The model:

Given a graph with vertices and undirected edges $(V, E)$ where the vertices are partitioned into interior vertices $I$ and boundary vertices $\mathcal{B}$ which are connected to a single interior vertex as a boundary spike.

- Boundary node
- Interior node

A particle is fed into the graph at one of the boundary vertices and is allowed to randomly walk starting there one vertex at a time, assuming equal probability to each available direction at any given time. The random walk ends when the particle reaches a boundary vertex. In this particular problem, we keep track of the number of steps the random walk takes and the boundary vertex it exits at.
This is intended to model the behaviour of particle introduced at the boundary node of a resistance network and allowed to diffuse as electron actually does, with the potential in the network off.

My model imagines the network as being built of unit resistors attached in series to model any resistor connecting to nodes.

For example, the electric network with such labeled resistances as these:

![Graph 1](image1)

might be modeled with the graph:

![Graph 2](image2)

The motivation for doing this is to create a metric for measuring the "distance" between boundary nodes that, for the purposes of inverse problems, might improve upon typical notions of shortest distance.
In Megan McCormick's paper of 2005, one of the problems she encountered with trying to solve the inverse problem she designed was that the shortest path metric only provided information about the shortest path.

In this graph, shortest distance information given between boundary nodes A, B, C would provide no clue about the presence of the true edges leading to vertex 1.

It is hoped that using my proposed metric instead of which we measure the expected number of steps a random walk would require to get from each boundary vertex to each other boundary vertex, the presence of the edges leading to vertex 1, could be detected as they would increase the expected number of steps to get from one boundary vertex to another.
Let us be more precise for a graph $G$.

With boundary vertices, $\{A, B, \ldots\}$.

Let the space of outcomes be all possible paths from boundary node $A$ to any boundary node (including $A$).

The probability of a given path that leaves a series of vertices $\{A, i_1, i_2, \ldots, i_k\}$ is

$$\frac{1}{\text{deg}_A(i_1)} \cdot \frac{1}{\text{deg}(i_2)} \cdot \ldots \cdot \frac{1}{\text{deg}(i_k)}$$

where $\text{deg}$ is the degree of vertex $i$ representing the number different option for the particle at that point.

Let $X_{A}(\text{path})$ be a random variable that represents the number of steps in a path leaving $A$ and exiting any where.

Let $X_{AB}(\text{path})$ be the number of step in a path leaving $A$ and exiting $B$ and $X_{AB}(\text{path})$ is zero for paths leaving $A$ and exiting somewhere other than $B$. 

\[ E[X_A] = \sum_{\text{all paths } P} X_A(p) \text{ prob}(p) \]

is the number of steps in a random walk starting at A ending at any boundary vertex.

\[ E[X_{AB}] \] is then the portion of the above sum that comes from paths that begin at A and end at B.

My proposed metric \( d(A, B) \) is a conditional expectation. Assuming a random walk gets from A to B, \( d(A, B) \) is the expected number of steps in such a walk. If \( A \neq B \)

\[ d(A, B) = \frac{E[X_{AB}]}{P_{AB}} \]

where \( P_{AB} \) is the probability and \( P_{BA} \) is the probability of a walk beginning at A will end at B.

\[ d(A, A) = 0 \]

(This additional part of the definition is needed for if we let a particle diffuse into the network at A it may take a long time to diffuse back out at A. This issue will come back to haunt us later.)

Claim:

\[ d(A, B) = d(B, A) \]

Any path from A to B leaves vertices \( \{A, \ldots, i, \ldots\} \) has probability \( \frac{1}{d_A} \cdot \frac{1}{d_B} \cdot \frac{1}{d_{i,\ldots}} \cdot \frac{1}{d_{i,\ldots}} \) and length \( k+1 \) steps reversing the path gives a path from B to A leaving vertices \( \{B, \ldots, i, \ldots\} \) and has probability \( \frac{1}{d_B} \cdot \frac{1}{d_{i,\ldots}} \cdot \frac{1}{d_{i,\ldots}} \) and length \( k+1 \) so \( P_{AB} = P_{BA} \) and \( E[X_{AB}] = E[X_{BA}] \)

so \( d(A, B) = d(B, A) \)
Claim: \( d(A, C) \leq d(A, B) + d(B, C) \)

