Talk 7/18/08 "Who's stealing my current?"

- Electrical network $\Gamma = (G, \gamma)$, $G = (V, E)$
  $\gamma : E \to \mathbb{R}_+$, have ∂V, $\text{int}\,V$ $\quad \partial V \cap \text{int}\,V = \{0\}$
  no multiple edges, no loops $\quad |\partial V| + |\text{int}\,V| = |V|$

where it is unclear use pi

↳ letters, like i, j will often denote vertices : $i, j \in V$
  (i,j) denotes edge btwn those vertices : $\gamma(i,j) = \gamma_{ij} = \gamma_{ji}$
- Labeling the graph's vertices – label the ~~first $|\partial V|$~~ boundary nodes 1 through $|\partial V|$. Label the interior vertices $|\partial V|+1$ through $|V|$

- When I put subscripts (i,j) on a matrix/vector, I mean the $i^{th} j^{th}$ component of that vector/matrix : Note this may not line up w/ the labeling of the network, but we can figure it out.
  thus, ex. $B_{ij} = K_{i,\, j+|\partial V|}$

- Form K as usual
- $(Ku)_i = \sum_{j \in N(i)} \gamma_{ij}(u_i - u_j)$ is total current out of node i
- Put a current source of strength α at node $k \in \text{int}\,V$
  then $$(Ku)_i = \begin{cases} 0 & i+|\partial V| \in \text{int}\,V,\ i+|\partial V| \neq k \\ \alpha & i+|\partial V| = k \\ (\varphi_\partial)_i & i \in \partial V \end{cases}$$
- For multiple sources of strength α_i at nodes $I \subset \text{int}\,V$

$$(Ku)_i = \begin{cases} 0 & i+|\partial V| \in \text{int}\,V \quad i+|\partial V| \notin I \\ \alpha_i & i+|\partial V| \in I \\ (\varphi_\partial)_i & i \in \partial V \end{cases}$$

- We still have $\sum_{i\in V}(Ku)_i = 1^t K u = 0$ since K is symmetric and $1 \in \ker K$

Thus

$$\sum_{i\in V}(Ku)_i = 0 = \sum_{i\in \partial V}(Ku)_i + \sum_{i\in \mathrm{int} V}(Ku)_i$$

$$= \sum_{i\in \partial V}(Ku)_i + \sum_{i\in I}\alpha_i$$

$$\Rightarrow \sum_{i\in \partial V}(Ku)_i = -\sum_{i\in I}\alpha_i$$ $=(-)$ Total extra internal current.

Dirichlet Problem: given ϕ_∂ on ∂V, look for a

$u = \begin{pmatrix}\phi_\partial \\ X\end{pmatrix}$ st $(Ku)_i = 0$ for $\cancel{i}\, i+|\partial V| \in \mathrm{int}\, V$, $i+|\partial V| \notin I$

$(Ku)_i = \alpha_i$ for $i+|\partial V| \in I$

so $Ku = \begin{pmatrix}A & B \\ B^t & C\end{pmatrix}\begin{pmatrix}\phi_\partial \\ X\end{pmatrix} = \begin{pmatrix}\psi_\partial \\ \psi_{int}\end{pmatrix}$ $(\psi_{int})_i = (Ku)_i$ $i+|\partial V| \in \mathrm{int}\, V$

$\Rightarrow B^t\phi + CX = \psi_{int} \Rightarrow X = C^{-1}\psi_{int} - C^{-1}B^t\phi$

then $\psi_\partial = A\phi_\partial + BC^{-1}\psi_{int} - BC^{-1}B^t\phi_\partial$

$\boxed{\psi_\partial = \Lambda_0\phi_\partial + BC^{-1}\psi_{int}}$ where Λ_0 is the response mx for the unaltered circuit.

- Suppose I know the network, i.e. know the response when there are no internal sources.

$\psi_\partial' = \Lambda_0\phi_\partial$. Then measure the response of the altered network + form the difference

$\psi_\partial - \psi_\partial' = \boxed{\Delta\psi = BC^{-1}\psi_{int}}$ Thus, we know $\Delta\psi$ from meas.,

- ~~If order~~ There are 2 sets of unknowns, the α_i and I. - The magnitudes of the sources & their locations.

- I will assume I know the magnitudes and am trying to figure out the locations.

