Talk ①

Review

We'll be dealing with

$$S_n := \{\text{permutations on } n \text{ letters}\}$$
$$= \langle s_1, s_2 \dots s_{n-1} \mid (s_i)^2 = id;\ (s_i s_j)^2 = id\ (|i-j|>1);\ (s_i s_j)^3 = id\ (|i-j|=1) \rangle$$

~~Every any permutation, we can write it as a product of these generators:~~

~~eg [illegible] 4231~~

Notation review: for a permutation $w \in S_n$, write one-line notation

$$w:\ 1 \longrightarrow 4,\ 2 \longrightarrow 2,\ 3 \longrightarrow 3,\ 4 \longrightarrow 1$$

$$w = [4231] \quad \rightsquigarrow \quad [w(1)\ w(2) \dots w(n)].$$

or, permutation matrix:

$$w = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \longleftarrow \text{watch out here! many different conventions!}$$

$$w \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 1 \end{bmatrix} \longrightarrow \text{multiplied by } [1234]^T \text{ gives one-line notation}$$

We can also represent w by a product of the generators. Eg.

$$[2134] = s_1$$
$$[4231] = s_3 s_2 s_1 s_2 s_3$$

~~We say that a permutation w has length (w) if it~~

We define the length of a permutation to be the number of

; $(s_i s_j)^3 = id$ >
$(|i-j|=1)$
~~generators~~ :

]

!

gens

Eg.

f it

vertex of

inversions in the one-line notation. ~~Inversions are places where~~

An inversion of a permutation is a pair which is out of order.

Eg: S_1 has length 1 : [1 2 1 3 4]

[4231] has length 5: [4 2 3 1]

A reduced expression for w is an expression for w as a pr
of ~~generators such that~~ [4 2 3 1] =
l(w) generators.

$$[4231] = s_3 s_2 s_1 s_2 s_3$$

is a reduced expression.

Bruhat order:

$V \leq W$ if for all reduced expressions s_{a_i} for w, $\exists$ a subexpression $\prod s_{a_{i_j}} = V$.

Proof: Exercise.

Pattern avoidance:

Kazhdan-Lusztig polynomials: indexed by two permutations, x,w.

Uniquely defined by following recurrence relations:

$P_{x,w} = 0$ if $x \nleq W$

$P_{w,w} = 1$

$\deg(P_{x,w}) \leq \frac{1}{2}(l(w) - l(x) - 1)$

If ~~[illegible]~~ s is a generator and $ws < w$,

$$P_{x,w} = q^c P_{x,ws} + q^{1-c} P_{xs,ws} - \sum_{\substack{x \leq z < ws \\ zs < z}} \mu(z,ws) q^{(l(w)-l(z))/2} P_{x,z}$$

$\mu(z,ws)$: coefficient of $q^{(l(ws)-l(z)-1)/2}$ in $P_{z,ws}$

$$c = \begin{cases} 1 & \text{if } xs < x \\ 0 & \text{else} \end{cases}$$

Facts about K-L polynomials:

1. $P_{x,w}(0)=1$ if $x \le w$
2. coefficients are non-negative integers.
3. If $y \le x \le w$, then for all k, the coefficient of q^k in $P_{y,w}$ is $\ge$ coefficient of q^k in $P_{x,w}$.

↳ Interpretation: as we travel down the Bruhat order, in the 1st term, coefficients increase.

<u>Corollary</u>: $P_{id,w}$ has the highest coefficients among $P_{x,w}$.

Problem: characterize $P_{id,w}(1)$ = sum of coefficients for all w.

Why do we care? To answer, we have to talk a little about <u>Schubert varieties</u>.

<u>Excursion</u>

What's an algebraic variety?

"<u>Def:</u>" Given a polynomial P in n variables, a variety $V(P)$ is the set of points $\vec{x}$ in $\mathbb{C}^n$ such that $P(\vec{x})=0$.

When you hear "variety", think "manifold".

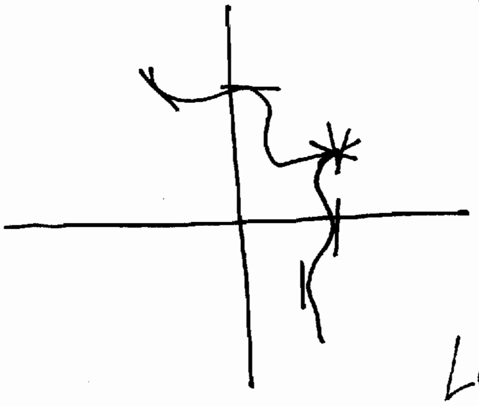
[not actually a manifold, but it's some surface sitting in $\mathbb{C}^n$]

Intuition: This turns <u>algebra</u> into <u>geometry</u>.

