Discussion: motivation for singular homology

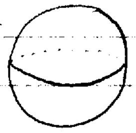
Idea: locate "Holes"

Examples

distinguish between topological spaces - tell how many "holes" of various dimensions are in the space and "what kind" of holes they are

examples:

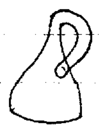
S_2



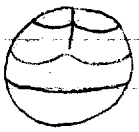
$T_{1,1}$

$H_*^S(S_2) = (\mathbb{Z}, 0, \mathbb{Z}, 0, ..)$ $\quad H_*^S(T_{1,1}) = (\mathbb{Z}, \mathbb{Z}\oplus\mathbb{Z}, \mathbb{Z}, 0, ...)$

K



RP2



$H_*^S(K) = (\mathbb{Z}, \mathbb{Z}\oplus\mathbb{Z}/2, 0, ...)$ $\quad H_*^S(\mathbb{R}P2) = (\mathbb{Z}, \mathbb{Z}/2, 0, ...)$

"loops" "Discs" generators

these are computed by looking at "loops" in the nth dimension which do not arise as the boundary of n+1 dimensional 'discs'

for instance, consider $\mathbb{R}^3$ and $\mathbb{R}^3 - D^3$

$\mathbb{R}^3, \mathbb{R}^3 - D^3$, $2S^2$ as loop generator

the surface $2S^2$ is a 2-dimensional loop in both spaces; however in $\mathbb{R}^3$ it is the boundary of a 3 disc but in $\mathbb{R}^3 - D^3$ it is not the boundary of a 3 disc because the region inside $2S^2$ is not a disc so this surface corresponds to a homology generator in $\mathbb{R}^3 - S^2$ and does not in $\mathbb{R}^3$

twisting, types of generators

the sort of abelian group for which the loop which is not a boundary of a disc is a generator is determined by the twisting that goes on in the space

example: RP2 via gluing

for instance, consider RP2 constructed as follows: glue the 2 disc into the 1 sphere (i.e. the circle) so that the the boundary of the disc traverses the circle twice

then the loop that is the circle does not bound a disc and so is a homology generator; however the loop which goes twice around the circle bounds the disc we glued in and so the loop which traverses the circle once has order 2 define n-simplex

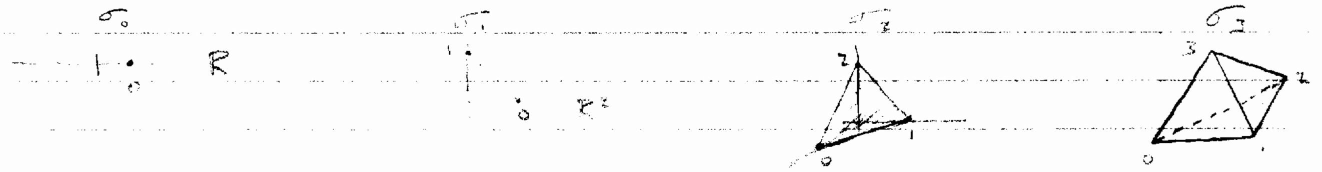
Definition: n-simplex

the subset σ_n of $\mathbb{R}^{n+1}$ which is the convex hull of $\{e_i \mid i \in \{1, \dots, n+1\}\}$

thus the set $\sigma_n = \{ v \in (\mathbb{R}^+)^{n+1} \mid \sum_{i=1}^{n+1} v_i = 1 \}$

these are illustrated as follows:

pictures



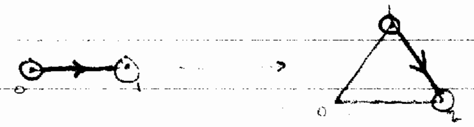
define face maps

Definition: ith face (of an n-simplex)

the map $\partial_i : \sigma_{n-1} \to \sigma_n$ which embeds the n-1-simplex in the n-simplex and preserves orientation

for example $\partial_0 : \sigma_1 \to \sigma_2$

picture



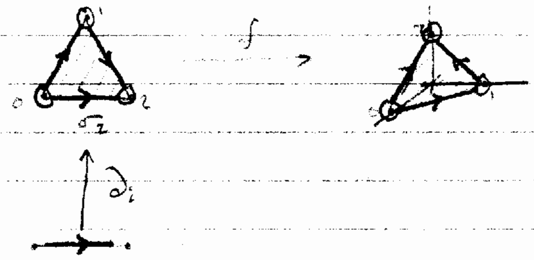
define boundary

Definition: boundary (of a map $f: \sigma_n \to X$)

