HOW TO CREATE VALID MEDIAL GRAPHS ON GENUS N RIEMANN SURFACES

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Abstract. I will show that a boundary circle with set of geodesics on a Riemann surface create regions that are two colorable if and only if each side of a 4n-gon that represents the surface of genus n has an even number of intersections with geodesics.(See section 2 of Nick Reichert's paper [3] for an explanation of how genus n surfaces can be represented as a 4n-gons).

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1. Introduction

It is necessary that the theorem gives a condition for two colorability which is independent of the planar representation of an embbedded graph. It might be hard to see that the following condition – the evenness of the number of geodesics-side intersections in a $4n$-gon planar representation – is indeed independent of the choice of representation. I will justify this claim briefly here (though it follows immediately from the theorem because two colorability is independent of planar representation, so anything equivalent to colorability must be independent of representation as well). Fix some $4n$-gon planar representation of a set of geodesics (where the boundary circle is assumed to be somewhere in the middle). As an example of how the representation can change, move the top side toward the boundary circle and the bottom side away from it. This is shown in Figure 1. If the new edge encounters a section of a geodesic that did not initially go through the side, then the geodesic must have initially looped back toward the middle too soon to intersect the side. Therefore, in the new representation it must intersect the translated side not once more, but twice more. The number of intersections would increase by two, which does not change the evenness. Likewise, if a translated side has one fewer geodesic intersection, it is actually has two fewer. This analysis works for every set of sides, so it works for any set of translations, so the claim is true.

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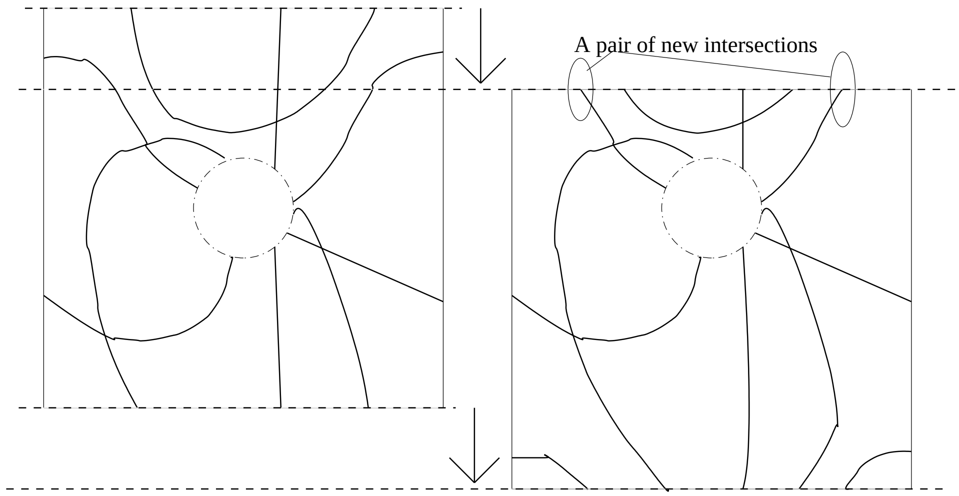

FIGURE 1. As the top edge is translated to the middle (so the bottom edge is translated away), new intersections with geodesics come in twos

2. Main Theorem and Useful Corrollary

Theorem 2.1. *Pseudo lines on a genus n Riemann surface that intersect at valence 4 verticies create a 2-colorable graph* $\iff$ *a representation of the surface as a 4n-gon shows each side intersecting pseudo-lines an even number of times.*

Proof. First, observe in Nick Reichart's paper that any genus n Riemann surface can be represented as a $4n$-gon with pairs of identified sides. He does it by taking n square representations of the torus, gluing them together at a corner, and opening the n squares to make a $4n$-gon. The 4 vertices of each square were identified, and each square has a vertex that ends up overlapping with a vertex of the square in the clockwise direction, so all verticies are identified.