All the paths that go from \( A \) to \( B \) and then from \( B \) to \( C \) are almost paths from \( A \) to \( C \), so

\[
d(A, B) + d(B, C) = E[X_{AB}] + E[X_{BC}]
\]

They just have an added trip from \( B \) to \( A \) and back, and so have probability reduced by \( \frac{1}{d_{AB}} \)

\[
P_{AB} \cdot P_{BC} < P_{AC}
\]

and paths from \( A \rightarrow B \rightarrow C \) are included in paths from \( A \rightarrow C \) so

\[
E[X_{AC}] \geq \left[ E[X_{AB}] + E[X_{BC}] \right] \frac{1}{d_{AB}}
\]

\[
d(A, B) + d(B, C) = \frac{E[X_{AB}]}{P_{AB}} + \frac{E[X_{BC}]}{P_{BC}}
\]

\[
= P_{BC} \cdot E[X_{AB}] + P_{AB} \cdot E[X_{BC}]
\]

since \( P_{BC} \cdot P_{AC} < 1 \)

\[
P_{BC} \cdot E[X_{AB}] + P_{BC} \cdot E[X_{BC}] \leq \left[ E[X_{AB}] + E[X_{BC}] \right] \frac{1}{d_{AB}} \leq E[X_{AC}]
\]

and since \( P_{AB} \cdot P_{BC} < P_{AC} \)

so

\[
\frac{P_{BC} \cdot E[X_{AB}] + P_{AB} \cdot E[X_{BC}]}{P_{AB} \cdot P_{BC}} < \frac{E[X_{AC}]}{P_{AC}}
\]

\[
d(A, B) + d(B, C) \leq d(B, C)
\]

Thus \( d(A, B) \) is a metric.
So is there an efficient way to calculate $d(A,B)$? Yes, it is a bit cumbersome. We need to calculate $P_{AB}$ and $E[X_{AB}]$.

$P_{AB}$ is calculated by solving the Dirichlet problem on the graph.

It is well known and is well discussed in *Random walks on Electric Networks* by Doyle and Snell.

that if we set boundary conditions one at B and zero at all the other boundary nodes.

The probability of a particle starting a random walk at $x$ and exiting at $B$ is the value of the solution to the Dirichlet problem at $x$, $P_{AB}$.

Since our particle diffuses with probability one from A to $B$, this is $P_{AB}$.

[denote the solution to this boundary value problem at vertex $B$ as $P_{AB}$]
$E[X_{AB}]$ is commonly calculated using Markov chains. I find this approach unintuitive and maybe not a good place to start an inverse problem so I approach finding $E[X_{AB}]$ differently, but Markov chains work well for computers so I will discuss the approach but I won't prove anything. Again, Doyle and Snell is a good reference.

Let $P$ be the transition matrix where $P_{ij}$ is the probability a particle at $i$ moves to $j$ after one step. In our case $P_{ij}$ is the equal to $1/|\text{nodes}|$ if $i$ and $j$ are connected by an edge. Except at $P_{i k}$ where $k$ is a boundary node, there $P_{i k} = 1$ for $|\text{nodes}| = k$.

In this way, in the particle stays at the $k$th vertex

$P^n = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}$

is the probability the particle is at the $i$th vertex after $n$ steps.

If $P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

and $N = (I - Q)^{-1}$
Then \( B = NR \)

and the probability of a particle at the \( i \)th vertex exiting eventually at the \( j \)th vertex on the boundary is \( B_{ij} \).

Thus \( P_{AB} = P_{iA}B = B_{iA} \).

And
\[
V = N^2 R
\]

the expected number:
\[
E[X_{A,B}] = r V_{iA} B
\]

Since the transition matrix \( P \) keeps particles on boundary vertices where they are we would need to start a random walk from \( A \) to \( B \) and \( i_A \) for the purposes of Markov Chains. Otherwise this approach works.
A Different Way

I began by thinking about a problem that turned up again and again on the graph:

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \]

Assuming you don't exit at 0, how many steps in a random walk do it take to go from 1 to 2, 2 to 3, 3 to 4, etc.

This is accomplished by summing geometric series.