the formal sum $\partial(f) = \sum_{i=0}^{n} (-1)^i f \circ \partial_i$

for example, with $f: \sigma_2 \to \mathbb{R}^3$ the inclusion, then $\partial f = \sum_{i=0}^{2} (-1)^i f \circ \partial_i$

picture



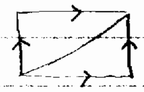
define simplicial complex

Definition: simplicial complex

a topological space $X = \bigsqcup_{i=1}^{k} \sigma_n / \sim$ where $\sim$ relates faces of simplices

for example $X = \sigma_2 \sqcup \sigma_2 / \sim$

example: $T_{1,1}$



define chain complex

Definition: chain complex

a sequence $C = (C_n)_{n \in \mathbb{N}}$ of abelian groups equipped with a boundary map $\partial : C_n \to C_{n-1}$ so that $\partial^2 = 0$

construct simplicial chain complex for simplicial complex

Discussion: chain complex for a simplicial complex
with $X = \bigsqcup_{i=0}^{k} \sigma_i / \sim$ a simplicial complex
for each n-simplex in X fix a homeomorphism $\bar{f}_i : \sigma_i \to \bigsqcup_{i=1}^{k} \sigma_i$
and let $f_i : \sigma_i \to X$ be the map $\bar{f}_i$ followed by the quotient
define $C_n^{\Delta}(X) = \langle f_i \rangle$ the free abelian group generated by
these maps
then define $C_{n+l}^{\Delta}(X) = 0 \quad \forall l \geq 1$
and define $C_{n-1}^{\Delta}(X) = \langle f_i \circ \partial_j \rangle$
and inductively define $C_{n-l}^{\Delta}(X) = \langle \mathcal{B}(C_{n-(l-1)}^{\Delta}) \circ \partial_j \rangle$

note that ∂ is a boundary map

then $(C_n)_{n \in \mathbb{N}}$ is a chain complex where ∂ is the boundary
map (that is $\partial^2 = 0$)

define cycle groups for a chain complex

Definition: $Z_*(C)$ where C a chain complex
the sequence $Z_*(C) = (Z_0(C), \dots)$ of abelian groups $Z_n(C) \subseteq C_n$
given by $Z_n(C) = \{ c \in C_n \mid \partial(c) = 0 \}$

define boundary groups for a chain complex

Definition: $B_*(C)$ where C a chain complex
the sequence $B_*(C) = (B_n(C))_{n \in \mathbb{N}}$ of abelian groups $B_n(C) \subseteq Z_n(C)$
given by $B_n(C) = \{ c \in C_n \mid \exists \hat{c} \in C_{n+1} \mid \partial(\hat{c}) = c \}$

define homology groups for a chain complex

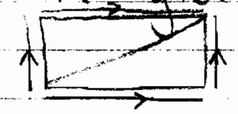
Definition: homology (of a chain complex C)
the sequence $H_*(C) = (H_n(C))$ of abelian groups $H_n(C)$ given
by $H_n(C) = Z_n(C) / B_n(C)$ (note that this is well defined
since $B_n(C) \subseteq Z_n(C)$ since $\partial^2 = 0$)

compute homology groups of torus

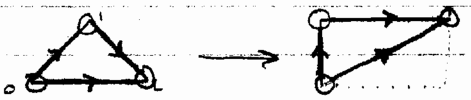
Discussion: simplicial homology for torus
so we can compute the homology of the chain complex
we constructed above for our simplicial complex X
we "compute" an example:

draw pictures

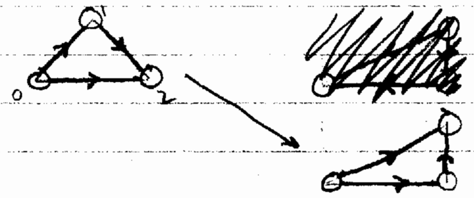
with X given by



we fix $f_1 : \sigma_2 \to X$ to be



and also $f_2 : \sigma_2 \to X$ to be



Discussion contd. simplicial homology for torus

the associated chain complex $C_*^\Delta(X)$ is given below

$\vdots$

write down chain groups

$C_3^\Delta(G) = 0$

$C_2^\Delta(G) = \langle f_1, f_2 \rangle$

$C_1^\Delta(G) = \langle f_1 \circ \partial_0, f_1 \circ \partial_1, f_1 \circ \partial_2 \rangle$ since $f_1 \circ \partial_0 = f_2 \circ \partial_2$ and $f_1 \circ \partial_1 = f_2 \circ \partial_1$ and $f_1 \circ \partial_2 = f_2 \circ \partial_0$

