SUFFICIENT CONDITIONS FOR AN ELECTRICAL NETWORK TO BE N-TO-ONE

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ABSTRACT. In this paper, we discuss the inverse problem for electrical networks, and give sufficient conditions for an electrical network to have an n-to-one correspondence for some positive integer n. We also discuss some of the problems that have come up so far in constructing an n-to-one correspondence and show how to avoid these problems.

1. DEFINITIONS

Convention. By a graph, we shall always mean a graph with no loops, but which may have multiple edges.

Definition 1. A *graph with boundary* is a graph $G = (V, E)$ with a set of vertices designated *boundary nodes,* the rest are called *interior nodes.* The set of boundary nodes will be denoted ∂V and the interior nodes will be denoted $intV$.

Definition 2. An *electrical network* $\Gamma = (G, \gamma)$ is a graph $G = (V, E)$ with boundary and a function $\gamma\colon E \to \mathbb{R}^+$ called the *conductivity function.* For any edge $e \in E$ the *conductance* of the edge e is $\gamma(e)$.

Definition 3. A *linear fractional transformation* is function from $\mathbb{R}$ to $\mathbb{R}$ which is a ratio of two linear functions.

Notation. If $\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m$ we will use $(\mathbf{x}, \mathbf{y})$ for $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$.

2. INTRODUCTION

The purpose of this paper is to explore the correspondence between the Kirchhoff matrix for an electrical network and its response matrix. For some graphs this correspondence is neither one-to-one nor infinite to one. We give sufficient conditions for a network to have an n-to-one correspondence for a natural number n.

2.1. The Forward Problem and The Inverse Problem.

We now discuss the forward problem for electrical networks, and introduce the inverse problem. A more detailed discussion is given in [2].

Suppose $\Gamma = (G, \gamma)$ is an electrical network, with n vertices, $1, 2, ..., n$ (we often use numbers to denote vertices) we shall use $\gamma_{i,j}$ for the sum of the conductances over all edges from vertex i to j. If i and j are not adjacent we say $\gamma_{i,j} = 0$.

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Definition 4. Suppose $\Gamma = (G, \gamma)$ is an electrical network with n vertices. The *Kirchhoff matrix* K is an $m \times m$ matrix defined as follows.

(1) $K_{i,j} = -\gamma_{i,j}$, *if* $i \neq j$
(2) $K_{i,i} = \sum_{i \neq j} \gamma_{i,j}$

Suppose we are given a function u defined on the vertices of G, after labeling the vertices we may regard u as a vector in the obvious way. The physical interpretation of K is that $(Ku)_i$ is the resulting current into the network due to u out of node i. Suppose now we are given an electrical network $\Gamma = (G, \gamma)$ with $G = (V, E)$ and a function $\phi \colon \partial V \to \mathbb{R}^+$. The forward problem is to find a function $u \colon V \to \mathbb{R}^+$ such that $u = \phi$ on ∂V and $(Ku)_i = 0$ for all $i \in intV$. As discussed in [2] the forward problem has a unique solution for every electrical network with a boundary node in each connected component. Let $\Gamma = (G, \gamma)$ be an electrical network and K its Kirchhoff matrix. Let A, B, C be sub-matrices corresponding to boundary to boundary connections, boundary to interior connections, and interior to interior connections, respectively. Then K has the following block structure

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}.$$

As proved in [2], the matrix C is always invertible and the solution to the forward problem is $(A - BC^{-1}B^T)u$. We will use Λ_γ to denote $A - BC^{-1}B^T$ and call it the network response. If G has no boundary vertices, then $\Lambda_\gamma = K$. The ij-th entry of Λ_γ will be denoted $\lambda_{i,j}$. The inverse problem is to recover K from Λ_γ, specifically we wish to see if the map sending γ to Λ_γ is one to one. For some networks, this correspondence is neither one-to-one nor infinite-to-one but is instead n-to-one for some $n \in \mathbb{N}$ with n bigger than 1. The first example discovered was in [3]. We now discuss the amalgamation and the star-K transformation, which will help us understand the possibility of an n-to-one correspondence.

2.2. **Graph Amalgamation.** We will often need to construct a graph by gluing it from smaller pieces, we usually do this by labeling vertices on each smaller piece and then gluing together vertices which have the same labels. We now define precisely what we mean by such a gluing process.

Definition 5. Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2), \ldots, G_n = (V_n, E_n)$ be graphs whose vertex sets are disjoint, Let

$$V = \bigcup_{i=1}^{n} V_i.$$

Suppose L is a set,whose elements we shall call *labels,* such that $|L| \leq |V|$ and that $\phi \colon V \to L$ is a surjective function satisfying $\phi(v) \neq \phi(w)$ if v is adjacent to w. We shall call ϕ a *labeling function* (or a labeling function on $G_1, G_2, \ldots, G_n$). We define a new graph G as follows, the vertex set is $\{\phi^{-1}(a) \colon a \in L\}$. For each $a, b \in V$ let $B_{a,b}$ be the set of all edges between v and w for all nodes $v \in \phi{-}1(a), w \in \phi^{-1}(b)$. We then declare that for each $a, b \in L$ there should be $|B_{a,b}|$ edges between $\phi^{-1}(a)$ and $\phi^{-1}(b)$ in G. We call G a the connected direct sum of the G_i with respect to ϕ and write

$$G = \left(\sum_{i=1}^{n} G_i \right)_\phi ,$$

Suppose G is a graph with boundary and $a \in L$. If $|\phi^{-1}(a)| = 1$ then we declare $|\phi^{-1}(a)|$ to be a boundary node if and only if the unique element in $|\phi^{-1}(a)|$ is. If $|\phi^{-1}(a)| > 1$ we shall have to declare whether $|\phi^{-1}(a)|$ is a boundary node or not.

Definition 6. Suppose $\Gamma_1 = (G_1, \gamma_1), \Gamma_2 = (G_2, \gamma_2), \dots, \Gamma_n = (G_n, \gamma_n)$, with $G_i = (V_i, E_i)$ for all $i \in 1, 2, \dots, n$, and $V_i \cap V_j =$ if $i \neq j$. For each i and for each edge $e \in E_i$ we shall use $\gamma(e)$ for $\gamma_i(e)$. Let L be a set such that $|L| \leq |\bigcup_{i=1}^{n} V_i|$, and let ϕ be a labeling function on $G_1, G_2, \dots, G_n$. Let G be the connected direct sum of the G_i with respect to ϕ. We define a new conductivity function $\widetilde{\gamma}$ as follows.