For example, in 2 to 3: To find \( P_{23} \)

\[
\text{prob}(2 \to 3) + \text{prob}(2 \to 1 \to 2 \to 3) + \text{prob}(2 \to 1 \to 2 \to 1 \to 2 \to 3) + \ldots
\]

\[
= \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \ldots
\]

\[
= \frac{1}{2} \left( 1 + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \ldots \right) = \frac{1}{2} \left( \frac{1 - \frac{1}{4^n}}{1 - \frac{1}{4}} \right) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}
\]

In this way we can find

\[
\text{prob}(2 \to 1) = \frac{1}{2}, \quad \text{prob}(1 \to 2) = \frac{1}{2}, \quad \text{prob}(2 \to 3) = \frac{2}{3}, \quad \text{prob}(3 \to 4) = \frac{3}{4}
\]

\[
P(n-1 \to n) = \frac{n-1}{n}
\]

To find \( E\left[X_{23}\right] \):

\[
E\left[X_{23}\right] = E\left[X_{2 \to 3}ight] + E\left[X_{2 \to 3 \to 2}ight] + E\left[X_{2 \to 3 \to 2 \to 3}ight] + \ldots
\]

\[
= \frac{1}{2} \left( 1 + \left(\frac{1}{4}\right) \right) + \frac{1}{2} \left( \frac{1}{4} \right)^2 \left( 2 \cdot 1 + 1 \right) + \frac{1}{2} \left( \frac{1}{4} \right)^3 \left( 2 \cdot 2 + 1 \right) + \frac{1}{2} \left( \frac{1}{4} \right)^4 \left( 2 \cdot 3 + 1 \right) + \ldots
\]

\[
= \frac{1}{2} \left( 1 + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \ldots \right) + \frac{1}{2} \left(\frac{1}{4}\right)^2 \left( 1 + 2 \cdot \left(\frac{1}{4}\right) + 3 \cdot \left(\frac{1}{4}\right)^2 + \ldots \right)
\]

\[
= \frac{1}{2} \left[ 1 + \frac{1}{4} \right] + \frac{1}{2} \left[ \frac{1}{1 - \frac{1}{4}} \right]^2 = \frac{1}{2} \cdot \frac{4}{3} + \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3} + \frac{4}{3} = \frac{10}{3}
\]

\[
d(2, 3) = \left( \frac{10}{3} \right)^{\frac{2}{3}} \approx 5.7
\]
Continuing in this way we get

\[ P_\infty = \frac{1}{2} \bigg( \frac{\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots}{2-\frac{1}{\phi}} \bigg) \]

\[ = \frac{1}{3} \bigg( \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \ldots \bigg) \]

\[ = \frac{1}{3} \bigg( \frac{2n-1}{3} \bigg) \]

\[ n \in \mathbb{N} \]

\[ d(0, A) = \frac{1}{3} \bigg[ n^2 + 2 \bigg] \quad \text{[sum of odds up to } 2n-1 \text{ is } n^2 \bigg] \]

Using these techniques we find

\[ \text{Where } C, D, E \text{ are boundary vertices} \]

The expected number of steps to go from A to B without going out at C, D, E is

\[ \frac{1}{3} \bigg[ s^2 + 2 - \frac{m+n+s+1}{m+n+s+1} \bigg] \]

Also if C, D, E are not boundary vertices

\[ \text{expected number of step from } \]

\[ \text{A to B is} \]

\[ \frac{s^2 + z(m+n+s)}{3} \]

or if \( s = 1 \)

\[ = 1 + 2 \left( \# \text{ of edges to the left of } A \right) \]

Turns out to be true for any graph G with no boundary vertices

\[ G \xrightarrow{A} B \]

The expected number of steps from A to B is

\[ 1 + z \left( \# \text{ of edges in back of } A \right) \]
Proof:

Consider an arbitrary edge.

Claim 1: The expected number of crossings in a random walk from A to B of that edge in the direction is equal to the number of crossings in the direction.

The expected number of crossings in the direction is

\[ \sum_{\text{path } P \text{ from } A \rightarrow B} \text{prob}(P) \left[ \text{crossing } \rightarrow \right] \]

for the direction:

\[ \sum_{\text{path } P \text{ from } A \rightarrow B} \text{prob}(P) \left[ \text{crossing } \leftarrow \right] \]

All the paths that cross the edge the same number of times in both directions contribute to both sums equally.

A path that crosses the edge in times in the direction and in times in the direction can be reversed and still be a path from A to B. By changing the starting point, for that path, we get in time in the direction and in times in the direction for both each way, so the expected number both ways is equal.
Again consider an arbitrary edge connecting vertices 1 and 2.