$C_0^\Delta(G) = \langle p \rangle$ where $p : \odot \longmapsto$ (corners of the square)

write down cycle groups

and $Z_*^\Delta(G)$ is given by

$Z_3^\Delta(G) = 0$

$Z_2^\Delta(G) = \langle f_1 - f_2 \rangle$

$Z_1^\Delta(G) = \langle f_1 \circ \partial_0, f_1 \circ \partial_1, f_1 \circ \partial_2 \rangle$

$Z_0^\Delta(G) = \langle p \rangle$

and $B_*^\Delta(G)$ is given by

$B_3^\Delta(G) = 0$

$B_2^\Delta(G) = 0$

write down boundary groups

$B_1^\Delta(G) = \langle f_1 \circ \partial_0 - f_1 \circ \partial_1 + f_1 \circ \partial_2 ,\ f_2 \circ \partial_0 - f_2 \circ \partial_1 + f_2 \circ \partial_2 \rangle$

$B_0^\Delta(G) = 0$

so $H_*^\Delta(G)$ is given by

write down homology groups

$H_3^\Delta(G) = 0$

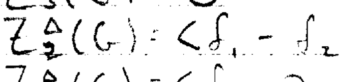
$H_2^\Delta(G) = \mathbb{Z} = \langle [f_1 - f_2] \rangle$

$H_1^\Delta(G) = \mathbb{Z} \oplus \mathbb{Z} = \langle [f_1 \circ \partial_0], [f_1 \circ \partial_2] \rangle$

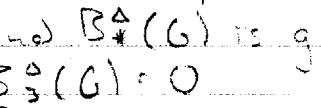
$H_0^\Delta(G) = \mathbb{Z} = \langle [p] \rangle$

interpret results

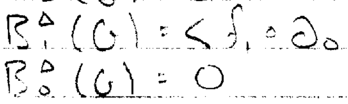
these correspond to the following "loops" in the space:



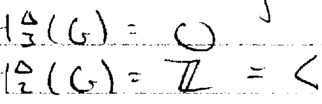
the generator of the second homology group is the 2-dimensional loop which is the surface of the torus



one generator of the first homology group is the 1-dimensional loop which goes through the hole



the other generator of the first homology group is the 1-dimensional loop which goes around the hole



The generator of the zeroth homology group is the 0-dimensional loop which is just any point on the torus set up parallel situation on graphs

Discussion: from simplicial homology of simplicial complexes to hush homology of hush complexes

we describe a way to produce a graph from a simplicial complex and a chain complex associated with the graph so that the homology groups of the rigid hush chain complex are the same as the homology groups of the simplicial chain complexes (in fact the two chain complexes will be naturally isomorphic)
in order to do this, we need to define hushes

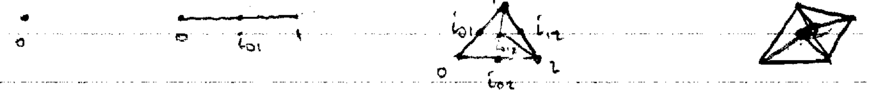
define n-hush

Definition: n-hush
a graph A_n with vertex set $V(A_n) = S(n) \cup \{i_s \mid S \in 2^{+}S(n)\}$
and edge system $E(A_n) = \{e_{ij} : i \sim j \mid i,j \in S(n)\} \cup \{e_{j i_s} : j \sim i_s \mid j \in S\}$

draw pictures

these are illustrated as follows

A_0 $\quad$ A_1 $\quad$ A_2 $\quad$ A_3



define face map

Definition: ith face (of an n-hush)
the map $\partial_i : A_{n-1} \to A_n$ which embeds the n-1-hush in the ith face of the n-hush and preserves orientation
for example $\partial_1 : A_1 \to A_2$

picture

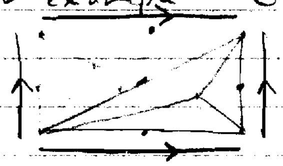
A_1 $\quad$ A_2

define boundary

Definition: boundary (of a map $f: A_n \to G$)
the formal sum $\partial f = \sum_{i=0}^{n} (-1)^i f \circ \partial_i$

define hush complex

Definition: hush complex
the graph $G = \bigsqcup_{i=1}^{k} A_n / \sim$ where $\sim$ relates faces of n-hushes
for example $G : \sigma_2 \sqcup \sigma_2 / \sim$



this will turn out to have the same homology as the torus define husk complex from a simplicial complex