For each pair of vertices $\phi^{-1}(v), \phi^{-1}(w)$ in G let $B_{v,w}$ be the set of all edges between $\phi^{-1}(v)$ and $\phi^{-1}(w)$ in G, and let $A_{v,w}$ be the set of all edges between elements of $\phi^{-1}(v), \phi^{-1}(w)$. By definition there exists a bijection $f : B_{v,w} \to A_{v,w}$. For each $e \in B_{v,w}$ we now define $\widetilde{\gamma}(e) = \gamma(f(e))$.We call $\Gamma = (G, \widetilde{\gamma})$ the *connected direct sum* of the Γ_i with respect to ϕ and write

$$\Gamma = \left(\sum_{i=1}^{n} \Gamma_i \right)_{\phi}.$$

We will sometimes drop any explicit reference to the labeling function if it is clear from the context. We thus sometimes speak of the connected direct sum of G_i and write $G = \sum_{i=1}^{n} G_i$, th similar conventions for networks. Whenever we talk about gluing graphs, or networks, together by labeling vertices we will mean in the sense of the preceding. This topic is discussed more in [1] and [8], there this process is called "amalgamation."

2.3. **The Star-K Transformation.** The star-K transformation allows us to regard the response matrix of an electrical network as an electrical network itself. The reader is encouraged to go to [9] for more details.

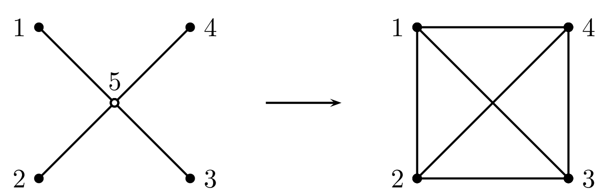


FIGURE 1. The four-star S_4 transformed into the complete graph on 4 vertices, K_4.

Given $n \in \mathbb{N}$ the n-star, denoted S_n, is a graph with one central vertex, which is adjacent to n other vertices, none of which are adjacent to each other. To regard S_n as an electrical network the central vertex is interior, every other vertex is a boundary vertex. The complete graph on n vertices, denoted K_n is a graph with n vertices which are all adjacent to each other. For K_n to be an electrical network we have all of its vertices be boundary vertices. The star-K transformation is a way to take a graph which is "composed" of star graphs, and replace them with complete graphs.

Suppose we start with S_n and a conductivity function γ on S_n , we then replace S_n with K_n and define a new conductivity function μ on K_n by $\mu_{i,j} = -\lambda_{i,j}$. In general we cannot reverse the operation, but in [9] it is proved we can under a simple condition, called the quadrilateral condition. Let $\mu_{i,j} = -\lambda_{i,j}$, then K_n comes from S_n if and only if for any four distinct vertices i, j, k, l

$$\mu_{i,j}\mu_{k,l} = \mu_{i,k}\mu_{j,l}. \tag{1}$$

That is, the products of conductances on opposite sides of a quadrilateral in a complete graph which came from a star graph are equal.

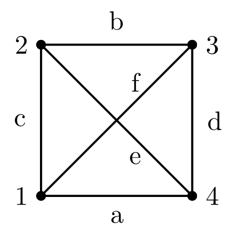


FIGURE 2. The quadrilateral condition is that $ab = cd = ef$.

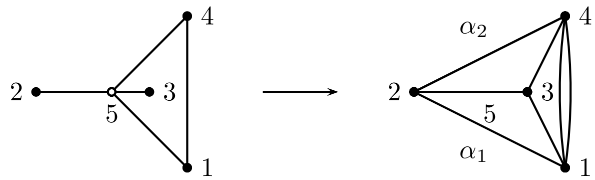
Suppose the quadrilateral condition is satisfied and let $\{1, 2, \ldots, n\}$ be the boundary vertices of S_n and denote the interior vertex by $n + 1$. Then

$$\gamma_{i,n+1} = \left(\sum_{i \neq j} \mu_{i,j}\right) + \frac{\mu_{i,j}\mu_{i,k}}{\mu_{j,k}} \tag{2}$$

this is well defined by the quadrilateral condition. (See [9]). It is also proved in [9] that if we know the original graph we can calculate the conductances on the new graph by

$$\mu_{i,j} = \frac{\gamma_{i,n+1}\gamma_{j,n+1}}{\sum_{k=1}^{n} \gamma_{k,n+1}} \tag{3}$$

We also describe the inversion, which is another graph we can apply the star-K transformation to. The inversion is drawn below on the left and its star-K transformed graph is drawn on the right.



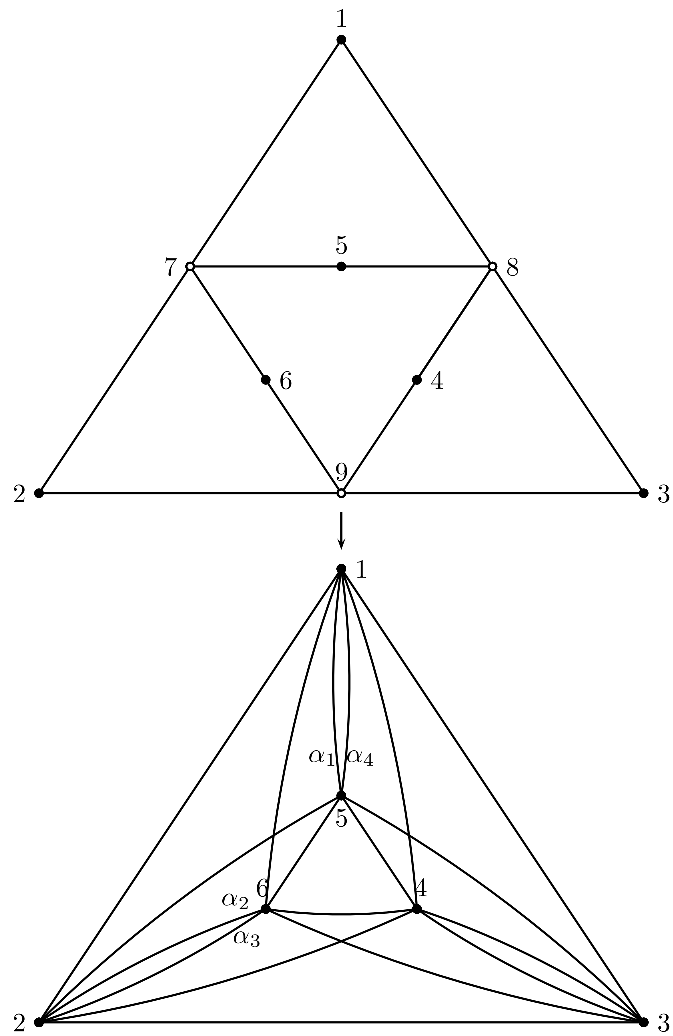
Using the quadrilateral rule, we deduce that $\alpha_2 = \frac{\mu_{3,4}\alpha_1}{\mu_{1,2}}$.

