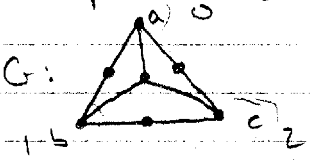
this motivates the following definition

Definition: singular n-hush

there is an explicit definition of an n-hush which requires a great deal of unpacking to do — it is in the paper

informally, a singular n-hush is a collection of trees on subsets of a set of select vertices Σ; in detail, there is an onto map $\text{Index}: \{0, \dots, n\} \longrightarrow \Sigma$ which we think of as labelling the vertices; then for each subset S of Σ we have a tree on S

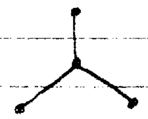
Example: a singular 3-hush

G: 

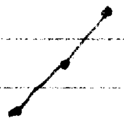
$\Sigma = \{a, b, c\}$

we have a tree for each subset $S \subset \Sigma$

for $S = \{a, b, c\}$, for $S = \{b, c\}$, for $S = \{a, c\}$

for $S = \{a, b\}$ for $S = \{a\}$, for $S = \{b\}$ for $S = \{c\}$

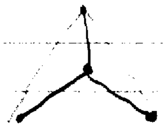
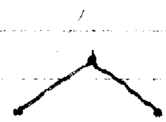
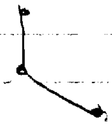


Example: a singular 4-hush

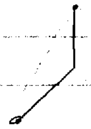
G as before 

$\Sigma = \{a, b, c\}$

for $S = \{a, b, c\}$, for $S = \{b, c\}$, for $S = \{a, c\}$

for $S = \{a, b\}$, for $S = \{a\}$, for $S = \{b\}$, for $S = \{c\}$

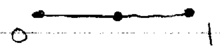
Definition: ith face of a singular n-hush

Formally, the ith face is the singular n-1-hush $\partial_i \Lambda$ given in the natural way

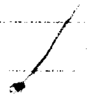
Example:

the 0th face of the first example

$\Sigma' = \{b, c\}$



for $S = \{b, c\}$, for $S = \{b\}$, for $S = \{c\}$

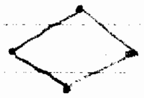
We can now define the chain complexes $C^S_*(G)$ for an arbitrary graph G and then define homology.
We denote by $A^S_n(G)$ the set of singular n-hushs in G.

Definition: singular chain complex of a graph G

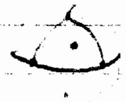
The sequence $C^S_*(G) = (C^S_n(G))_{n \in \mathbb{N}}$ of abelian groups, where $C^S_n(G) = \langle A^S_n(G) \rangle$ is the free abelian generated by all the singular n-hushs of G, equipped with the boundary map $\partial : C^S_n(G) \to C^S_{n-1}(G)$ which is the alternating sum of face maps given on basis elements by $\partial \Lambda = \sum (-1)^i \partial_i \Lambda$

Definition: singular homology of a graph G
the homology $H^s_*(G) = H_*(C^s_*(G))$ of the singular chain complex

Example:

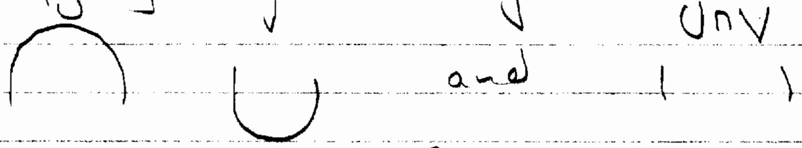
G: • G: 

$H^s_*(G) = (\mathbb{Z}, 0, \ldots)$ $H^s_*(G) = (\mathbb{Z}, \mathbb{Z}, 0, \ldots)$

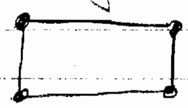
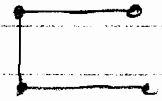
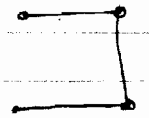
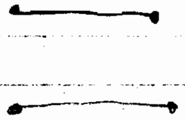
G: 

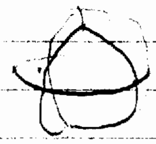
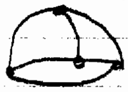
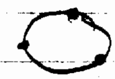
$H^s_*(G) = (\mathbb{Z}, 0, \mathbb{Z}, 0, \ldots)$

Discussion: Mayer-Vietoris
In general, these things are hard to compute since there are singular husks of all-dimensions so the chain group is non-trivial in every dimension.
In the case of singular homology, there is a result known as Mayer-Vietoris which allows us to compute the homology of a space by computing the homology of pieces. For example we can compute the homology of a circle by computing the homologies of the following spaces:



There are examples of this being possible with certain simple graphs:

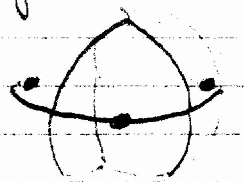
G:  U:  V:  U∩V: 

G:  U:  V:  U∩V 

However, I have not proven this in general. It is hard to see how to use the technique used to

Discussion: contd.
prove Mayer-Vietoris in the case of topological spaces which proceeds by barycentric subdivision - a process which requires "continuous space".

Discussion: connection between singular homology and rigid homology in the category of hush complexes
It would be nice to find that, for every hush complex G, $H_*^S(G) \cong H_*^R(G)$ - that is the rigid homology is isomorphic to the singular homology.
There are lots of examples of this, but I have not proven it. For example, the singular homology of the n-hush is trivial. As another example, the singular homology of the following graph

G: 

is the same as the rigid homology (in particular $H_*^R(G) = H_*^S(G) = (\mathbb{Z}, 0, \mathbb{Z}, 0, \dots)$

Fin.