- Some properties of B, C^{-1} + BC^{-1}
  - The sum of the entries of a particular column of BC^{-1} is equal to -1

    Pf:

    We have $\psi_\partial = \Lambda_0 \varphi_\partial + BC^{-1}\psi_{int}$

    apply 1^t

    $$1^t \psi_\partial = 1^t \Lambda_0 \varphi_\partial + 1^t BC^{-1}\psi_{int}$$

    but let $\psi_{int} = e_j$ a unit source at node $j+|\partial V|$

    $$\Rightarrow 1^t \psi_\partial = 1^t (BC^{-1})_j = \text{sum of current out of circuit} = -1$$

    ~~If Multiple~~ ~~sources~~

    ~~$-(\alpha_1+\alpha_2+\alpha_3+\cdots) =$~~

    ~~$-\sum_{i\in I}\alpha_i/ =$~~

  - Interpretation of C^{-1}

    $C(\varphi_{int}) = \psi_{int}$ so $C^{-1}(\psi_{int}) = \varphi_{int}$

    ~~so this gives~~

    $(C^{-1})_{i,j} = \cancel{u_i(j)}$ $u_{i+|\partial V|}(j+|\partial V|)$ is the potential at interior node $i+|\partial V|$ due to a unit current source at interior node $j+|\partial V|$

- This gives the following form for BC^{-1}

$$(BC^{-1})_{i,j} = \sum_{k=1}^{|\text{int}V|} b_{ik}\, u_{k+|\partial V|}(j+|\partial V|)$$

$$= -\sum_{k=1}^{|\text{int}V|} \gamma_{i,k+|\partial V|}\, u_{k+|\partial V|}(j+|\partial V|)$$

$$(BC^{-1})_{i,j} = -\sum_{m=|\partial V|}^{|V|} \gamma_{i,m}\, u_m(j+|\partial V|)$$

Note BC^{-1} is $|\partial V| \times |\text{int}\, V|$
note $\gamma_{i,k+|\partial V|}$ are conductivities between boundary nodes and interior nodes only.

- All elements of C^{-1} are positive (for a connected graph)

Pf: min/max values of u occur on ∂V. Consider the current sources as boundary nodes w/ $u>0$. ~~Then all interior nodes~~ Consider setting $\phi_\partial = 0$ (other than current sources)

$$K\begin{pmatrix} 0 \\ \phi_{int} \end{pmatrix} = \begin{pmatrix} \psi_\partial \\ \psi_{int} \end{pmatrix} \Rightarrow C\phi_{int} = \psi_{int}$$

$$\phi_{int} = C^{-1}\psi_{int}$$

then we know ~~$(\phi_{int})_i \geq 0$~~ $0 \leq (\phi_{int})_i \leq V$
~~but~~ so $C^{-1}\psi_{int} = (C^{-1})_i > 0$

- All elements of B are $\leq 0 \Rightarrow$ all elements of BC^{-1} are ≤ 0

40-4?

- Uniqueness of Sources - - Can we ^given a set of magnitudes $\{\alpha_i\}$ have multiple distributions that give the same boundary measurements?

  - consider one source w/ strength α
    $\Rightarrow \Delta\psi = BC^{-1}\alpha e_k$

    then $\Delta\psi/\alpha = BC^{-1}e_k$.= the k^{th} column of BC^{-1}
    Thus, if we can find which column of BC^{-1} is $\Delta\psi/\alpha$, we know where the source is. This boils down to asking the question: are the columns of BC^{-1} distinct?

    Call $C^{-1}e_j = \varphi_j$ where $(\varphi_j)_i = u_{i+|\partial V|}(j+|\partial V|)$

    The same question is: Suppose you have 2 networks, one with a source of strength α at interior node $j' = j + |\partial V|$ and one ~~a~~ source of strength α at node $k' = k + |\partial V|$. ~~Set boundary voltages to zero. We have $B\varphi_{int} = \psi_\partial$ We hope that we do not get the same~~ boundary ~~so~~ ~~currents if $j \neq k$, so we consider the difference $\alpha B\varphi_j - \alpha B\varphi_k = \Delta\psi_1 - \Delta\psi_2$~~
    This gives us two difference measurements, $\Delta\psi = \alpha B\varphi_j$ + $\Delta\psi' = \alpha B\varphi_k$. The requirement

uniqueness of sources → that we can distinguish the sources is equivalent to $\Delta\psi - \Delta\psi' = \alpha B(\varphi_j - \varphi_k) = 0 \iff j = k$ or equivalently, $\varphi_j - \varphi_k \notin \ker B$ for $j \neq k$.

  - Consider multiple sources $\{\alpha_j\}$ $j \in [1, n]$ $n \leq |\text{int} V|$

Suppose we have 2 arrangements of those sources$_n$

$$\varphi_{1,\text{int}} = \sum_{j=1} \alpha_j \varphi_{a_j} \qquad \varphi_{2,\text{int}} = \sum_{j=1}^{n} \alpha_j \varphi_{b_j}$$

~~we consider the difference~~

where $\{a_j\} \subset \text{int} V$ $\quad \{b_j\} \subset \text{int} V$

assume $a_i \neq a_j$ for $i \neq j$, $b_i \neq b_j$ for $i \neq j$. Thus, we want

$$\boxed{B\left(\sum_{j=1}^{n} \alpha_j (\varphi_{a_j} - \varphi_{b_j})\right) = 0 \Rightarrow a_j = b_j \quad \text{for } j \in [1, n]}$$

