WHEN IS A FUNCTION EVERYWHERE INVERTIBLE?

Contents

1. Introduction

Immense benefits can be derived by applying mathematics to life sciences. Gregor Mendel has already demonstrated the use of inverse problems in his famous pea-plant experiment, where he deduced, from the consistent 9:3:3:1 ratio he observed in the traits, that there are heritable factors (what are now called genes) inherited according to the laws of probability. There are still several significant challenges that can be addressed by applying mathematics, particularly the inverse conductivity problem, to biology. By modeling the chemical reactions occurring in a complex biological system, such as a cell or organism, as a network of resistors (reversible reactions), capacitors (enzyme catalyzed reactions), and transistors (active transport coupled reactions through organelle membranes), and applying the techniques of a generalized inverse conductivity problem to the equivalent electrical network, we could infer the internal behavior of the system based on externally observable data. However, the biological criteria for data many not be the same as the internal and external behavior of an electrical network (of resistors, capacitors, inductors, and transistors), so that the boundary function that we have to recover the conductivities from is not necessarily the same as the electrical Dirichlet-to-Neumann map. This paper on global invertibility of functions from $\mathbf{R}^n$ to $\mathbf{R}^n$ (or more exactly, to an n-dimensional submanifold of a higher-dimensional set) will examine all $C1$ maps from $\mathbf{R}^n$ to $\mathbf{R}^n$ that satisfy certain conditions and prove existence of global inverses of these maps. (The domain is supposed to represent the conductivities of the bio-network.)

2. Fixed Point Criteria, C^1-ness of the inverse.

The motivation for most of the theorems in this paper arises from the following proposition. Theorem 1 provides a characterization of nonsingular matrices that also applies to nonlinear $C1$-maps.

Date: June 21, 2003.

Theorem 1: *Let H be an $n \times n$ matrix, and suppose there exists a matrix Y such that for every vector $\mathbf{u}$ and $\mathbf{v} \in \mathbf{R}^n$, there exists a constant $c < 1$ such that*

$$|(I - YH)(\mathbf{u} - \mathbf{v})| = |\mathbf{u} - YH\mathbf{u} - \mathbf{v} + YH\mathbf{v}| \leqslant c|\mathbf{u} - \mathbf{v}|$$

Then the original matrix H is nonsingular.

Proof: Suppose on the contrary that H is singular. Let $\mathbf{u}$,$\mathbf{v}$ belong to the nullspace of H. Then for every matrix Y, $|(\mathbf{u} - \mathbf{v}) - Y \cdot (H\mathbf{u} - H\mathbf{v})| = |\mathbf{u} - \mathbf{v}|$, leading to a contradiction.

The version of this theorem for nonlinear C1-maps is as follows:

Theorem 2. *Let f be a function on an open connected set $U \subset \mathbf{R}^n$ and suppose there exists a matrix A such that for every u, v in U,*

$$|(I - Af)(u - v)| < c|u - v|$$

for c ¡ 1. Then a)*the Jacobian of f is never zero, and* b) *the function f is globally invertible.*

Proof: a) suppose that for some value x, $Jf(x) \cdot \sigma = 0$ for a vector σ. Therefore $|\frac{\partial}{\partial \sigma}(I - Af)| = 1$. Pick ϵ. There exists δ such that: For every u, v with $|u - v| < \delta$, $|\frac{|(I-Af)(u-v)|}{|u-v|} - 1| > -\epsilon$. By the triangle inequality, $|\frac{|(I-Af)(u-v)|}{|u-v|}| > 1 - \epsilon$. This is a contradiction.

b) For any vector y, $I - Af + Ay$ is a contraction, so it has a unique fixed point. This fixed point solves $f(x) = y$.

2.1. Conditions on derivatives, in the special case of finite-dimensional spaces.

The global inverse is also a C1-map (by the inverse mapping theorem). Theorem II also provides an iterated-function algorithm to calculate the global inverse of a function. In the one-dimensional case, theorem II implies that any function with nonzero bounded derivative is globally invertible. However, in multiple dimensions the theorem implies that on an open connected set, there exist vectors $\vec{A}_i$ such that the one-dimensional values $x_i - \vec{A}_i \cdot f$ are contractions with capacity c less than $1/\sqrt{n}$. That is, the functions $\vec{A}_i \cdot f$ have a derivative with respect to x_i between $1 + \frac{1}{\sqrt{n}}$ and $1 - \frac{1}{\sqrt{n}}$. The above theorems (but not the corollary just stated) can be generalized to any complete normed vector space, including an infinite dimensional space like the set of all continuous functions.

2.2. Algorithms for finding global inverses.

These are yet to be determined.

3. Applications to the Inverse Conductivity Problem

Theorem I implies that the inverse conductivity problem can be solved if the voltage-to-current map Λ_γ being zero implies that all the conductivities are zero. The reference [1] gives conditions on recoverability of such a graph. However, if the voltage-to-current map is of a different form, but satisfies the conditions of Theorem II, (more examples are yet to be added - square conductivities, polynomial conductivities, etc.) the original values can be recovered. This may have some significance in biology.

References

[1] Jeff Russell, *Star and K Solve the Inverse Problem*, 2003.

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