Suppose we have networks $\Gamma_1 = (G_1, \gamma_1), \Gamma_2 = (G_2, \gamma_2), \ldots, \Gamma_n = (G_n, \gamma_n)$ where each G_i is a star or an inversion for $i \in \{1, 2, 3 \ldots, n\}$ and let $V = \bigcup_{i=1}^{n} V_i$. For each i, let $\hat{\Gamma}_i = (\hat{G}_i, \hat{\gamma}_i)$ be the star-K transformed network and let ϕ be a labeling function on $G_1, G_2, \ldots, G_n$, such that $|\phi^{-1}(\phi(a))| = 1$ for all interior nodes a (i.e. we are not gluing interior nodes together). We can apply the star-K transformation to $\Gamma = (\sum_{i=1}^{n} \Gamma_i)_\phi$, as follows. For each i, let $\hat{G}_i = (\hat{V}_i, \hat{E}_i)$ and let $\hat{V} = \bigcup_{i=1}^{n} \hat{V}_i$. We may regard $\hat{V}_i$ as a subset of the vertex set for G_i ,we then restrict the domain of ϕ to $\hat{V}$, we also restrict the codomain of ϕ, if necessary, to obtain a surjective function $\hat{\phi}$. We then set $\hat{\Gamma} = \left(\sum_{i=1}^{n} \hat{\Gamma}_i\right)_{\hat{\phi}}$ .Then $\hat{\Gamma}$ is the star-K transformed network. Let $\hat{\gamma}$ be the conductivity function for $\hat{\Gamma}$ and letΛ be the response matrix for Γ. Under these conditions it is still true that $\hat{\gamma}_{i,j} = -\Lambda_{i,j}$, by construction the quadrilateral rule holds for each K.

A graph G with boundary which can be constructed by gluing stars and inversions together as above, will be called *star-based*, and the collection $\{G_i\}$ will be called a *star-basis* for G.

Note that in general this transformation leads to multiple edges. Suppose we do not know the original Kirchhoff matrix, and that for two nodes i, j there are n edges between i and j we then pick n variables $\alpha_1, \alpha_2, \ldots, \alpha_n$ and place one on each edge in the transformed graph and call them partial conductances. We are then required to solve for the α_1 requiring that $\sum_{i=1}^{n} \alpha_i = \mu_{i,j}$, and that they are all positive. If we do know the original Kirchhoff matrix, then for any edge e between i and j we can calculate the partial conductance on e by $\hat{\gamma}(e)$.

Example 1. Triangle – in – Triangle Graph



Using the quadrilateral rule, $\alpha_2 = (\mu_{1,2}\mu_{5,6})/(\alpha_1)$. Using the double edge $\alpha_3 = \mu_{2,6} - (\mu_{1,2}\mu_{5,6})/(\alpha_1)$. Continuing, we see that the α_4 must be

$$\frac{\mu_{1,4}\mu_{3,5}}{\mu_{2,3} - \cfrac{\mu_{3,6}\mu_{2,4}}{\mu_{2,6} - \cfrac{\mu_{1,2}\mu_{5,6}}{\alpha_1}}}.$$

Therefore

$$\mu_{1,5} = \alpha_1 + \cfrac{\mu_{1,4}\mu_{3,5}}{\mu_{2,3} - \cfrac{\mu_{3,6}\mu_{2,4}}{\mu_{2,6} - \cfrac{\mu_{1,2}\mu_{5,6}}{\alpha_1}}}. \tag{4}$$

If we clear denominators in (4) the continued fraction can be written as a linear fractional transformation. We then clear denominators again to see that α_1 satisfies a quadratic polynomial, this can lead to a two-to-one correspondence for appropriate values of conductances. Note that this polynomial is only determined by our choice of starting with α_1. If instead we start α_4 and determined α_1 by going clockwise around the graph our equation would be

$$\mu_{1,5} = \alpha_4 + \cfrac{\mu_{1,6}\mu_{2,5}}{\mu_{2,6} - \cfrac{\mu_{3,6}\mu_{2,4}}{\mu_{3,4} - \cfrac{\mu_{1,4}\mu_{3,5}}{\alpha_1}}} \tag{5}$$

If we do the same process with (5) as we did with (4) we will in general have a different quadratic equation.

There are other examples of n-gon in n-gon networks which have the same behavior, see [5].

In order to be valid partial conductances, it is necessary and sufficient for each partial conductance to be positive and to satisfy (1). Condition (1) has already been discussed. To see that it is sufficient that each partial conductance be positive we use (2); for necessity (3).

Suppose $\Gamma = (G, \gamma)$ is an electrical network constructed by gluing stars together as described above, let N be the number of edges in G and let M be the number of edges in its star-K transformed graph, not counting multiple edges. We may regard the conductivity function γ as an element of $\mathbb{R}^N$ and the response matrix as an element of $\mathbb{R}^M$.

Fix a response matrix $\Lambda = (\lambda_{i,j})$ for Γ. Suppose β and α are partial conductances, and that for some linear fractional transformation l_i, we have $\beta = l_i(\alpha)$. We thus have a function $g_\beta\colon (\mathbb{R}^+)^{N+1} \to \mathbb{R}$, continuous in a neighborhood of (α, Λ), so that $\beta = g_\beta(\alpha, \Lambda)$. Each of the entries in the response matrix may be written as rational functions of the conductances in the original graph. Thus for each partial conductance β we can find a function $f_\beta : \mathbb{R}^{N+1} \to \mathbb{R}$ continuous in a neighborhood (α, γ), so that $\beta = f_\beta(\alpha, \gamma)$.

3. Polynomial Relations and Multiple Solutions

As seen above, sometimes we can use the quadrilateral conditions and partial conductances to write an entry in the response matrix $\mu_{i,j}$ as a function of a partial conductance α. We wish to discuss this situation in general, so we must come up with a class of graphs we can apply this analysis to. We first define the notion of a multiplexer. Intuitively, a multiplexer is what allows us to write some partial conductances in a graph as a function of others.