Let \( V_1 \) and \( V_2 \) be the expected number of visits to vertices 1 and 2 respectively in a random walk from A to B. \( d_1, d_2 \) are the degrees of 1 and 2.

Since \( V_1 \) is the expected number of visits to 1, the expected number of crossing from 1 to 2 should be \( \frac{V_1}{d_1} \). Similarly, the expected number of crossings from 2 to 1 is \( \frac{V_2}{d_2} \).

By the last part, these are equal, so \( \frac{V_1}{d_1} = \frac{V_2}{d_2} \) holds for any connected pair of vertices, so the \( V \)'s are proportional to the \( d \)'s throughout the graph.

Now let us calculate \( V_A \) directly:

The probability of going \( A \to B \) directly is \( \frac{1}{d_A} \)

The probability of instead going from \( A \) into the rest of the graph is \( \frac{d_A - 1}{d_A} \) since there are no boundary vertices in the graph, the path will return to \( A \) so the expected number of visits to \( A \) is

\[
V_A = \frac{1}{d_A} \left( 1 + \frac{d_A - 1}{d_A} \cdot 2 + \left( \frac{d_A - 1}{d_A} \right)^2 \cdot 3 + \cdots \right) = \frac{1}{d_A} \left( \frac{1}{1 - \frac{d_A - 1}{d_A}} \right)^2 = \frac{1}{d_A} \left( \frac{d_A}{1} \right)^2 = d_A
\]
So $V_A = d_A$ so the factor of proportionality of the $V$'s and the $d$'s is one so $V_i = d_i$.

Thus for any arbitrary edge the expected crossings in each direction is $\frac{V_1}{d_1} = \frac{d_1}{d_1} = 1$ and $\frac{V_2}{d_2} = \frac{d_2}{d_2} = 1$

so the expected number of crossing of each edge in a random walk from $A$ to $B$ is 2. Adding the step to go from $A$ to $B$ $E[X_{AB}] = 1 + 2 \times (\text{# of edges behind } A)$

$P_{AB} = 1$ since there are no other exits.

$\ell(A, B) = 1 + 2 \times (\text{# of edges behind } A)$

We can write in terms of vertices as

$\ell(A, B) = \sum_{\text{verts.}} V_i = \sum_{\text{verts.}} d_i$

We can extend this argument to the more complicated case where there are other boundary vertices aside from $B$ and get that

$E[X_{AB}] = \sum_{\text{verts.}} V_i = \sum_{\text{Verts}} d_i P_{iB}^2$

where $P_{iB}$ is the probability of a particle at vertex $i$ existing at $B$. The solution to the Dirichlet problem with boundary value one at $B$ and zero at all other boundary nodes.

Unhappily, the argument is fallacious, but the result is
So we have to prove it by other means.
And let's shift our labeling a bit in the process.

Let $G$ be a graph with more than one boundary vertex.
Consider a random walk starting at $0$, exiting at any boundary vertex.

Let $v_i$ be expected visits to vertex $i$ in the random walk beginning at 0, exiting at any boundary vertex $A, B, C, \ldots$.

The $v_i$'s are related to their neighboring interior vertices.

That is

$$v_i = \sum_{j \in \text{interior}} \frac{1}{d_j} v_j$$

This true for all interior vertices except for 0 because as the particle was put there from the outside it has one visit that did not come from other interior vertices.

So

$$v_0 = \left[ \sum_{j \in \text{interior}} \frac{1}{d_j} v_j \right] + 1$$

Thus we have a matrix relating interior vertices

$$\begin{bmatrix}
Q \\
v_1 \\
v_2 \\
\vdots \\
v_n
\end{bmatrix} =
\begin{bmatrix}
v_0 - 1 \\
v_1 \\
v_2 \\
\vdots \\
v_n
\end{bmatrix}$$

The column of $Q$ total one or less

less when the column is for a vertex that is next to a boundary vertex.
\[
\begin{bmatrix}
I - Q
\end{bmatrix}
\begin{bmatrix}
V_0 \\
V_1 \\
\vdots \\
V_n
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

I - Q is diagonal dominant so is invertible so there is a unique vector \[\begin{bmatrix}
V_0 \\
V_1 \\
\vdots \\
V_n
\end{bmatrix}\] that solves the relations claimed.

\[V_i = d_i \cdot P_{iA}\] where \(P_{iA}\) is the probability of exiting at \(A\) if starting at \(i\).