Nice varieties: <u>smooth</u>.

We are going to look at ~~singular varieties~~ singular varieties: variation where the differential structure breaks down:

How do we tell if a point in singular



Tangent spaces:

at singular points, tangent space has dimension which is too high.

Leave on board.

Singularity = High dimension of tangent s

~~Grassmanian as manifold~~ Special example of a ~~manifold~~ variety

Schubert varieties:

Def: A ~~flag~~ complete flag of a vector space V is a sequence of subspaces

$$V_1 \subset V_2 \subset \dots \subset V_i \subset \dots V_n = V,$$

where $\dim(V_i) = i$.

We will be looking at complete flags of $\mathbb{C}^n$.

Eg. if $\langle e_1, \dots e_n \rangle$ is a standard basis for $\mathbb{C}^n$,

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \dots \subset \langle e_1 \dots e_n \rangle = \mathbb{C}^n$$

is a complete flag.

For any complete flag ~~For any other complete flag~~ $F_1 \subset F_2 \subset \dots \subset F_n$, we can define a corresponding ~~[illegible]~~ permutation as follows: write down basis vectors in a column for the basis vectors of F_i.

for all k,

g^k in $P_{x,w}$.

at order j_n

many $P_{x,w}$.

all w.

$\cong$

$v(P)$

$$\begin{bmatrix} 9 & 5 & 9 & 7 \\ 6 & 2 & 4 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 1 & 0 & 0 \\ 2 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ same as } [2413]$$

We call this the position of g w/r/t the base flag.

So given a permutation $w \in S_n$, define

$$C_w = \{\text{flags } F \mid \text{pos}(F) = w\}.$$

Then define the Schubert variety corresponding to w to be

$$X_w := \{ C_z \mid z \leq w \text{ in Bruhat order}\}.$$

Fact: This is actually a projective variety.

Talk, Part 2

Clarification from yesterday

1. dimension of V, what's a tangent space?

easy: informally, longest chain of varieties w/i V, each ~~properly~~ properly contained in the last. $\dim(X_w) = \text{length}(w)$

easy

~~[illegible]~~ tangent space → Jacobian

2. Huge lie: singular points are ~~isolated~~ finite only looking at points fixed by maximal torus. Just isn't true → in fact, we're ~~[illegible]~~

In fact, singular points of $X_w = \bigcup X_v$.

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Back to problem:

Characterize ~~polynomials s.t.~~

$\{ w \mid P_{id,w}(1) \le k \} = F_k$

in terms of pattern avoidance.

~~What does this mean?~~ Remark!

~~pattern avoiding?~~

~~Well it's not a question of whether we can use~~

pattern avoidance

Thm ~~[illegible]~~

$\exists$ a list of patterns B_k such that $w \in F_k$ iff w avoids B_k.

So we're done!

No $\longrightarrow$ ~~[illegible]~~ we're interested in whether $|B_K|$ is infinite or not.

If B_K is a finite list of patterns, what are they?

If B_K is infinite, what does it look like?

If B_K

Baby steps

$K=1$.

$$F_1 = \{w \mid P_{id,w}(1) \leq 1\} = \{w \mid P_{id,w}(1) = 1\}$$

How to characterize these permutations?

Thm [Carrell-~~Peterson~~] 1991

If $P_{id,w}(1) = 1$, then X_w is smooth.

How do we tell if something is smooth?

Tangent spaces!

~~Fact~~: Define, for $x \leq w$, $R(x) = \{t \in T \mid xt \leq w\}$

Fact: Tangent space to X_w at point e_x has a basis indexed by $R(x)$.

So, recall $\dim(X_w) = \text{length}(w)$,

w is smooth at x iff $|R(x)| = \text{length}(w)$.

Proof is pretty easy, takes place in Schubert world.

Lakshmibai – Sandhya

Claim (~~Billey~~ ?):

X_w is smooth iff w avoids [4231] and [3412]

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1973! Demazure

"Proof" The ~~Bruhat~~ graph for $w \in S_n$ equate to length.

w "factors nicely" iff w or w^{-1} ends in a decreasing end sequence ~~of length with n~~.

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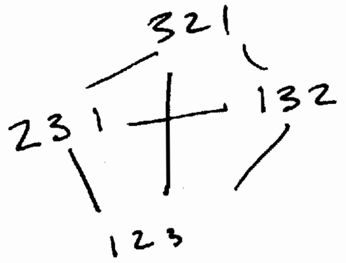
e.g. if $w = [3612\underline{7}\underline{5}\underline{4}]$, w "factors nicely".