$\Leftarrow$ If a planar representation of the surface shows pseudo-lines going through each side an even number of times, then since the endpoints of each side are identified the pseudo-lines create an even number of intervals. Therefore, the boundary of the $4n$-gon can be two colored such that identified sides match. Now we can ignore the fact that the $4n$-gon represents a Riemann surface and simply try to two color a polygon in the plane that has a two-colorable boundary and is divided up into regions by curves that intersect at valence four vertices and end only at the boundary. Following Morrow and Curtis's logic [4] on page 125, the curves can be made into pseudo-lines again by connecting adjacent endpoints. (See Figure 2 part B). By [1] Theorem 6.1.3, these region are two colorable, so the original surface is two colorable.

$\Rightarrow$ See Figure 3 for an example of an odd number of intersections. If the pseudo lines create a two colorable graph, then every planar representation of the graph is two colorable, so the boundary of a 4n-gon planar representation is two colorable, so each side on the 4n-gon must be two colorable, so each side must be divided into an even number of intervals, so pseudo lines must go through each side an even number of times. $\square$

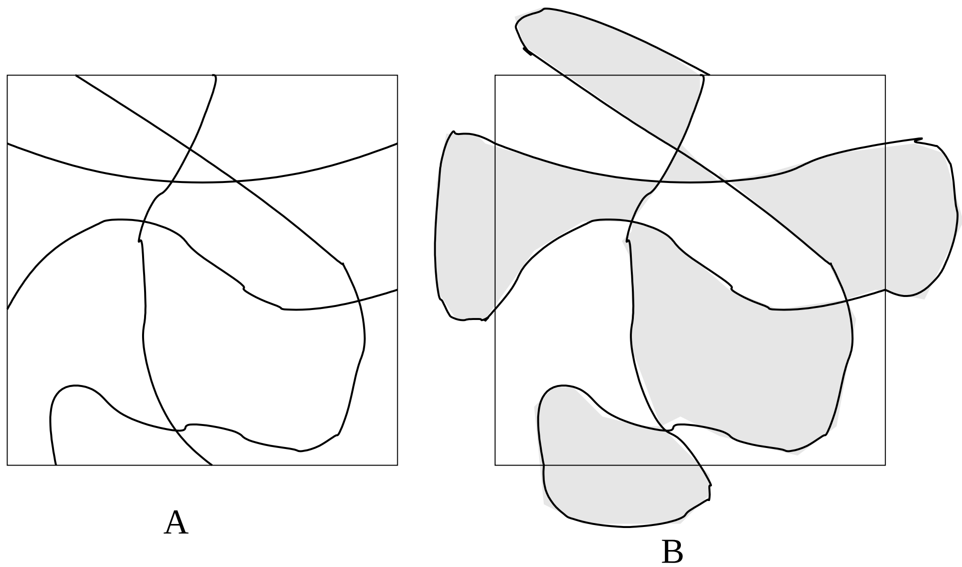

FIGURE 2. By [1], anything like figure B can be 2-colored

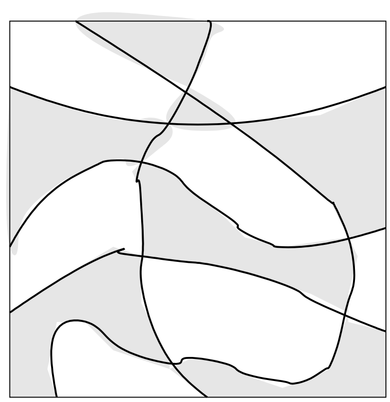

FIGURE 3. This looks like a two-coloring, but the top edge and the bottom edge are supposed to be the same. In fact this toroidal set of geodesics create non-two-colorable regions.

Corollary 2.2. *Every 4n-gon representation of a medial graph of a graph that is circularly embedded on a genus n Riemann surface contains geodesics that go through each side an even number of times (as long as the boundary circle is somewhere in the middle of the 4n-gon).*

Proof. Geodesics touch the boundary circle an even number of times: two for each boundary node. Just as in Figure 2, connecting each intersection to an agacent intersection makes the geodesics into pseudo-lines that intersect that boundary circle at valence 4 intersections. The boundary circle is also a pseudo-line. The original medial graph is two colorable, so the new set of pseudo lines is two colorable.