\(P_{iA}\) is the solution to a Dirichlet problem and so is harmonic at all interior points where boundary conditions are all zero except for \(A\) which is one.

So for \(i \neq 0\),

\[P_{iA} = \sum_{i, j} \frac{P_{ij}}{d_i}\] at interior.

So \(d_i P_{iA} = \sum_{i, j} P_{ij} \left(\frac{d_j}{d_i}\right)\) at interior.

So \(V_0 = \sum_{i, j} \frac{V_j}{d_j}\) at interior.

For \(i = 0\),

\[P_{EA} = \sum_{i, j} \frac{P_{ij}}{d_i} + P_{AA}\] \(P_{AA} = 1\).

So \(V_0 = \sum_{i, j} \frac{V_j}{d_j} + 1\).
so the expected number of step of a random walk beginning at 0 exiting any where is the sum of all the visits to all the interior vertices [note the extra visit at 0 is the extra step needed to exit at a boundary vertex]

so \( E[X_0] = \sum \text{interior vertices} d_i \cdot P_{i\rightarrow a} \)

If we let \( V_a = P_{AA} \cdot d_A = 1 \)
and \( V_a = P_{KA} \cdot d_A = 0 \) for \( k \neq A \) boundary vertex

so \( E[X_A] = \sum \text{interior vertices} d_i \cdot P_{i\rightarrow a} + 1 = \sum \text{all vertices} d_i \cdot P_{i\rightarrow a} \)

Now suppose the particle starts at \( A \), what is the expected number of step to exit at \( B \) \( d(A, B) = \frac{E[X_{AB}]}{P_{AB}} \)

\( P_{AB} = P_{BA} \) which we can find by solving the Dirichlet problem with one at \( B \) and zero at other boundary vertices.

We will calculate \( E[X_{AB}] \) from the visit function.

Let \( P_{i\rightarrow a} \) be the probability of going from \( A \) to \( a \) directly (not returning once)

Let \( P_{a\rightarrow b} \) be the probability of going from \( b \) to any boundary vertex directly (not returning to \( b \))

Let \( r_i \) be the probability of starting at \( i \) and returning once
\[ V_i = P_{A_i} P_{B_i} + P_{A_0} \Gamma_i P_{B_0} \cdot 2 + P_{A_0} \Gamma_i^2 P_{B_0} \cdot 3 + \ldots \]

If we want \( V_i \) to be the portion of the above sum that exits at \( B \), then if \( P_{iB}^* \) be the probability of going directly to \( B \) from \( i \), then
\[ V_i' = P_{A_i} P_{iB}^* \cdot 1 + P_{A_0} \Gamma_i P_{iB}^* \cdot 2 + P_{A_0} \Gamma_i^2 P_{iB}^* \cdot 3 + \ldots \]

so if we can convert \( V_i \) to \( V_i' \) by multiplying by \( \frac{P_{iB}^*}{P_{iB}} \). We can exit from \( i \) directly or return that is the entire range of options so \( P_{iB}^* + r = 1 \)

or \( P_{iB}^* = 1 - r \)

also \( P_{iB} = P_{iB}^* + P_{iB} \Gamma_i + P_{iB} \Gamma_i^2 + P_{iB} \Gamma_i^3 + \ldots \)

\[ P_{iB} = P_{iB}^* \left[ \frac{1}{1 - \Gamma_i} \right] \]

\[ P_{DB} = P_{DB} (1 - r) \text{ so } \frac{P_{DB}^*}{P_{DB}} = P_{DB} \]

so \( V_i' = V_i' P_{DB} = \delta_i P_{DA} P_{DB} \)

so \( E \left[ X_{AB} \right] = \sum_{i \in \text{vertices}} \delta_i P_{DA} P_{DB} \)
So we have a way of calculating the expected number of steps getting from boundary vertex $A$ to boundary vertex $B$ in terms of two solutions to the Dirichlet problem: one setting $A$ to one other boundary vertex to zero, the other setting $B$ to one and the other boundary vertices to zero.

\[ d(A, B) = \frac{\sum_{\text{all vert.}} d \cdot P_{\mathcal{U}} A \cdot P_{\mathcal{U}} B}{P_{\mathcal{U}, B}} \]

where $\mathcal{U}$ is the interior vertex by $A$.

