Disallowed Connections on a Genus-g surface

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Abstract

The Jordan Curve Theorem is an indispensable tool when dealing with graphs on a planar, or genus zero, surface. In an attempt to acquire a similarly useful tool for a genus-g surface, the set of paths between two sets of 2(g+1) points are studied. It is found that some paths cannot occur on the genus-g surface.

1 Genus Zero

Theorem 1 *Any simple closed curve C divides the points of the plane not on C into two disjoint domains of which C is a common boundary.*

This is known as the Jordan Curve Theorem. In generalizing to surfaces of higher genus, however, the theorem will be formulated as follows:

Theorem 2 *Let B be a circle inscribed on a sphere, D the disk enclosed by the circle, and X, Y disjoint, open intervals of B. Let $x_1 < x_2 < y_1 < y_2 < x_1$ be points in circular ordering on B such that $x_1, x_2 \in X$, $y_1, y_2 \in Y$. If $C_1 \in D$ is a curve that joins x_1 to y_1 and $C_2 \in D$ is a curve that joins x_2 to y_2, then C_1 intersects C_2.*

In other words, if $\{x_1, x_2\}$ is connected to $\{y_1, y_2\}$ with disjoint arcs, then $x_1 \leftrightarrow y_2$ and $x_2 \leftrightarrow y_1$. With interest in the recovery of electrical networks, we could now proceed to prove the Cut Point Lemma and study the recovery of genus 0, circular planar, graphs. The search for an analog to the Jordan Curve Theorem in higher genus surfaces takes this shape of nonintersecting curves join the curves on a circle.

2 Disallowed Connections

2.1 Preliminaries

Let R be a genus-g, orientable surface. Let D be a disk in R, with $B = \partial D$, and let $2d$ points be placed in some order on the circle, B. Let X and Y be disjoint, connected subsets of B such that d of the points are in X and the other d are in Y. We are interested in when the d points in X may be connected to the d points in Y by d disjoint arcs through $R - D$, giving an allowed connection, and when they may not, giving a disallowed connection.

There are many angles from which to view the problem.

1. Imagine the circle B enclosing a region with g handles, with the simply connected region outside of the circle. This visualization is useful for drawing the connection and for seeing how "entangled" the curves between points are.

2. Imagine B enclosing D, while outside is the surface with g holes. Often it is easier to draw the connection with this viewpoint, though it is harder to see the significance of the connection.

3. Do away with the circle B and imagine all $2d$ points arranged in a circle around an additional point which is connected to all of them. Since B encloses a simply connected region on one side, this can always be done. In effect, the circle B is diminished to a point with $2d$ rays coming out. This visualization is the best way to see if the connection is disallowed or not, as well as the only method of proof, using the Euler Characteristic.

2.2 Methods of Proof

It is simple to prove that a connection is allowed on a genus-g surface; just draw the connection. Less obvious is the method to prove that a connection is disallowed. The method used in this paper is tracing faces and using the Euler Characteristic to give the genus of a graph. In order to use this formula, however, we must first ignore the surface R, and concentrate locally on D. Think of the $2d$ points, with the additional point in D, and the total of $3d$ edges as a graph. There is a defined rotation system for each point, so the graph has a combinatorial, cellular embedding in some orientable