Intuition?

Fact you can check! [4231] and [3412] are the only 2 permutations w in S_4 such that neither w, w^{-1} has a decreasing endsequence.

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So it's "clear" that 4231 & 3412 are ~~the~~ singular.

Open question:

Explain. Why is it enough to just check up to S_4?

h(w).

$w \in F_1 \iff w$ avoids [4231], [3412].

usplue

What else do we know?

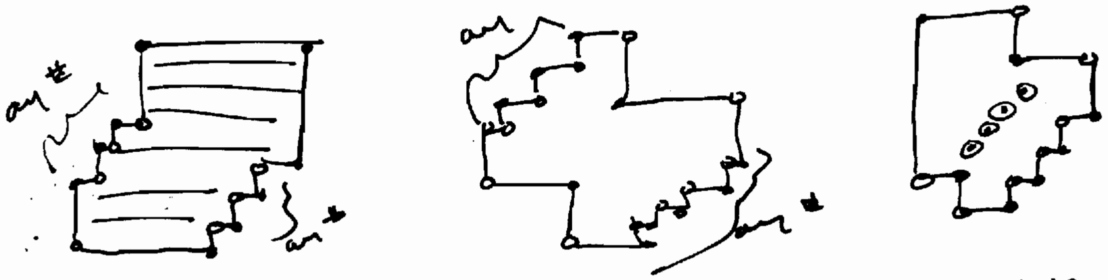
~~Thm~~ Good thm for telling whether $x \in \text{maxsing}(w)$

Thm [Billey–Warrington, 2001] –
$x \in \text{maxsing } w$ iff.

1) $x = w \circ$ cycle.

2) picture criterion:

$w =$



4231 3412 ~~8~~45312

Yay!

— As of 6 months ago, this is about all ~~[illegible]~~ finally establish that we had towards the F_k question.

Problem Sara posed:

$$F_2 = \{w \in S \mid P_{id,w}(1) \leq 2\} \quad ?$$

She thought it was hard, turns out it's easy.

Thm: Woo (2004?)

$P_{id,w}(1) = 2$ iff $|\text{maxsing}(w)| = 1$

and w avoids . . .

theory (w)

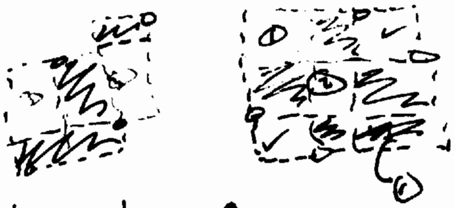
(526413) (54213) (463152)

(465132) (632541) (653421)

Proof: ??

Lemma: If $|\mathrm{Sing}\, w| \geq k$, then $\exists v \leq w$ with at most $4k$ entries s.t. $|\mathrm{Sing}(v)| \geq k$.

Proof: Reinterpret pictures!



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Corollary: If $|\mathrm{Sing}\, w| \geq 2$, then it contains a pattern v with at most 8 entries s.t. $|\mathrm{Sing}(v)| \geq 2$.

So F_2 is characterized by a finite number of patterns: all our "minimal conditions" in S_8. □

all establish

In fact: 66 patterns characterize F_2.

What's next?

F_3, of course.

Right now, just 8 things to characterize necessary conditions for $P_{id,w}=1$

=1

Approaches:

Thm: (Billey–Warrington, 2001) If $x \in \mathrm{maxsing}(w)$, $P_{x,w}$ has one of the following 2 forms: $1+q^a$, $1+q+q^2+\cdots q^a$.

more importantly → each corresponds to a certain type of ~~product~~ singular element.

~~Ideal type~~

(11)

If $P_{id,w}(1)=3 \Rightarrow P_{id,w} = 2q^a+1, \; q^a+q^b+1$.

For each term, $\text{coeff}(P_{x,w}) \leq \text{coeff}(P_{id,w})$.

So we can characterize explicitly the type of singular element. (No word yet on #).

Conjecture: $|Sing| \leq 3$.

$$P_{id,w}(1) \overset{=3}{\Rightarrow}$$

in general

$$P_{id,w}(1) = k \Rightarrow |Sing| \leq \binom{k}{2}$$

Unfortunately, no shortcut to finding K-L polynomials in general.

Note: With our lemma, getting even a very loose bound on $|Sing|$ will suffice to prove finiteness

END OF TALK

Look up Deligne's proof for finite Riemann hypothesis.