Definition 7. Given a graph with boundary G, a *multiplexer* is a partitioning of all the subsets of size two of ∂V into two sets U and K, where U is called the *unknown set* and K is called the *known set.* We require that

(1) For any function $f\colon U \to \mathbb{R}^+$, if there exists a response matrix Λ_1 such that $(\Lambda_1)_{i,j} = -f(\{i,j\})$ for all $\{i,j\} \in U$ then there also exists a response matrix $\Lambda_2 \neq \Lambda_1$ with $(\Lambda_2)_{i,j} = -f(\{i,j\})$ for all $\{i,j\} \in U$.

(2) For all $\{i,j\} \in K$, and for any negative real number λ there is at most one response matrix Λ on G such that $(\Lambda)_{i,j} = \lambda$.

If $|U| = n$ we call this multiplexer an n-plexer.

Note that by (2) the graph is recoverable. Intuitively, (1) says that if we only know entries corresponding to pairs in the unknown set we cannot recover all the conductivities on the graph, while (2) says that if we known any entry corresponding to a pair in the known set we can recover all the conductivities in the graph. Multiplexers are described in more detail in [4].

We are now able to describe general conditions where we can generalize the analysis of the triangle-in-triangle graph.

Definition 8. Let G be a star-based graph and let G_i be a star-basis for G, and for each i let P_i be a multiplexer for G_i. Let $\hat{G}$ be the the star-K transformed graph of G an suppose there exist two nodes a, b in $\hat{G}$ which have n edges $e_1, e_2, \ldots, e_n$ between them which satisfy the following properties.

(1) For any positive real number β there is at most one conductivity function γ on G such that $\hat{\gamma}(e_1) = \beta$.

(2) For each $j \in \{2, 3, \ldots, n\}$ there exists a sequence

$$e_1 = e_{j,0}, e_{j,1}, \ldots, e_{j,m_j} = e_j$$

of edges which satisfying the following. Let $e_{j,k} = \{a_k, b_k\}$ and $e_{j,k+1} = \{a_{k+1}, b_{k+1}\}$. Either $\{a_{k+1}, b_{k+1}\} = \{a_k, b_k\}$ and $e_{i,k}, e_{i,k+1}$ are the only edges between a_k and b_k, or $\{a_k, b_k\}$ is an element in the known set of one of the multiplexers P_l and $a_{k+1} \neq a_k, b_{k+1} \neq b_k$, and a_k, b_k are vertices in G_l.

By (1) knowing a valid partial conductance on e_1 allows us to recover all the conductivities in G. By (2) for each $k+1 \in \{1, 2, \ldots, m_j\}$ there exist a unique linear fractional transformation $l_{j,k+1}$ such that for any valid partial conductance α on $e_{j,k}$ a valid partial conductance on $e_{j,k+1}$ is given by $l_{j,k+1}(\alpha)$. Let

$$l_j = l_{j,m_j} \circ l_{j,m_j-1} \circ \cdots \circ l_{j,1}.$$

By the preceding, if α is a partial conductance on e_1 we conclude that $\mu_{a,b}$ satisfies

$$\mu_{a,b} = \alpha + l_1(\alpha) + l_2(\alpha) + \ldots + l_n(\alpha),$$

for $i = 1, \ldots, n$, moreover if we know a root of this equation we can recover all the conductivities in the graph. Clearing denominators and subtracting we see that α must satisfy what is, in general, a polynomial p of degree n. We call p the conductance polynomial of the original electrical network with respect to the edge e_1.

Note that the conductance polynomial depends on our choice of starting edge, (this was noted in Example 1). However, if the choice of edge is irrelevant we may abuse terminology and call p the conductance polynomial.

The reader should see how the steps involved in this definition work in the example of the triangle-in-triangle graph, or indeed, any of the n-gon in n-gon graphs.

The reader may be worried that a valid partial conductance α may correspond to a point where this denominator vanishes, but this cannot happen. Each l_i may be written as a composition of linear fractional transformations $l_i = S_1 \circ S_2 \cdots \circ S_n$. Since α is a valid partial conductance implies that $(S_i \circ S_{i+1} \cdots \circ S_n)(\alpha) > 0$ and finite for all i. For any two linear fractional transformations $S_1(x) = \frac{a_1x+b_1}{c_1x+d1}$ and $S_2(x) = \frac{a_2x+b_2}{c_2x+d2}$,

$$(S_1 \circ S_2)(x) = \frac{a_1(\frac{a_2x+b_2}{c_2x+d_2}) + b_1}{c_1(\frac{a_2x+b_2}{c_2x+d_2}) + d_1}.$$

If the denominator is not 0, then

$$(S_1 \circ S_2)(x) = \frac{(a_1a_2 + c_2b_1)x + b_2a_1 + d_2b_1}{(a_2c_11 + d_1c_2)x + b2c_1 + d_1d_2}.$$

If this expression is finite the new denominator cannot be 0. In our case we know all the intermediate linear fractional transformations are positive and finite so the denominator of l_i has to be nonzero for a valid partial conductance. Thus we can clear denominators and conclude that a valid partial conductance must satisfy the conductance polynomial.

This polynomial can have more than one root, of course, when this happens we may have more than one choice for the partial conductance α. Each of these choices will determine a sequence of partial conductances, either by the quadrilateral rule, or by the use of a multiple edge. If all the partial conductances obtained this way are positive, then we have a correspondence which is not one-to-one, but is still finite-to-one. We must assume that in every star we cannot determine α from the other entries in Λ, because if we know α we could determine the other entries and the correspondence would be one-to-one. This is why we need to use a multiplexer at every stage.

We now give sufficient conditions for all the partial conductances obtained from every root of a conductance polynomial p to be positive. Our idea will be to find values of conductances so that the polynomial p has a root of multiplicity n; we will show that for coefficients "nearby" our original polynomial there are conductances which give an n-to-one correspondence. To make this precise associate to every real polynomial $p(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0$ its vector of coefficients $\mathbf{v}_p = (a_0, a_1, \ldots, a_n)$, we can then make the notion of "nearness" precise by using standard Euclidean distance. Recall that for any $\varepsilon \in \mathbb{R}^+$ and $\mathbf{x} \in \mathbb{R}^{n+1}$ we use $B_\varepsilon(\mathbf{x})$ for the ball of radius epsilon around $\mathbf{x}$, i.e. $B_\varepsilon(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^{n+1} : |\mathbf{x}-\mathbf{y}| < \varepsilon\}$.