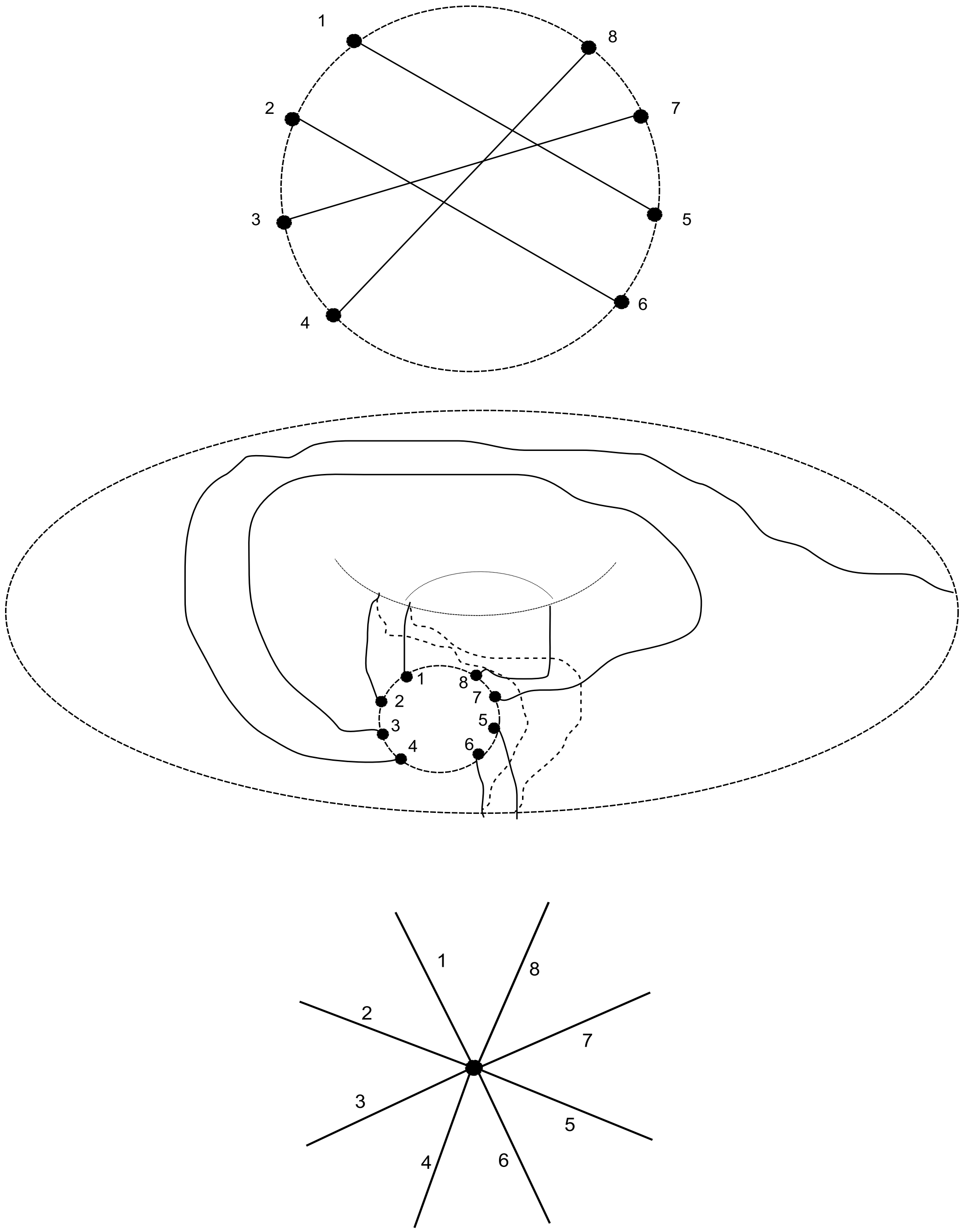

Figure 1: Three visualizations

Riemann surface, S, of genus h. This genus may be found using the Euler Characteristic, but first the number of faces is needed.

Draw the connection in the third manner listed above, and label the rays coming out of the center point in the following way: starting with the most clockwise point in the set X, label the rays counterclockwise 1 to d, then in Y, label the ray linked by an arc to i in the first set by $d+i$. Let the permutation $\sigma(d+1, d+2, \ldots, 2d)$ be the counterclockwise order of the points in Y. Now starting at the corner clockwise of 1, trace 1 out, on the clockwise side. Since the surface is orientable, the trace will be on the counterclockwise side when it comes back in by the $d+1$ ray. Reaching the angle, continue to the edge counterclockwise of $d+1$ and repeat. Upon returning to the angle clockwise of 1, a face has been traced. Proceed to the next face until all faces have been traced, all angles touched.

With the number of faces, the Euler Characteristic can now be calculated. The number of vertices is $2d$ plus 1 for the added center node. The number of edges is $3d$.

$$V - E + F = 2 - 2h$$

$$2d + 1 - 3d + F = 2 - 2h$$

$$F = 1 + d - 2h$$

Of course, there is at least one face.

$$1 \leq 1 + d - 2h$$

$$d \geq 2h$$

Now since we are concerned with the connections that cannot exist on a surface of genus g, the number of edges d which will be of interest are those for which the graph may be embedded on a genus higher, $h = g + 1$, so $d \geq 2(g + 1)$. Take $d = 2(g + 1)$, the least number of points for which there will be disallowed connections. Then if there is one face, the connection is disallowed. If there is more than one face, then there must be at least three.

$$\begin{aligned} 1 < F = 1 + 2(g + 1) - 2h &\Rightarrow 0 < 2(g + 1 - h) \\ &\Rightarrow 2 \leq 2(g + 1 - h) = F - 1 \Rightarrow F \geq 3 \end{aligned}$$

Thus if the number of faces is three or greater, a connection is allowed. This counting of faces is the method by which a connection is proven to be disallowed.

