THE NEUMANN-TO-DIRICHLET MAP DETERMINES CONNECTION STRUCTURE
FOR CIRCULAR PLANAR NETWORKS

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Fix a circular planar network \( \Gamma = (G, \gamma) \), where the underlying graph \( G \) is an ordered pair \((V, E)\), where \( V \) is disjointly partitioned as \( \partial V \cup \text{int} \ V \). Let \( H \) denote the Neumann-to-Dirichlet map, which maps a boundary voltage to the unique normalized boundary current. Consider the two following theorems.

**Theorem 1.** Let \( P = (p_1, \ldots, p_k) \) and \( Q = (q_1, \ldots, q_k) \) be two sequences of boundary nodes. Define

\[
L = \begin{pmatrix}
(\eta_{p_1,q_1} - \eta_{p_1,q_k}) & \cdots & (\eta_{p_1,q_{k-1}} - \eta_{p_1,q_k}) & (\eta_{p_k,q_1} - \eta_{p_k,q_k}) \\
(\eta_{p_1,q_1} - \eta_{p_1,q_k}) & \cdots & (\eta_{p_1,q_{k-1}} - \eta_{p_1,q_k}) & (\eta_{p_k,q_1} - \eta_{p_k,q_k}) \\
\vdots & \ddots & \ddots & \vdots \\
(\eta_{p_1,q_1} - \eta_{p_1,q_k}) & \cdots & (\eta_{p_1,q_{k-1}} - \eta_{p_1,q_k}) & (\eta_{p_k,q_1} - \eta_{p_k,q_k}) \\
0 & \cdots & 0 & \eta_{p_k,q_1} - \eta_{p_k,q_k}
\end{pmatrix}
\]

If \( L \) is singular, there does not exist a connection between \( P \) and \( Q \).

**Proof.** To start off, some notation: let \( S = \partial V \setminus (P \cup Q) \). Assume that \( L \) is singular. Then there exists \( x = (x_1 \ldots x_k) \) such that \( Lx = 0 \). It follows that

\[
\begin{pmatrix}
(\eta_{p_1,q_1} - \eta_{p_1,q_k}) & \cdots & (\eta_{p_1,q_{k-1}} - \eta_{p_1,q_k}) & (\eta_{p_k,q_1} - \eta_{p_k,q_k}) \\
(\eta_{p_1,q_1} - \eta_{p_1,q_k}) & \cdots & (\eta_{p_1,q_{k-1}} - \eta_{p_1,q_k}) & (\eta_{p_k,q_1} - \eta_{p_k,q_k}) \\
\vdots & \ddots & \ddots & \vdots \\
(\eta_{p_1,q_1} - \eta_{p_1,q_k}) & \cdots & (\eta_{p_1,q_{k-1}} - \eta_{p_1,q_k}) & (\eta_{p_k,q_1} - \eta_{p_k,q_k}) \\
0 & \cdots & 0 & \eta_{p_k,q_1} - \eta_{p_k,q_k}
\end{pmatrix}x = 0.
\]

Next, it follows that

\[
\begin{pmatrix}
\eta_{p_1,q_1} & \cdots & \eta_{p_1,q_{k-1}} & \eta_{p_1,q_k} \\
\vdots & \ddots & \ddots & \vdots \\
\eta_{p_k,q_1} & \cdots & \eta_{p_k,q_{k-1}} & \eta_{p_k,q_k}
\end{pmatrix}x = \alpha e,
\]

where \( \alpha = \begin{pmatrix} \eta_{p_k,q_1} - \eta_{p_k,q_k} & \cdots & \eta_{p_k,q_{k-1}} - \eta_{p_k,q_k} \end{pmatrix} \) and \( e \) is the ones vector of appropriate length. Finally, it follows that

\[
\begin{pmatrix}
\eta_{p_1,q_1} & \cdots & \eta_{p_1,q_k} \\
\vdots & \ddots & \vdots \\
\eta_{p_k,q_1} & \cdots & \eta_{p_k,q_k}
\end{pmatrix}y = \alpha e,
\]

where

\[
y = \begin{pmatrix} x_1 \\
\vdots \\
x_{k-1} \\
-\sum_{j=1}^{k-1} x_j \end{pmatrix}.
\]

It follows that there exists \( y \neq 0 \) such that \( H(P; Q) y = \alpha e \), with the property that the element sum over \( y \) is zero.

This work implies that

\[
\begin{pmatrix}
H(P; P) & H(P; S) & H(P; Q)
\end{pmatrix} \begin{pmatrix} 0 \\
0 \\
y \end{pmatrix} = \alpha e.
\]
This equation in turn implies that

\[
\begin{pmatrix}
H(P;P) & H(P;S) & H(P;Q) \\
H(P;S)^T & H(S;S) & H(S;Q) \\
H(P;Q)^T & H(S;Q)^T & H(Q;Q)
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
y
\end{pmatrix}
= \begin{pmatrix}
0 \\
z_1 \\
z_2
\end{pmatrix} + \alpha e,
\]

where \(z_1 = H(S;Q)x - \alpha e\) and \(z_2 = H(Q;Q) - \alpha e\). That is,

\[
H\begin{pmatrix}
0 \\
y
\end{pmatrix}
= \begin{pmatrix}
0 \\
z_1 \\
z_2
\end{pmatrix} + \alpha e.
\]

Note that since the element sum over \(y\) is zero, we have shown that the potential \((0 \ z_1 \ z_2)^T + \alpha e\) solves the Neumann problem for the input current \((0 \ 0 \ y)^T\). Since no normalization is required for the Dirichlet problem, it follows that the potential \((0 \ z_1 \ z_2)^T\) induces the current \((0 \ 0 \ y)^T\). It follows that the vector \((z_1 \ z_2)^T\) is nonzero: if it was the zero vector, the input potential \((0 \ z_1 \ z_2)^T\) would be constant, which would imply that the output current \((0 \ 0 \ y)^T\) is constant, which would imply that \(y\) is the zero vector.

This work shows that the equation

\[
\Lambda \begin{pmatrix}
0 \\
z_1 \\
z_2
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
y
\end{pmatrix}
\]

holds, which implies that

\[
\Lambda (P,S;S,Q) \begin{pmatrix}
z_1 \\
z_2
\end{pmatrix} = 0.
\]

Since \((z_1 \ z_2)^T\) is nonzero, it follows that \(\Lambda (P,S;S,Q)\) is singular. The connection-determinant formula implies that there does not exist a connection between \(P\) and \(Q\).

**Theorem 2.** Let \(P = (p_1, \ldots, p_k)\) and \(Q = (q_1, \ldots, q_k)\) be two sequences of boundary nodes. Define \(L\) in the previous fashion. If \(L\) is nonsingular, there exists a connection between \(P\) and \(Q\).

**Proof.** Let’s prove the contrapositive of the claim. Assume that there does not exist a connection between \(P\) and \(Q\). Note that the proof of the previous claim is reversible; that is, under the assumption that there does not exist a connection between \(P\) and \(Q\), the previous argument implies that \(L\) is singular. The contrapositive follows.

The title of this paper is now justified.