Theorem 1. *Let p be a polynomial of degree n with real coefficients, and suppose p has a root of multiplicity n. For any positive real number ε there exists $\mathbf{v}_q \in B_\varepsilon(\mathbf{v}_p)$ so that q is a real polynomial with n distinct real roots, furthermore the roots of q can be made arbitrarily close to the roots of p by making ε small enough.*

Proof. Let ε be a given positive real number. By hypotheses p may be written as $p(x) = a_n(x-a)^n$ for some $a \in \mathbb{R}$. For any $\mathbf{y} \in \mathbb{R}^n$ with $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ let

$p_{\mathbf{y}}(x) = a_n(x-(a+y_1))(x-(a+y_2))\cdots(x-(a+y_n))$. Let σ_i be the polynomial

$$\sigma_i(x_1, x_2, \ldots, x_n) = \sum_{j_1<j_2<\ldots<j_i} x_{j_1}x_{j_2}\cdots x_{j_i},$$

and $\sigma_0 = 1$.

The i-th coefficient of p is $(-1)^{n-i}a_n\sigma_{n-i}(a, a, \ldots, a)$ and the i-th coefficient of $p_{\mathbf{y}}$ is $(-1)^{n-i}a_n\sigma_{n-i}(a-y_1, a-y_2, \ldots, a-y_n)$. Since σ_{n-i} is continuous it follows that

$$(-1)^{n-i}a_n\sigma_{n-i}(a-y_1, a-y_2, \ldots, a-y_n) \to (-1)^{n-i}a_n\sigma_{n-i}(a, a, \ldots, a)$$

as $\mathbf{y} \to 0$. Therefore $\mathbf{v}_{p_{\mathbf{y}}} \to \mathbf{v}_p$ as $\mathbf{y} \to 0$. So we can find a δ such that $\mathbf{y} \in B_\delta(0)$ implies that $\mathbf{v}_{p_{\mathbf{y}}} \in B_\varepsilon(\mathbf{v}_p)$. Choosing $\eta \in B_\delta(0)$ with distinct coordinates it follows that $q = p_\eta$ has the desired properties. That the roots of q can be made close to the roots of p follows. □

We now recall some terminology from mutltivariable calculus which we will need in our next result.

Definition 9. Suppose U is an open subset of $\mathbb{R}^n$ and V is an open subset of $\mathbb{R}^m$ and let $T\colon U \to V$. Suppose all the first partial derivatives of T exist. For any vector $\mathbf{x} \in \mathbb{R}^n$, we use x_i for the i-th coordinate of $\mathbf{x}$, and T_j for the function defined by $T_j(\mathbf{x}) = (T(\mathbf{x}))_j$. The *differential* of T, denoted, dT is a $n \times m$ variable matrix defined by

$$(dT)_{ij} = \frac{\partial T_j}{\partial x_i}.$$

Define $[dT(\mathbf{x})]_{i,j} = \frac{\partial T_j}{\partial x_i}(\mathbf{x})$.

The coefficients of our associated polynomial may be written as a rational function of entries in the response matrix. We also know that entries in the response matrix may be written as rational functions of the conductivities in our network. Composing we have a map which sends conductances on the network to coefficients of the polynomial, for which we can calculate the differential. We use this fact in then next result.

Theorem 2. *Suppose we have an electrical network $\Gamma = (G, \gamma)$,which satisfies the conditions of Definition 8 and suppose its conductance polynomial has degree n. Let N be the number of edges in G. We may then regard $\gamma \in \mathbb{R}^N$. Suppose we have a vector of conductivities γ for which the conductance polynomial p has a root r of multiplicity n. Let $T\colon (\mathbb{R}^+)^N \to \mathbb{R}^n$ be the map which sends conductances on G to coefficients of the polynomial, assume that $dT(\gamma)$ has rank n+1. Then there exists a response matrix Λ which corresponds to n distinct Kirchhoff matrices.*

Proof. Since dT has rank $n+1$ there are $n+1$ columns which are linearly independent, let $e_1, e_2, \ldots, e_n, e_{n+1}$ be the edges which correspond to these $n+1$ columns. We fix the other conductivities on Γ and let the conductivities on $e_1, e_2, \ldots, e_n$ vary. This gives a new map T' which is locally invertible at $\gamma' = (\gamma(e_1), \gamma(e_2), \ldots, \gamma(e_n), \gamma(e_{n+1}))$. By the Inverse Function Theorem, we may select an open subset U of $(\mathbb{R}^+)^{n+1}$ containing γ' which maps homeomorphically onto an open subset V of $\mathbb{R}^{n+1}$ containing $\mathbf{v}_p$. By Theorem 1 we may select a sequence $\{\mathbf{v}_m\}$ so that the polynomial $p_{\mathbf{v}_m}$ corresponding to $\mathbf{v}_m$ has n distinct positive real roots, and $\mathbf{v}_m \to \mathbf{v}_p$ as $m \to \infty$. Since V is open we may choose $\mathbf{v}_m \in V$ for all m.

Let $\{r_{1,m}, \{r_{2,m}\}, \dots, \{r_{n,m}\}$ be the sequences of distinct roots of the polynomial $p_{\mathbf{v}_m}$. Theorem 1 implies that $r_{i,m} \to r$ as $m \to \infty$.

Since T' is invertible, for each m we can find γ_m so that $T'(\gamma_m) = \mathbf{v}_m$. Thus $\gamma_m = T'^{-1}(\mathbf{v}_m)$, and since $\mathbf{v}_m \to \mathbf{v}_p$ as $m \to \infty$ we conclude that $\gamma_{\mathbf{m}} \to \gamma'$ as $m \to \infty$, since T'^{-1} is continuous. One of the partial conductances in the transformed network must satisfy the original polynomial p, since p has only one root the root of this polynomial must be a partial conductance on the network. As described in section 2.3 for each β we can find a function $f_\beta \colon \mathbb{R}^{n+2} \to \mathbb{R}$, continuous in a neighborhood of (r, γ'), so that $\beta = f_\beta(r, \gamma')$. Since $f_\beta(r, \gamma') > 0$, we can find an open set $W \subseteq (\mathbb{R}^+)^{n+2}$ containing (r, γ') so that $(x, y_1, y_2, \dots, y_{n+1}) \in W$ implies that $f_\beta(x, y_1, y_2, \dots, y_{n+1}) > 0$.

Since $\gamma_m \to \gamma'$ as $m \to \infty$, and $r_{i,m} \to r$ as $m \to \infty$, we conclude that for each i there is a large enough N_i so that $m \geq N_i$ implies that each partial conductance which is a function of $r_{i,m}$ is positive. Picking m larger than all the N_i we conclude $r_{1,m}, r_{2,m}, \dots, r_{n,m}$ correspond to all positive partial conductances. Since all the roots are distinct we have constructed n different positive Kirchhoff matrices which correspond to the same response matrix.