2.3 Example

Proof that the (1234) $\leftrightarrow$ (5678) four connection is disallowed on the torus with a (12345678) circular order.

Upon tracing the faces, it is found that one face is 15-62-37-84-51-26-73-48. This is the only face, since every edge has been traversed twice. Thus this connection can only occur on genus-2 or above. It is disallowed on the torus.

3 Results

A program has been written to trace the faces explained above. It is similar to the programs written in [1] and [2], but attuned to this specific graph for speed. Results of the calculation are as follows:

Genus	0	1	2	3	4	5
Total Connections	2	24	720	40,320	3,628,800	479,001,600
Disallowed Connections	1	8	180	8,064	604,800	68,428,800
Disallowed over Total	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
Computation Time	–	–	–	3.3 s	7.2 min	14.3 hr

Of course, the fourth row is curious. Does it hold that for a genus-g surface, the disallowed connections will make up $\frac{1}{g+2}$ of the total connections, or $\frac{(2(g+1))!}{g+2}$ connections? This is an open question right now, though the next tool should help to answer.

4 Cyclic Elements

Let $d = 2(g+1)$ be half the number of points on the circle, so that $\sigma(d+i)$ is the ith counterclockwise point in Y. Define the following permutations.

$$p = \begin{pmatrix} 1 & 2 & \dots & d & \sigma(d+1) & \dots & \sigma(2d-1) & \sigma(2d) \\ 2 & 3 & \dots & \sigma(d+1) & \sigma(d+2) & \dots & \sigma(2d) & 1 \end{pmatrix} \tag{1}$$

$$q = \begin{pmatrix} 1 & 2 & \dots & d & d+1 & \dots & 2d-1 & 2d \\ d+1 & d+2 & \dots & 2d & 1 & \dots & d-1 & d \end{pmatrix}$$

The permutation p takes a point to the next point in the counterclockwise order on the circle, while q takes a point to the point to which it is connected. Thus tracing a face of the graph from the angle 34 can be represented as finding the smallest m such that $(q \circ p)^m(3) = 3$. Then m is the number of angles on the face traced. In general, a connection is allowed if

$$\exists x, \exists m \text{ s.t. } (q \circ p)^m(x) = x \quad \text{with } 0 < m < 4(g+1)$$

This is because if there is a cyclic element of order less than $4(g+1)$, then there is a face which does not cover everything, and thus there is more than one face. However, if there is more than one face, then there are at least three of them. So one face will have at most one third of the total angles.

$$\exists x, \exists m \text{ s.t. } (q \circ p)^m(x) = x \Rightarrow F > 1 \Rightarrow F \geq 3$$

$$\exists x, \exists m \text{ s.t. } (q \circ p)^m(x) = x \quad \text{with } 0 < m \leq \frac{4}{3}(g+1) \tag{2}$$

The condition for a disallowed connection is the negation.

$$\forall x, \forall m, \ (q \circ p)^m(x) \neq x \quad \text{if } 0 < m \leq \frac{4}{3}(g+1) \tag{3}$$

Of interest is $q \circ p$, obviously. Writing it explicitly, we have

$$q \circ p = \begin{pmatrix} 1 & 2 & \ldots & d & \sigma(d+1) & \ldots & \sigma(2d-1) & \sigma(2d) \\ d+2 & d+3 & \ldots & q(\sigma(d+1)) & q(\sigma(d+2)) & \ldots & q(\sigma(2d)) & d+1 \end{pmatrix} \tag{4}$$

5 Open Questions

1. Does the relation between the number of disallowed paths and the number of total paths hold for arbitrary genus, and if so, why?

2. Is there a way to simplify the relation $q \circ p$, so that a connection can easily be seen to be allowed or disallowed just from the permutation σ?

3. Can these disallowed connections be used in some way to help acquire a property analogous to the Cut Point Lemma for genus-g graphs?

Answers, especially to the third item, are eagerly awaited.

6 Thanks

I'd like to thank David Ingerman for giving me the idea of thinking about connecting opposite points on the boundary circle, Jaime for giving counterexamples to a couple of my conjectures, Owen and Nick for the method of proof, Tom for the permutation program, and Dr. Morrow helping with the paper and giving me this opportunity.

References

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[2] N. Reickert. Generalized Circular Medial Graphs. University of Washington, August 2004.

[3] B. Curtis, and J.Morrow. Inverse Problems for Electrical Networks. Series on applied mathematics Vol. 13. World Scientific, 2000.