- The case that B is injective; $\alpha_j > 0$
  - Let B be injective. Then $Bv = 0 \Rightarrow v = 0$

So $B\left(\sum_{j=1} \alpha_j (\varphi_{a_j} - \varphi_{b_j})\right) = 0 \Rightarrow \sum_{j=1} \alpha_j (\varphi_{a_j} - \varphi_{b_j}) = 0$

$$\Rightarrow \sum_{j=1}^{n} \alpha_j \varphi_{a_j} = \sum_{j=1}^{n} \alpha_j \varphi_{b_j}$$

(I'm leaving out other sets of α's that are the same)

possibly reorder $\Rightarrow$ $$\sum_{j=1}^{l} \alpha_j \varphi_{a_j} + \sum_{j=l+1}^{n} \alpha_j \varphi_{a_j} = \sum_{j=1}^{l} \alpha_j \varphi_{b_j} + \sum_{j=l+1}^{n} \alpha_j \varphi_{b_j}$$

where $\alpha_i = \alpha_j = \alpha$ for $i, j \in [1, l]$

$\alpha_i \neq \alpha_j$ for $i, j \in [l+1, n]$ $\quad i \neq j$

$$\Rightarrow \quad \alpha \sum_{j=1}^{l} (\varphi_{a_j} - \varphi_{b_j}) + \sum_{j=l+1}^{n} \alpha_j (\varphi_{a_j} - \varphi_{b_j}) = 0$$

apply C to both sides since $\varphi_i = C^{-1} e_i$

$$\Rightarrow \alpha \sum_{j=1}^{l} (e_{a_j} - e_{b_j}) + \sum_{j=l+1}^{n} \alpha_j (e_{a_j} - e_{b_j}) = 0$$

20-65

$i \neq j$

Since the α_j are distinct + since $a_i \neq a_j$ + $b_i \neq b_j$ then we must have

$$\sum_{j=l+1}^{n} \alpha_j (e_{a_j} - e_{b_j}) = 0 \quad \text{and} \quad e_{a_j} = e_{b_j} \quad \text{for } j \in [l+1, n]$$

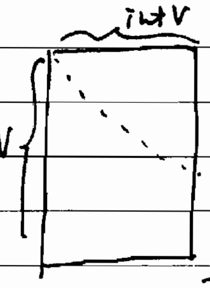
we also must have

$$\alpha \sum_{j=1}^{l} (e_{a_j} - e_{b_j}) = 0 \Rightarrow e_{a_j} = e_{b_i} \quad \text{for } i, j \in [1, l]$$

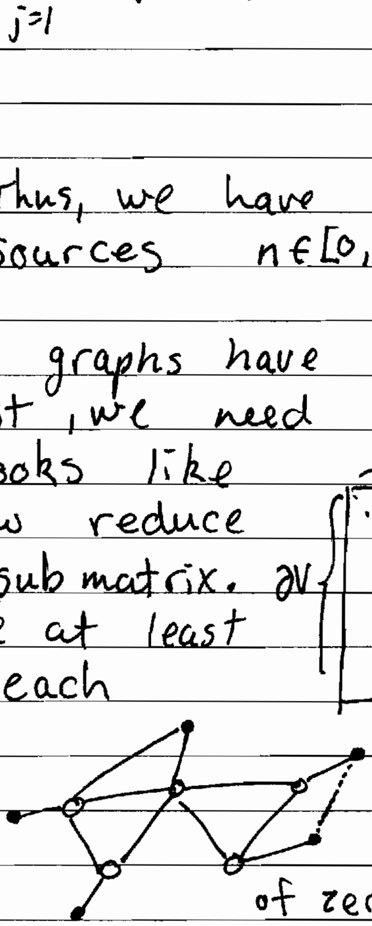
this is enough since the sources are the same magnitude.

Thus, we have uniqueness of ~~sources~~ n sources $n \in [0, |\text{int}V|]$ when B is injective.

- What graphs have B injective?
  - First, we need $|\partial V| \geq |\text{int}V|$
  - B looks like



and ~~since~~ we can row reduce to a diagonal $|\text{int}V| \times |\text{int}V|$ submatrix. This shows that there is at least 1 distinct boundary node to each interior node.



If there were an interior node not connected to a boundary node, there would be a column of zeroes in B and it wouldn't be injective.

- When B is not injective
  - one source: we consider $\alpha B(\varphi_j - \varphi_k) = \alpha B C^{-1}(e_j - e_k)$ $\ker B \neq \{0\}$

1) If $\varphi_j - \varphi_k \notin \ker B$, then I can tell the difference between a source at j + a source at k.

2) If $\varphi_j - \varphi_k \in \ker B$ then I cannot tell the difference between a source at k + a source at j.