Ordinary Differential Equations on Networks

1: THE DIRICHLET PROBLEM.

Definition: An ***n*th order network** N is a triple (V, V_B, E) together with an n+1-tuple of functions $\{h_0, h_1, \ldots, h_n\}$ satisfying: For all edges e in E, there exists an i such that $h_i(e) > 0$. The i-Kirchhoff matrix of an nth-order network is:

$$K_{ij}^i = \begin{cases} h_i(ij) & i \overset{i}{\sim} j \\ 0 & i \neq j, i \,!\overset{i}{\sim} j \\ -\sum\limits_{\ell \overset{i}{\sim} i} h_i(\ell i) & i = j \end{cases}$$

Since K^i is always symmetric, it can be represented in block-matrix form:

$$K^i = \begin{pmatrix} A^i & B^i \\ B^{iT} & C^i \end{pmatrix}$$

A C^n function $\boldsymbol{u} = (\boldsymbol{u}_{Boundary}, \boldsymbol{u}_{Interior}) : \mathbb{R} \to \mathbb{R}^{\#(V)}$ is a **potential** if it satisfies:

$$\sum_i K^i \frac{d^i \boldsymbol{u}}{dt^i} = \boldsymbol{x}_{Boundary}$$

where $\boldsymbol{x}_{Boundary}$ can be any vector whose interior values are always zero. A **network of resistors and capacitors** is a 1st-order network. (This can also be generalized to directed graphs, where the K^i's may not be symmetric: this is useful when converting higher-order networks to simpler 1st-order ones.)

The notation for first-order networks will be simplified as follows:

$$A^0 \Rightarrow A, B^0 \Rightarrow B, C^0 \Rightarrow C, A^1 \Rightarrow D, B^1 \Rightarrow E, C^1 \Rightarrow F$$

Therefore, the equation for Kirchhoff's law can be written as:

$$A\boldsymbol{u}_{Boundary} + B\boldsymbol{u}_{Interior} + D\boldsymbol{u}'_{Boundary} + E\boldsymbol{u}'_{Interior} = \boldsymbol{x}_{Boundary}$$

$$B^T\boldsymbol{u}_{Boundary} + C\boldsymbol{u}_{Interior} + E^T\boldsymbol{u}'_{Boundary} + F\boldsymbol{u}'_{Interior} = \boldsymbol{0}$$

The second equation yields: $\boldsymbol{u}'_{Interior} + F^{-1}C\boldsymbol{u}_{Interior} = -F^{-1}B^T\boldsymbol{u}_{Boundary} - F^{-1}E^T\boldsymbol{u}'_{Boundary}$ if F is invertible. Uniqueness of the Dirichlet problem can be considered by looking at the initial value problem $\boldsymbol{u}'_{Interior} + F^{-1}C\boldsymbol{u}_{Interior} = -F^{-1}B^T\boldsymbol{u}_{Boundary} - F^{-1}E^T\boldsymbol{u}'_{Boundary}$ with $\boldsymbol{u}_{Interior}(0)$ being the h_0-harmonic extension of $\boldsymbol{u}_{\mathrm{Boundary}}$. The solution will be unique if $F^{-1}C$ is a diagonalizable matrix.

1) Networks with Only Capacitors in the Interior

F is nonsingular if at least one node is connected by a capacitor to a boundary node. C is nonsingular if all interior nodes are connected by at least one resistor to a boundary node. See the special case in section 2.

2) Networks with Some Resistors and Some Capacitors

Case I: Suppose that the boundary nodes of the graphs are connected by resistors alone to the interior nodes, and the boundary nodes are not connected to other boundary nodes.

Definitions: 1) V_0, E_0 are the vertices that are connected to a resistor, and the edges that have zero h_0 value. (V_1, E_1 are defined similarly). The associated graphs are called G_0, G_1. By definition of network of resistors and capacitors, $G_0 \cup G_1 = N$. However, $G_0 \cap G_1$ does not have to be the null set, since an edge can be a resistor parallel to a capacitor.
2) The adjoint network of (V, V_B, E, h_0, h_1) is $N_{adj} = (V, V_B, E, h_1, h_0)$.

Note that G_0 does not have to be a connected graph. Let n denote the number of connected components of G_0, ordering them $\mu_1, ..., \mu_n$. Order the vertices of N as follows:

$$V_B \cap V(\mu_1), ..., V_B \cap V(\mu_n), V_B \cap V \setminus V_0, V_{\text{int}} \cap V(\mu_1), ..., V_{\text{int}} \cap V(\mu_n), V_{\text{int}} \cap V \setminus V_0.$$

This is called the *standard order*. If $V_{\text{int}} \cap V \setminus V_0 = \varnothing$, then all the diagonal terms in C are nonzero. Also note that the matrix C can be written in block matrix form:

$$C = \begin{array}{cc} & \begin{array}{cccc} \mathbf{1} & \cdots & n & \textbf{extras} \end{array} \\ \begin{array}{c} \mathbf{1} \\ \cdots \\ n \\ \textbf{extras} \end{array} & \begin{bmatrix} M_1 & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \ddots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & M_n & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{bmatrix} \end{array} = \begin{array}{cc} & \begin{array}{cccc} \mathbf{1} & \cdots & n & \textbf{extras} \end{array} \\ \begin{array}{c} \mathbf{1} \\ \cdots \\ n \\ \textbf{extras} \end{array} & \begin{bmatrix} \Sigma_1 \Lambda_1 \Sigma_1^T & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \ddots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \Sigma_n \Lambda_n \Sigma_n^T & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{bmatrix} \end{array}$$

where each Σ_i is orthogonal and each Λ_i is a real diagonal matrix. Note that if a μ_n is purely inside a network, then the associated matrix M_n is singular because it is a Kirchhoff matrix. Also note that $\sigma(C) = \bigcup_i \sigma(M_i)$. Therefore, if it is impossible to find a Λ_i, Λ_j with $i \neq j$ that have common eigenvalues, then C is diagonalizable (the spectral theorem). Since there are no capacitors connecting the interior vertices with the boundary vertices, F is a Kirchhoff matrix and therefore singular. There may or may not be time dependence in the solution. However, if all interior nodes are connected to the boundary by at least one resistor and one capacitor, then $F^{-1}C$ will be nonsingular. (This only supposes that there are no zero eigenvalues and doesn't take diagonalization into account - the product of two diagonalizable matrices does not have to be diagonalizable.)

In the special case of a network with only resistors on the inside and a fixed number of capacitors connected from all interior vertices to a boundary vertex, C is diagonalizable (since it is symmetric) and F is a multiple of the identity matrix, so $F^{-1}C$ is diagonalizable. The same is true for a network with only capacitors on the inside and a fixed number of resistors connected from all interior vertices to a boundary vertex. (What if F is diagonal?)