Definition: husk complex (corresponding to a simplicial complex $X = \bigsqcup_{i=1}^{n} \sigma_n /\sim$)

The husk complex $G(X) = \bigsqcup_{i=1}^{n} A_n/\sim$ where $\sim$ relates "the same things"

for example with X given by

$G(X)$ is given by

define rigid husk chain complex

Discussion: chain complex for a husk complex

with $G = \bigsqcup_{i=1}^{n} A_n/\sim$ a husk complex

we define a chain complex $C^R_*(G)$ almost exactly in the same way as before

for each n-husk in G fix a map $\tilde{f}_i : A_n \to \bigsqcup_{i=1}^{n} A_n$ and define $f_i : A_n \to G$ to be $q \circ \tilde{f}_i$ where q the quotient

define $C^R_{n+l}(G) = 0 \quad \forall l > 0$

define $C^R_n(G) = \langle f_i \rangle$

define $C^R_{n-1}(G) = \langle f_i \circ \partial_j \rangle$

and define $C^R_{n-l}(G) = \langle \mathcal{B}(C^R_{n-l+1}(G)) \circ \partial_j$

show that $\partial^2 = 0$ — then $(C_n)_{n\in\mathbb{N}}$ is a chain complex where ∂ is the boundary map (so again $\partial^2 = 0$)

Observation:

note fact about rigid homology of husk complexes

with X a simplicial complex and $G(X)$ the corresponding husk complex then $H^\Delta_*(X) = H^R_*(G(X))$

with G a simplicial complex and $S(G)$ the simplicial complex then $H^R_*(G) = H^\Delta_*(S(G))$

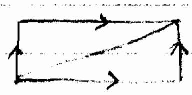
Proof: (sketch)

this follows from an isomorphism between the chain complexes for the correct choice of maps $f_i : A_n \to G(X)$ given a choice of maps $f_i : \sigma_n \to X$

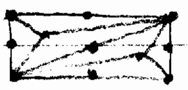
Example:

do torus computation

with X the torus as before



then $G(X)$ is the graph



Example: contd.
so $H_*^R(G(X)) = (\mathbb{Z}, \mathbb{Z}\oplus\mathbb{Z}, \mathbb{Z}, 0, \dots)$

sad fact about rigid homology of hush complexes with down 1) and 2)

Discussion:
unfortunately, this observation means that in a strong sense the rigid homology of a hush complex is entirely uninteresting (because it is just the same as the homology of the corresponding simplicial complex
in addition, this does not work for anything like all graphs — hush complexes are very special
these two facts illuminate the underlying truth that essentially we are just emulating the simplicial homology with graphs — this may be useful computationally, but that's about it
this calls for looking in new directions

singular homology

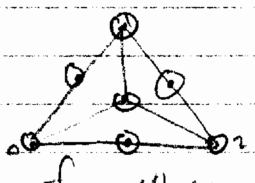
Discussion: new direction: singular homology
one direction to pursue is to address fact number 2) — that is to make the definitions work for all graphs
in order to do this we define a singular homology theory on graphs

singular chain complex, singular maps

this means that we define a chain complex associated to a graph to be generated by 'singular' maps from the hushes to the graph where a singular map is a series of "levelling maps" followed by an injective graph homomorphism

levelling maps

a levelling map is a map $L^{ij}: A_n \to A_{n-1}$ which collapses the nodes i and j along with other structure into a single node, for instance the following picture shows the fibers of the vertices of A_1 under L^{02}



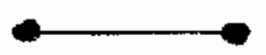
0 homology generators connected comp

so far, this route has proved interesting — it is easy to see that the number of generators of the 0^{th} homology group is the number of connected components
it is also easy to compute the homology for some small

Discussion: contd. new direction: singular homology

graphs — the series and a small loop

ompute segment — with G given by



then $H_*^s(G) = (\mathbb{Z}, 0, \ldots)$

ompute triangle — with G given by

then $H_*^s(G) = (\mathbb{Z}, \mathbb{Z}, 0, \ldots)$

ed more machinery — with bigger graphs, though, the generators of the singular chain groups become too numerous and complicated to work with easily. for this reason we need to understand this machinery better in order to make computations efficiently — in particular, we need a better understanding of the relationship between levelling maps and boundary maps

implicial, cosimplicial sets — Jeff Jean-Sorocuza has alerted me to the fact that the system of face and levelling maps and hooks will generate simplicial and/or cosimplicial sets and this point of view may allow us to use more general theorems from that theory in our case