□

We close this section with some conditions that guarantee a polynomial with exactly one root, recall that if p is a polynomial of degree n and if p has a root of multiplicity n, then $p'|p$; thus we may hope for conditions that guarantee a root of multiplicity n by using the division algorithm.

Theorem 3. *Let $n \in \mathbb{N}$ and let $p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots a_1 x + a_0$ be a polynomial of degree n whose leading term is not zero. Then p has a root of multiplicity n if and only if for all $i \in \{0, 1, \dots, n-2\}$ it is true that*

$$n(n-i)a_i a_n - (i+1)a_{n-1}a_{i+1} = 0. \tag{6}$$

Proof. First suppose that p has a root of multiplicity n, so $p'|p$. By the division algorithm,

$$p = \left(\frac{1}{n}x + \frac{a_{n-1}}{n^2 a_n}\right)p' + \sum_{i=0}^{n-2}\left(\frac{n-i}{n}a_i - \frac{(i+1)(a_{n-1}a_{i+1})}{n^2 a_n}\right)x^i.$$

Therefore $\left(\frac{n-i}{n}a_i - \frac{(i+1)(a_{n-1}a_{i+1})}{n^2 a_n}\right) = 0$ for all $i \leq n-2$ so equation (6) holds. Conversely, suppose equation (6) holds. Then since $a_n \neq 0$, we know

$$a_i = \frac{(i+1)a_{n-1}a_{i+1}}{n(n-i)a_n}, \tag{7}$$

Let $\widetilde{p(x)} = a_n(x + \frac{a_{n-1}}{na_n})^n$, and denote its i-th coefficent by $\widetilde{a_i}$, we claim that $p = \widetilde{p}$. We have that $\widetilde{a_i}$ is

$$\binom{n}{i}a_n \frac{a_{n-1}^{n-i}}{a_n^{n-i}n^{n-i}} = \frac{\binom{n}{i}a_{n-1}^{n-i}}{a_n^{n-(i+1)}n^{n-i}}. \tag{8}$$

By (7) we see $a_{n-2} = \frac{(n-1)a_{n-1}^2}{2na_n} = \frac{\binom{n}{n-2}a_{n-1}^2}{n^2 a_n}$ which agrees with $\widetilde{a_{n-2}}$. Assuming that $a_{i+1} = \widetilde{a_{i+1}}$, equations (7) and (8) imply that

$$a_i = \frac{(i+1)a_{n-1}a_{i+1}}{n(n-i)a_n} = \frac{(i+1)a_{n-1}}{n(n-i)a_n}\left(\frac{\binom{n}{i+1}a_{n-1}}{a_n^{n-(i-2)}n^{n-i-1}}\right) = \frac{\binom{n}{i+1}(i+1)a_{n-1}^{n-i}}{n^{n-i}(n-i)a_n^{n-(i+1)}}.$$

It is easy to see that $\frac{\binom{n}{i+1}(i+1)}{(n-i)} = \binom{n}{i}$ so $a_i = \widetilde{a_i}$. The claim holds by induction. $\square$

4. Linear Fractional Transformations and How to Control Signs of The Derivative

The method above was developed because the author had hoped to use it to show that a certain network was three-to-one. This turned out to be false. Recall from Section 2.3 that by repeated application of the Star-K transformation we end up with an equation

$$\mu_{i,j} = \alpha + l_1(\alpha) + l_2(\alpha) + \ldots + l_n(\alpha), \tag{9}$$

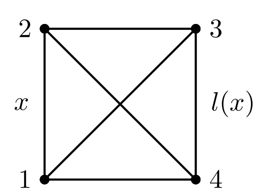
for some linear fractional transformations $l_1, l_2, \ldots, l_n$, and some $\mu_{i,j}$ which is the negative of an entry in the response matrix. Chad Klumb discovered (see [7]) that all the solutions to this equation must lie in an interval where the right hand side of (9) is continuous. So if each l_i has a positive derivative it cannot have n distinct roots in any interval where the right hand side of (9) is continuous. We now show how to control this behavior. For any 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

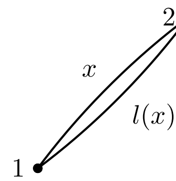
associate the linear fractional transformation $S_A(x) = \frac{ax+b}{cx+d}$. It follows from Section 3 that $S_A \circ S_B = S_{AB}$. So we have a homomorphism from the group of 2×2 invertible real matrices to the group of real linear fractional transformations. Also

$$S'_A(x) = \frac{a(cx+d) - c(ax+b)}{(cx+d)^2} = \frac{\det A}{(cx+d)^2}.$$

Therefore $\operatorname{sign}(S'_A) = \operatorname{sign}(\det A)$. The continued fractions we have can be written as compositions of linear fractional transformations. Using the homomorphism we see that $\operatorname{sign}[(S_{A_1} \circ S_{A_2} \cdots \circ S_{A_n})'] = \operatorname{sign}(\det A_1 \det A_2 \cdots \det A_n)$. This allows us to calculate the signs of the derivatives of our linear fractional transformations without actually computing the composition. We simply account for the sign at each stage. Suppose we express a partial conduction x in a four-star as a function of the partial on an opposite edge $l(x)$. This will take the form $l(x) = C/x$ for some positive constant C, and has a negative derivative.

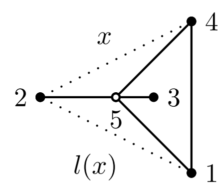


Expressing one conductance as a function of another across a double edge is a



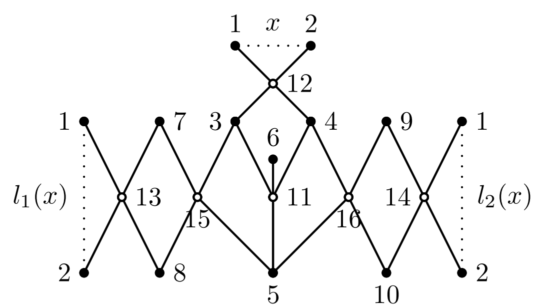
transformation of the form $l(x) = C-x$ and has a negative derivative.

Writing one partial conductance as function of another across an inversion can be written as $l(x) = Cx$ for some postive constant C, (in our figure below $l(x) = \frac{\mu_{1,3}x}{\mu_{3,4}}$)



and thus has a positive derivative.