Remember we defined $d(A, A) = 0$ but

\[ \sum_{\text{all vert.}} d \cdot P_{\mathcal{U}} A \]

\[ \frac{P_{\mathcal{U}, A}}{P_{\mathcal{U}, A}} \]

is the expected number of steps for a particle leaving $A$ to return and it is definitely non-zero.

Perhaps we don't have so much as metric as something else.
We have an inner product over the vector space of solutions to the Dirichlet boundary value problem which is a perfectly good inner product over the vector space of all functions (vectors) on the graph.

One of course, can design any sort of inner product one likes, but I'm going to argue that this might be special.

First, a random walk is sensitive to geometric properties of space it walks through. In particular, dimension, in a flat d-dimensional space the return probability $P(#\text{step}) \propto \frac{1}{(#\text{steps})^{d/2}}$ so perhaps an inner product that falls out of a random walk product would make appropriate volumetric adjustment to an integral (in this case a sum). This inner product might be imagined as the standard $L^2$ inner product adjusted for the inhomogeneities of the graph - by multiplying by the degree $d_i$ at each vertex.

Perhaps it would be worthwhile to look at selected sets of orthogonal function under this inner product.

Second, this inner product is closely related to the Dirichlet norm.
Recall that the Dirichlet norm for an electrical network is
\[ W_y(u) = \frac{1}{2} \sum_{\text{vertices}} (u_i - u_j)^2 = \frac{\Delta V^2}{R} = \text{Power} \]

Where \( y \) and \( u \) are equal on boundary nodes and \( u \) is \( \delta \)-harmonic

\[ W_y(u) \leq W_y(y) \]

In other words, solution to the Dirichlet problem minimize the Dirichlet norm (minim Power loss)

\( W_y(u) \) is a norm (or a semi-norm) over edges we can convert it into a norm of vertices

First, assume that \( \delta = 1 \)

\[ W_y(u) = \frac{1}{2} \sum_{\text{edges}} (u_0^2 - 2u_0u_y - u_y^2) \]

\[ = \frac{1}{2} \sum_{\text{edges}} (u_0^2 - 2u_0u_y - u_y^2) \]

\[ = \frac{1}{2} \left( \sum_{\text{edges}} \partial_0 u_0^2 - 2u_0 \sum_{\text{edges}} u_y + \sum_{\text{edges}} u_y^2 \right) \]

give the last term to their respective vertices and you get

\[ = \frac{1}{2} \sum_{\text{edges}} \partial_0 u_0^2 - 2u_0 \sum_{\text{edges}} \frac{\partial u_0}{\partial \delta} \]

\[ = \sum_{\text{edges}} \partial_0 u_0 \left[ u_0 - \frac{\partial u_0}{\partial \delta} \right] \]

We recognize this as our inner product of \( u \) with a function measuring how different \( u \) is from harmonic.
One may wonder if we redesign the random walk to run directly on networks where \( \Theta \) varies, will the random walk-inner product and Dirichlet norm still be closely related?

Let's convert the inner product to an inner product over edges just to see what it looks like.

Restricting our space to solutions to the Dirichlet problem on a graph with boundary vertices \( A, B, C, D \ldots \)

the functions on the vertices \( P_0A, P_0B, P_0C \ldots \)

form a basis for that space.

\[
\langle A, B \rangle = \sum_{i \in \text{vertices}} \Phi_i P_0A P_0B = \sum_{i \in \text{vertices}} \sum_{j \in \text{edges}} \phi_i e((i \rightarrow j))
\]

\( e((i \rightarrow j)) = \text{expected number of times edge from } i \text{ to } j \text{ is crossed in that direction only passing the start at } A \text{ and end at } B \)

\[
V_i' = \sum_{j \rightarrow i} e((j \rightarrow i))
\]

\[
\langle A, B \rangle = \sum_{i \in \text{vertices}} \phi_i P_0A \left( \sum_{j \in \text{edges}} P_0B \right) = \sum_{i \in \text{vertices}} P_0A \sum_{j \in \text{edges}} P_0B
\]

The same argument shows:

\[
V_i' = \sum_{j \rightarrow i} e((j \rightarrow i)) = \sum_{j \rightarrow i} P_0A P_0B
\]