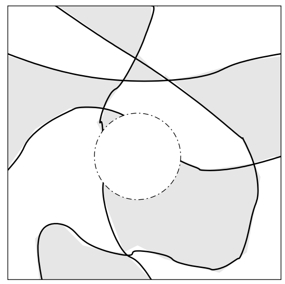

FIGURE 4. This two colorable circularly embedded set of geodesics is two-colorable and both black and white supercellular, so it comes from two different graphs (which are dual to each other).

By Theorem 2.1, pseudo-lines must intersect each side at an even number of places. As long as the boundary circle is somewhere in the middle of the 4n-gon, adding line segments in the middle of the boundary circle has no effect on the number of times geodesics or pseudo-lines intersect each side. Therefore, the original geodesics must also intersect each side an even number of times. □

3. Real live medial graphs

Two colorability is not a sufficient condition for a set of geodesics to correspond to a real medial graph. The only other condition is that one set of boundary regions (either the blacks or whites) must contain regions that are all homeomorphic to discs and which do not touch the boundary circle more than once. In their papers, Rachel[6] and Ming[5] call this $black - supercellular$. If this holds, Curtis and Morrow's description in section 8.2 [4] of how to draw a graph given a medial graph will give a well defined graph whose medial graph is indeed the original set of geodesics. (Here's my quick description of recovering a graph from a medial graph (you can try it on Figure 4): Place a boundary node in every boundary region, and an interior node in every interior face. Then consider each node in succession. Draw a line segment from the node to each vertex of the face that surrounds it, omitting verticies that are on the boundary circle. In this way, you will draw complete edges that go through all the intersection points of the medial graph.)

If a boundary region touches the boundary circle twice (see Figure 5), the graph will be poorly defined because the boundary node could be placed in one of two intervals. Worse, none of the possible graphs will have a medial graph that is the original set of geodesics. If a boundary region is a sort of cylinder, then the geodesics cannot be a medial graph of any graph because medial graphs are constructed so that they fence off a region around each boundary node with line segments that stay on a region of the Riemann surface that is homeomorphic to a disc.

References

[1] O. Ore, *The Four-color problem*, Academic Press, N. Y. (1965)
[2] Sam, Jeff, Ernie. "A LaTeX Starter File." 2003.

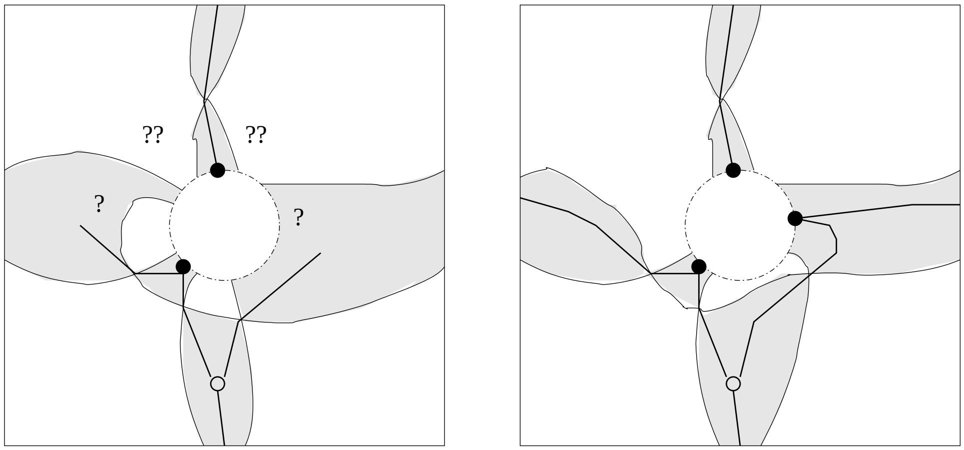

Figure 5. This set of geodesics is not black or white supercellular because the ?? region and ? region each touch the boundary circle more than once. It is unclear where the boundary node should be placed in reconstructing either the black or white graph. Worse, after picking the right side for the boundary node, the graph's medial graph is not the original set of geodesics!!

[3] Reichert, Nick. "Generalized Circular Medial Graphs." 2004.
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[5] Li, Ming. "Thoughts on Cut Point Lemma." 2006 .
[6] Bayless, Rachel. "Analyzing Non-planar Medial Graphs." 2006 .

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