These are the basic transformations we use so we can keep track of the signs of the derivative easily. For instance, the linear fractional transformations l_1, l_2 in the graph below (note that nodes numbered the same should be identified) always have positive derivatives, this was the graph that the author worked on; which is not three-to-one.



One simply counts the four-stars and double edges in pairs to see that we end up with all positive derivatives. In fact if we start with any multiplexer for which we can write all partial conductances in the unknown set as a linear function of one partial conductance in the known set, and then attach four-stars, the same reasoning applies and shows that we always get positive derivatives. So we know why we need the inversions that Ilya used in [6], replacing a four-star with an inversion switches the sign of the derivatives and allows negative derivatives, this shows us why Ilya's graph without inversion is not three-to-one. For each integer $n \geq 3$ we can now give examples of graphs G_n that have the possibility of being n-to-one and avoid these problems with the derivatives. For simplicity we indicate boundary nodes with solid dots, and interior nodes with open circles. We start with an $n+1$-star in the middle, its boundary nodes numbered $1, 2, \ldots, n+1$ the interior node to be numbered later. let X_i be the graph drawn in figure 3. For each $i-1 \in \{2, \ldots (n-1)/2\}$ let X_i be the graph drawn in figure 4, the other nodes to be numbered later. For all $i-1 \in \{(n-1)/2, (n-1)/2+1, \ldots, n\}$ let X_i be the graph drawn in figure 5.

For each of these graphs we number the nodes as in figure 3,4,5. Let N be the number of remaining unnumbered boundary nodes and M the number of interior nodes. Label the rest of the boundary nodes $n+4, n+5, \ldots, n+3+N$ and the interior nodes $n+4+N, \ldots, n+3+N+M$. We then set

$$G_n = \sum_{i=1}^{n} X_i.$$

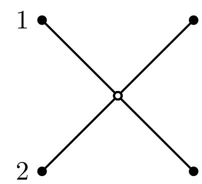


FIGURE 3.

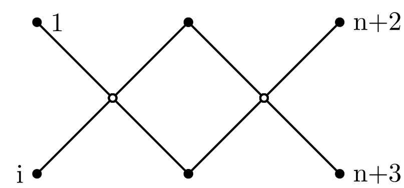


FIGURE 4. Two four-stars glued together.

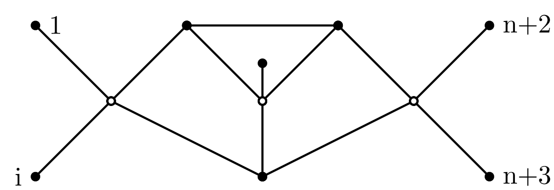


FIGURE 5. Two four-stars with an inversion.

If n is even we do the same but instead have one more graph which has an inversion.

We need at least two four-stars on every wing except one to prevent an edge in series. Note that we use a complete bipartite graph for the multiplexer, that is because it is the only n-plexer known to exist for all n , as discussed in [4]. It is not hard to see that once we fix a partial conductance x on one of the $\{n+2, n+3\}$

edges all the partial conductances on the $\{n+2, n+3\}$ can be written as $l_i(x)$ for some linear fractional transformation l_i. It is not hard to verify that in the case of odd n half the l_i is have positive derivatives, and the other half have negative derivatives. For n even there is one more negative derivative. It may be the case that this is not the best choice for the signs of the derivatives. This can be easily changed by either deleting one of the inversions or by replacing one of the four-stars with an inversion.

5. Future Research

One obvious direction for research is to try and find methods for deciding whether or not the graphs described above are n-to-one. Another is to investigate the behavior of the map T which sends conductances on the original graph to coefficients of its associated polynomial. In order to prove Theorem 2 we needed the assumption that dT had full rank. This assumption may be already satisfied, there does not seem to be enough dependence of the equations on each other for the rank of dT to not be full. One could also make refine Theorem 1 and find out which way to move the coefficients to guarantee n distinct real roots. This may make the rank hypothesis of Theorem 2 superfluous, because it is not necessary to move the coefficients in an arbitrary direction just in the right direction.

If one can associate a polynomial p to an electrical network Γ which has the properties that a root of p can be used to determine all the partial conductances in $\hat{\Gamma}$ and that the other partial conductances depend continuously on such a root, then one can make an obvious generalization of Theorem 2. Therefore one further direction for research would be to try and define the conductance polynomial in further generality.

The idea for Theorem 2 came from Ernie Esser, and from our technique to try to prove our graph was three-to-one. We put random conductivities on most of the edges of our original graph but set some others to be variable. We then try to solve equations to prove that it had values for which the polynomial had a root of multiplicity three. In doing this we are studying a polynomial whose coefficients vary, but in a continuous manner. One possibility for future research is to try to understand more about how the roots of a polynomial vary when the coefficients vary continuously. What can one say about the behavior of the roots in this situation? All the author is aware of is the result of Theorem 1 and that if the leading term never vanishes the roots vary continuously. This last fact can be seen by using Rouche's Theorem from Complex Analysis.

Theorems 2 and 1 has obvious generalizations, one is that if you have a polynomial with all real roots, and no assumption on the multiplicities, there are polynomials which have distinct roots, and whose coefficients are nearby the original polynomial. One simply applies Theorem 1 and uses the fact that coefficients of products of polynomials depend continuously on the original coefficients. One can generalize Theorem 2 to say that if you have conductances for which all the roots of the associated polynomial (whether they are distinct or not) correspond to positive Kirchhoff matrices, and if dT has full rank; then there are conductances nearby which correspond to n-distinct positive Kirchhoff matrices. The proof is the same as before.

6. Acknowledgments

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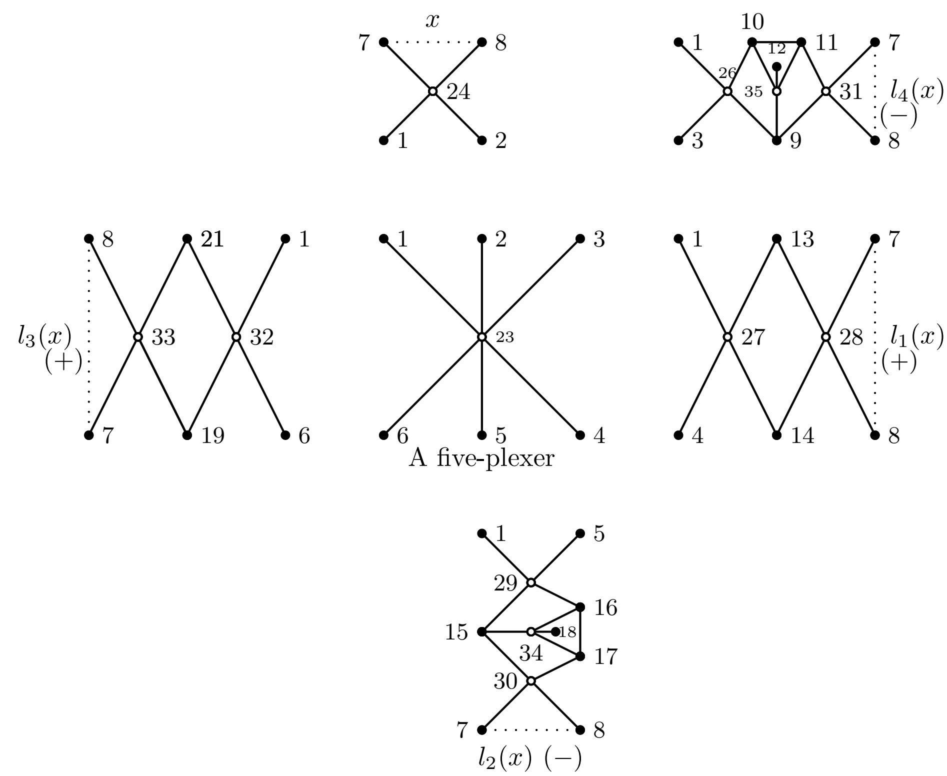

FIGURE 6. A potentially five-to-one graph, nodes numbered the same should be considered identical. Indicated are the signs of the derivatives of the partial conductances on the $(7, 8)$ edge.

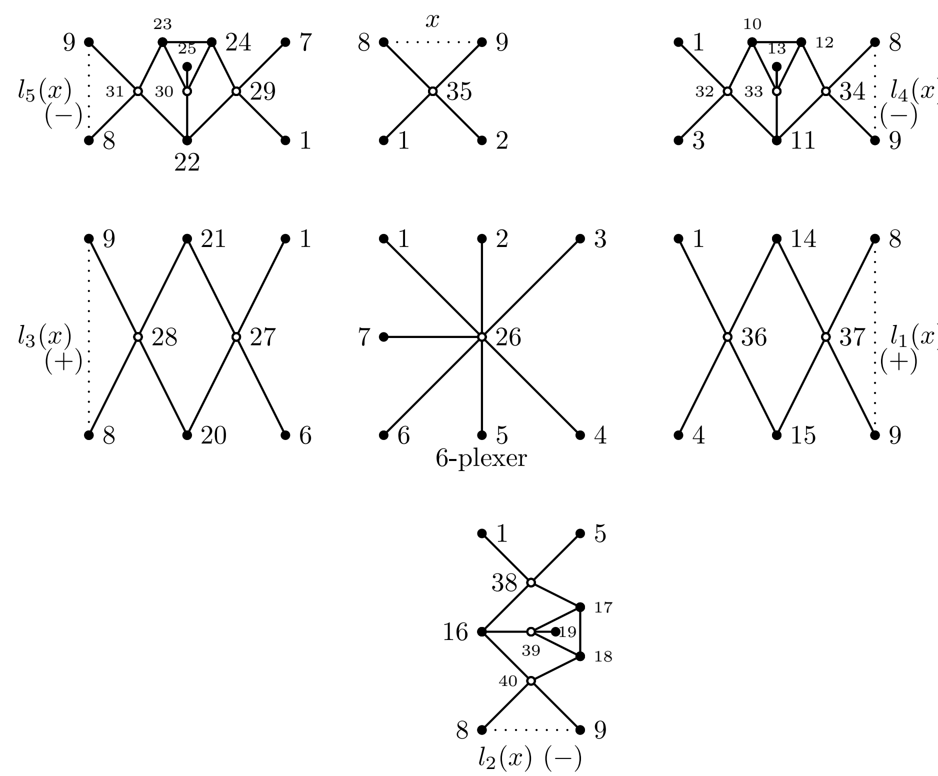

FIGURE 7. A possibly six-to-one graph

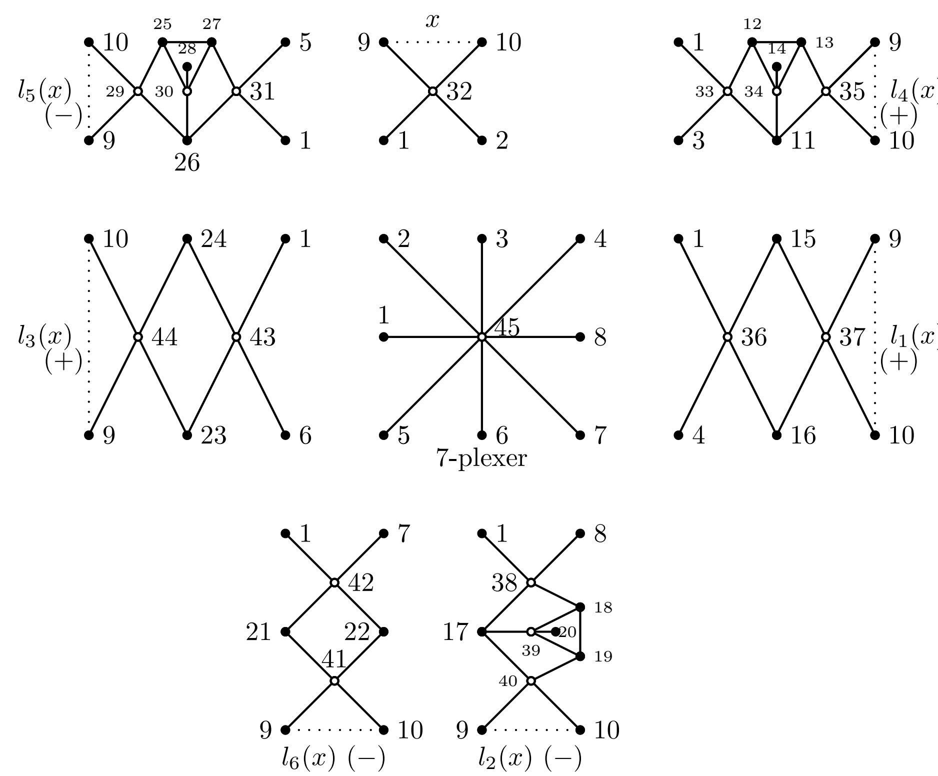

Figure 8. A potentially seven-to-one graph.